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Editors

Arithmetic and Geometry Around Galois Theory

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Preface

This Lecture Notes volume is a fruit of two research-level summer schools jointly organized by the GTEM node at Lille University and the team of Galatasaray University (Istanbul). Both took place in Galatasaray University: "Geometry and Arithmetic of Moduli Spaces of Coverings" which was held between 09–20 June, 2008 and "Geometry and Arithmetic around Galois Theory" which was held between 08–19 June 2009. The second summer school was preceded by preparatory lectures that were delivered in TÜBİTAK Feza Gürsey Institute.

A group of seventy graduate students and young researchers from diverse countries attended the school.

The full schedules of talks for the two years appear on the next pages.

The schools were mainly funded by the FP6 Research and Training Network Galois Theory and Explicit Methods (GTEM) and the Scientific and Technological Research Council of Turkey (TÜBİTAK). Funding provided by the International Mathematical Union (IMU) and the International Center for Theoretical Physics (ICTP) have been used to support participants from some neighbouring countries of Turkey. We are also thankful to Galatasaray University and to University of Lille 1 for their support. Feza Gürsey Institute gave funding for the preparatory part of the summer school. The last named editor has been funded by TÜBİTAK grants 104T136 and 110T690 and a GSU Research Fund Grant during the summer school and the ensuing editorial process.

This volume focuses on geometric methods in Galois theory. The choice of the editors is to provide a complete and comprehensive account of modern points of view on Galois theory and related moduli problems, using stacks, gerbes and groupoids. It contains lecture notes on étale fundamental group and fundamental group scheme, and moduli stacks of curves and covers. Research articles complete the collection.

J. Bertin's paper, "Algebraic stacks with a view toward moduli stacks of covers", is an introduction to algebraic stacks, which focuses on Hurwitz schemes and their compactifications. It intends to make available to a large public the use of stacks gathering in a unified presentation most of the elements of the theory. Its goal is to study the moduli stacks of curves and of covers, which is the central theme of this collection of articles.

M. Romagny's article on "Models of curves" is a detailed account of the proof of Deligne-Mumford on semi-stable reduction of curves with an application to the

study of Galois covers of algebraic curves. The author provides all the concepts and necessary ground making possible for the reader to understand the proof of the main theorem, supplying some complementary arguments, which are stated without proof in Deligne-Mumford's paper. The last part of the article is devoted to the problem of reduction of tamely ramified covers of smooth projective curves.

In her article on "Galois categories", A. Cadoret aims at giving an outline of the theory of the étale fundamental group that is accessible to graduate students. Her choice is to present the Grothendieck's theory of Galois categories in full generality, giving a detailed and self-contained proof of the main theorem not relying on Grothendiecks pro-representability result of covariant lim-compatible functors on artinian categories. The main example is that of the category of étale finite covers of a connected scheme, to which the rest of the article is devoted. All main theorems of the subject are proved in the paper, which contains also a complete description of the fundamental group of abelian varieties. Let us mention a very useful digest of descent theory given in appendix.

As a Galois category is equivalent to the category of continuous finite Π-sets for some profinite group Π, a Tannaka category is equivalent to the category of finite-dimensional representations of some affine pro-algebraic group. M. Emsalem's article on "Fundamental groupoid scheme" is an overview of the original construction by Nori of the fundamental group scheme as the Galois group of some Tannaka category $EF(X)$ (the category of essentially finite vector bundles) with a special stress on the correspondence between fiber functors and torsors. Basic definitions and duality theorem in Tannaka categories are stated, making the material accessible to non specialists. A paragraph is devoted to the characteristic 0 case and to a reformulation of Grothendieck's section conjecture in terms of fiber functors on $EF(X)$. Although this formulation is known from specialists, no complete reference was available.

Classically the structure theorem on the étale fundamental group of a curve is obtained by comparison with the topological fundamental group over $\mathbf{C}$.N. Borne's article on "Extension of Galois groups by solvable groups, and application to fundamental groups of curves" gives an account of the description of the pro-solvable p'-part of the étale fundamental group on an affine curve by purely algebraic means. The method inspired by Serre's work on Abhyankar's conjecture for the affine line relies on cohomological arguments, which are completely explained in the article, with a special stress on the Grothendieck-Ogg-Shafarevich formula.

The fundamental group scheme of a scheme X is an inverse limit of torsors under finite group schemes. In the context of Galois theory of étale fundamental group, a finite étale morphism $Y \to X$ has a Galois closure. The question addressed by M. Garuti in his article on "Galois Closure for finite morphism" is to characterize, in the case of positive characteristic, which finite morphisms are dominated by a torsor under a finite group scheme, thus what finite morphisms benefit from a "Galois Closure" in the context of Nori's fundamental group scheme. The article, which gives a complete satisfactory answer, recalls all the necessary material to get to the main theorem.

Cohomology which was a main tool in Borne's paper, is the core of J.-C. Douai article "Hasse principle and cohomology of groups". But here occurs non abelian cohomology: precisely, H^1 and H^2 of semi-simple groups defined over $K = k(X)$, where k is a pseudo-algebraically closed field and X a proper smooth curve over k. The main result is the fact that the non-abelian H^2 of a semi-simple simply connected group whose center has an order prime to the characteristic of k consists in neutral classes.

With the article "Periods of mixed Tate motives, examples, ℓ-adic side" by Z. Wojtkowiak, it is the motivic side of the area that comes into play. One hopes that the $\mathbf{Q}$-algebra of periods of mixed Tate motives over $\mathrm{Spec}(\mathbf{Z})$ is generated by values of iterated integrals on $\mathbf{P}^1(\mathbf{C}) \setminus \{0, 1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ (some numbers also called multiple zeta values). Assuming the motivic formalism, some variant of this is proved, and is then further studied in the ℓ-adic Galois setting. Numerous examples are given that provide some ground for future research in this direction.

The article "On totally ramified extensions of discrete valued fields" of L. Bary-Soroker and E. Paran is devoted to a more arithmetical aspect. In the context of Artin-Schreier field extensions, they revisit and simplify a criterion for a discrete valuation of a Galois extension E/F of fields of characteristic $p > 0$ to totally ramify. Interesting examples illustrate this criterion.

R.P. Holzapfel and M. Penkava's paper "An Octahedral Galois-Reflection Tower of Picard Modular Congruence Subgroups" studies a subgroup $\Gamma(2)$ of the Picard modular group Γ. The quotient of the complex 2-ball under this group becomes the projective plane after compactification. $\Gamma(2)$ has an infinite chain of subgroups that leads to an infinite Galois-tower of ball-quotient surfaces, making it possible to work with algebraic equations for Shimura curves, which is of importance in coding theory.

This volume has benefited very much from the precious and anonymous work of the referees. We are very grateful to them.

Finally we wish to thank all the members of the scientific committees and of the organization committees for their collaboration in the organization of the two events: Kürsat Aker (Feza Gürsey Institute), José Bertin (Institut Fourier), Özgür Kişisel (METU), Pierre Lochak (Paris 6), Hurşit Önsiper (METU), Meral Tosun (Galatasaray University), Sinan Ünver (Koç University), Zdzisław Wojtkowiak (Nice) and Stephan Wewers (Hannover). And we would like to extend our thanks to Celal Cem Sarıoğlu, Ayberk Zeytin, Neşe Yaman who also contributed at various levels to the organization during the long preparation process before and during the summer school.

October 6, 2012 Istanbul, Lille and Paris

The Editors

2008 Summer School Schedule

"Geometry and Arithmetic of Moduli Spaces of Coverings"

Lectures

Lecturer	Minicourse
Bertin, José	Introduction to stacks
Cadoret, Anna	Galois categories
Dèbes, Pierre	Foundations of modular towers, inverse Galois theory and abelian varieties
Emsalem, Michel	On the fundamental groupoid scheme
Fried, Michael	Modular towers
Garuti, Marco	p-adic representations of the fundamental group scheme
Korkmaz, Mustafa	Mapping class groups
Litcanu, Razvan	Intersection theory on algebraic stacks
Lochak, Pierre	Profinite complexes of curves and another geometric view of the GT group
Romagny, Matthieu	Models of curves
Schneps, Leila	Grothendieck-Teichmuller theory
Türkelli, Seyfi	Connected components of Hurwitz schemes and Malle's conjecture
Tossici, Dajano	Weak and strong extension of torsors
Ünver, Sinan	Multi-zeta values and the Grothendieck-Teichmuller group
Wewers, Stefan	Algebraic patching and covers of curves

2009 Summer School Schedule

"Geometry and Arithmetic around Galois Theory"

Lectures

Lecturer	Minicourse
Aker, Kürşat	Hurwitz Schemes (at FGI)
Borne, Niels	Extensions of Galois groups by solvable groups, and application to fundamental groups of curves
Cadoret, Anna	Descent theory for covers
Çakçak, Emrah	An Introduction to Algebraic Fundamental Groups (at FGI)
Dèbes, Pierre	Geometric Galois Theory: an Introduction (at FGI)
Dettweiler, Michael	Middle convolution and the Inverse Galois Problem
Feyzioğlu, Ahmet	Infinite Galois Theory (at FGI)

Fehm, Arno	Ample Fields IV
Geyer, Wulf-Dieter	Ample Fields III
Haran, Dan	Ample Fields II
İkeda, İlhan	Higher-dimensional Langlands correspondence
Jarden, Moshe	Ample Fields I
Özden, Şafak	Fields of Norms (at FGI)
Ramero, Lorenzo	Lectures on logarithmic algebraic geometry
Türkelli, Sefyi	Malle's conjecture and number of points on a Hurwitz space
Wojtkowiak, Zdzisław	Galois actions on fundamental groups and on torsors of paths

Research Talks

Speaker	**Talk Title**
Antei, Marco	On the fundamental group scheme of a family of curves
Bary-Soroker, Lior	Frobenius automorphism and irreducible specializations
Cadoret, Anna	A uniform open image theorem for ℓ-adic representations
Cau, Orlando	Irreducible components of Hurwitz spaces
Collas, Benjamin	Action on torsion-elements of mapping class groups by cohomological methods
Dettweiler, Michael	On the automorphy of hypergeometric local systems
Douai, Jean-Claude	Principe de Hasse et cohomologie des groupes
Hatami, Omid	A short talk on Class field theory
Holzapfel, Rolf-Peter	Galois reflection towers
Kim, Minyong	Diophantine geometry and fundamental groups
Mendes, Sergio	Class field theory and the principal series of SL(2)
Neftin, Danny	On arithmetic field equivalences and crossed product division
Pal, Ambrus	The real section conjecture and Smith's fixed point theorem
Paran, Elad	Power series over generalized Krull domains
Petersen, Sebastian Poineau, Jérôme	Inverse Galois problem for convergent arithmetic power series
Schmidt, Johannes	Rigid G_2 Representations and Motives of Type G_2
Türkelli, Seyfi	Homological stability of Hurwitz schemes
Yafaev, Andrei	Andre-Oort and Manin-Mumford conjectures: a unified approach

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Algebraic Stacks with a View Toward Moduli Stacks of Covers

José Bertin

Abstract. Stacks arise naturally in moduli problems. This fact was brilliantly foreseen by Mumford in his wonderful paper about Picard groups of moduli problems [47] and further amplified by Deligne and Mumford in their seminal work about the moduli space of stable curves [15]. Even if the theory of stacks is somewhat technical due to the predominance of a functorial language, it is important to be able to use stacks without a complete knowledge of all intricacies of the theory. In these notes our aim is to explain the fundamental ideas about stacks in rather concrete terms. As we will try to demonstrate in these notes, the use of stacks is a powerful tool when dealing with curves, or covers, or more generally when we are trying to classify objects with non-trivial automorphisms, abelian varieties, vector bundles etc. Many people think that stacks should be considered as basic objects of algebraic geometry, like schemes, and [62] is an example of a convincing and heavy set of notes toward this goal. We hope to show how to use them in various concrete examples, especially the moduli stack of stable pointed curves of fixed genus $g \geq 2$, with a view toward the moduli stack of covers between curves of fixed genera, the so-called *Hurwitz stacks*. Hurwitz stacks appear basically as correspondences between moduli stacks of pointed curves.

Mathematics Subject Classification (2010). 14A20, 14H10, 14H30, 14H37.

Keywords. Algebraic stack, category, covering, cover, curve, elliptic curve, groupoid, Hurwitz, node, stack, moduli space, stack.

I would like to express my warm thanks to the referee who patiently read the consecutive versions of these notes. His pertinent and constructive criticism helped me to transform a rough text into what I hope is a readable paper. I want also to thank the organizers of the school, especially M. Emsalem, for patiently waiting for the final form of the present paper.

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1. Stacks

1.1. Introduction

It is well known that schemes can be seen as covariant functors from the category of commutative rings to sets, the so-called functors of points. Indeed this formalism tells us that the functor of points defines a fully faithful embedding

$$\{\text{Schemes}\} \hookrightarrow \text{Fun}(\mathbf{Aff}^{\text{op}}, \mathbf{Set})$$

where **Aff** denotes the category of affine schemes. If such a functor is given, it is in general hard to decide whether or not it comes from a scheme. This is the so-called **representability problem**. In order to be representable, a functor must fulfill strong conditions. For example it needs to be local for the Zariski topology, in other words a Zariski sheaf, and also it must be locally representable, see Subsection 1.2.5 for the precise conditions. A basic result of Grothendieck is the fact that the functor of points of a scheme is a sheaf for a finer topology than the Zariski topology: the **fpqc topology**, and this discovery opens the path to new techniques of construction of geometric objects. A first step in this path was Artin's introduction of algebraic spaces, a class of geometric objects larger than the class of schemes but sufficiently close to deal with moduli problems. Soon after it was realized[1] that stacks, originally introduced in the setting of non-abelian cohomology, once algebraized by Deligne-Mumford and later by Artin, were genuine and useful geometric objects.

The natural functors encountered in Algebraic Geometry are often modelled on the pattern

$$A \mapsto \{\text{isomorphism classes of} \ldots \text{ over} A\}$$

but in most cases they are not representable – not even Zariski sheaves. If you take for "…" the set of projective modules of rank 1 (line bundles), then the presheaf that you obtain is not a sheaf in the Zariski sense: indeed, its stalks are all trivial. Algebraic stacks can be defined in a similar way, but now keeping the objects together with their automorphisms. The big difference is that the functor (sheaf) of points must be replaced by a sheaf in groupoids. This subtlety is due to the fact that isomorphic objects are definitely *not* identified. There is an alternative and important way to think about stacks with perhaps a more geometric flavour. A scheme in its primary definition is obtained by gluing affine schemes along local isomorphisms. Similarly, as we shall see, an algebraic stack can be defined as a quotient of a scheme by an equivalence relation, taken in a generalized sense (Section 2).

As we said before, the moduli stacks we are interested in are kind of "functors" in a sense explained below. The categorical language is obviously necessary to deal properly with these geometric objects. Basic concepts about categories and functors will be used freely, with a brief glossary in the first section to fix the notations. A stack is a category, and stacks are the objects of a 2-category, meaning

[1] On the occasion of the Deligne-Mumford proof of the irreducibility of the moduli space of genus g curves [47].

that besides 1-morphisms of stacks we will encounter homotopies, or 2-morphisms between 1-morphisms.

The prerequisite of this course is a standard knowledge in algebraic geometry, for example the first half of Hartshorne's book [33], together with some elementary facts about algebraic groups. In the second part of this course, we will freely use some basic notions about curves and covers of curves. Chapter IV of Hartshorne's book, among many others, is a very good reference for all this material. Finally almost all algebraic groups occurring in these notes are finite constant, one notable exception being the multiplicative group $\mathbf{G}_m$. It should be noted that recent and very good sets of lectures notes treat with more or less details various aspects of the recent story of stacks, the most advanced one being de Jong's rapidly growing encyclopedic online Stacks Project [62]. This will be one of our main references throughout the text.

Let us fix our conventions. Unless otherwise stated, schemes are assumed separated. Our notation for the category of schemes is **Sch**, or $\mathbf{Sch}/S$ for schemes over a base S. Working in the setting $\mathbf{Sch}/S$, the base S will often be assumed locally noetherian. In the second part of these notes dealing with curves and covers, it will be convenient to fix a ground field k (often algebraically closed), then a scheme will be a scheme over Spec k, and the corresponding category will be denoted $\mathbf{Sch}_k$. A further bit of conventions: **Ann** is the category of commutative rings, and $\mathbf{Alg}_k$ the category of finitely generated k-algebras.

I apologize in advance to a potential reader that even if the definitions presented in these notes are essentially general, our aim is a balance between general concepts and applications. The applications we have in mind focus on Deligne-Mumford stacks, especially moduli stacks of curves, and their relatives, the Hurwitz stacks. This explains why many interesting things about algebraic stacks are ignored.

1.2. Background on categories and topologies

1.2.1. Reminder on categories. For the main part, this subsection will be a glossary. All set-theoretic issues will be ignored. We refer to the chapters "Set theory" and "Categories" in [62], or to [45], for a serious discussion. Our conventions are as follows: categories will be denoted by calligraphic or bold face letters, and functors by capital letters.

A category consists of a class (a set) $\mathrm{Ob}\,\mathcal{C}$, the objects of $\mathcal{C}$, and for each $X, Y \in \mathrm{Ob}\,\mathcal{C}$, a set $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, the morphisms from X to Y. For any triple X, Y, Z of objects, a composition map

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Z) \tag{1.1}$$

denoted $(f, g) \mapsto g \circ f$. The composition map is assumed associative. For each X there exists $1_X \in \mathrm{Hom}_{\mathcal{C}}(X, X)$ such that $f \circ 1_X = f$, $1_Y \circ g = g$.

In the sequel, the composition of $f : X \to Y$ and $g : Y \to Z$ will be denoted gf. A morphism $f : X \to Y$ is a **monomorphism** (resp. **epimorphism**) if for any morphisms g_1, g_2, $fg_1 = fg_2 \Longrightarrow g_1 = g_2$ (resp. $g_1 f = g_2 f \Longrightarrow g_1 = g_2$).

The **opposite category** $\mathcal{C}^{\mathrm{op}}$ is obtained by reversing the arrows of $\mathcal{C}$, i.e., $\mathrm{Ob}\,\mathcal{C}^{\mathrm{op}} = \mathrm{Ob}\,\mathcal{C}$, and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(Y,X)$, the composition being the obvious one. We write **Set** for the category of sets, and $\mathbf{Vect}_k$ for the category of k-vector spaces with linear maps as morphisms.

A category is **discrete** or a set if the only morphisms are the identity morphisms 1_X. Let $S \in \mathrm{Ob}\,\mathcal{C}$ be an object. The category of objects over S, denoted $\mathcal{C}/S$, is the one with objects the morphisms $X \to S$ with target S, and morphisms $(X \to S) \longrightarrow (Y \to S)$ the S-morphisms, i.e., morphisms $X \to Y$ making the obvious triangle commutative.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories. A **(covariant) functor** $F : \mathcal{C} \to \mathcal{D}$ is the data of a map $F : \mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$ and for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$ of a map still denoted F

$$F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y)) \tag{1.2}$$

such that $F(1_X) = 1_{F(X)}$, and $F(fg) = F(f)F(g)$. A **contravariant functor** $F : \mathcal{C} \to \mathcal{D}$ is a covariant functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. Given morphisms $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, there is a naturally defined composition $G \circ F : \mathcal{C} \to \mathcal{E}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is **fully faithful** if for all $X, Y \in \mathrm{Ob}\,\mathcal{C}$ the map

$$F : \mathrm{Hom}_{\mathcal{C}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X),F(Y))$$

is bijective. Say F is **essentially surjective** if for any $Y \in \mathrm{Ob}\,\mathcal{D}$ there is an object $X \in \mathrm{Ob}\,\mathcal{C}$ such that $F(X) \cong Y$.

Recall now the definition of a morphism of functors.

Definition 1.1. Let F_1, F_2 are two functors from $\mathcal{C}$ to $\mathcal{D}$. A **morphism of functors** or **natural transformation** $\theta : F_1 \to F_2$ is the data for all $X \in \mathrm{Ob}\,\mathcal{C}$ of a morphism $\theta(X) : F_1(X) \to F_2(X)$ such that for all $f \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$, the diagram

$$\begin{tikzcd} F_1(X) \arrow[r, "\theta(X)"] \arrow[d, "F_1(X)"'] & F_2(X) \arrow[d, "F_2(Y)"] \\ F_1(Y) \arrow[r, "\theta(Y)"] & F_2(Y) \end{tikzcd} \tag{1.3}$$

commutes.

A morphism of functors will be visualized as a diagram like this:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \end{tikzcd} .$$

There are obvious composition laws of morphisms of functors which we picture by diagrams

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \end{tikzcd} \circ \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_2"] \arrow[r, bend right, "F_3"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{D} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_3"'] \arrow[r, phantom, "\Downarrow\eta.\theta"] & \mathcal{D} \end{tikzcd}$$

and

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\theta"] & \mathcal{D} \end{tikzcd} \circ \begin{tikzcd} \mathcal{D} \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathcal{E} \end{tikzcd} = \begin{tikzcd} \mathcal{C} \arrow[r, bend left, "G_1F_1"] \arrow[r, bend right, "G_2F_2"'] \arrow[r, phantom, "\Downarrow\eta.\theta"] & \mathcal{E} \end{tikzcd} .$$

As a consequence there is a natural notion of isomorphism of functors $F \xrightarrow{\sim} G$. This notion leads to the definition of an equivalence of categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then F is an equivalence if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ together with two isomorphisms $G \circ F \xrightarrow{\sim} 1_{\mathcal{C}}$, $F \circ G \xrightarrow{\sim} 1_{\mathcal{D}}$. We shall not give the proof of the well-known but important result that follows:

Proposition 1.2 ([62], Lemma 02C3). *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if F is fully faithful and essentially surjective.*

The functors $\mathcal{C} \to \mathcal{D}$ together with their natural transformations define a category $\mathbf{Fun}(\mathcal{C}, \mathcal{D})$. Let $X \in \operatorname{Ob}\mathcal{C}$ be an object of $\mathcal{C}$. Recall how one defines the category of objects over X, denoted $\mathcal{C}/X$: the objects are the morphisms $u : S \to X$, the morphisms are the commutative triangles

$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "u"'] & & T \arrow[dl, "v"] \\ & X, & \end{tikzcd}$$

and the composition law is the obvious one. There is an obvious forgetful functor $\mathcal{C}/X \to \mathcal{C}$.

Any $X \in \operatorname{Ob}\mathcal{C}$ defines a contravariant functor $h_X : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$, according to the rule $f_X(S) = \operatorname{Hom}_{\mathcal{C}}(S, X)$. This yields a functor $h : \mathcal{C} \to \mathbf{Fun}(\mathcal{C}^{\mathrm{op}}, \mathbf{Set})$. Yoneda's lemma ([62], Lemma 001P) states that $\phi \mapsto \phi(1_X)$ yields a one-to-one correspondence between $\operatorname{Hom}(h_X, F)$ and $F(X)$. In particular h defined above is fully faithful.

We are interested in a very particular class of categories, the groupoids as a substitute of the sets.

Definition 1.3. A **groupoid** is a category $\mathcal{G}$ in which all morphisms are isomorphisms. Thus $\operatorname{Hom}_{\mathcal{G}}(x, x) = \operatorname{Isom}_{\mathcal{G}}(x)$ (or $\operatorname{Aut}(x)$) is a group. We write $[\mathcal{G}]$ the set[2] of isomorphism classes of objects.

A **discrete groupoid** is a groupoid in which for all objects x, y, the set $\operatorname{Hom}(x, y)$ is either empty or consists of a single element. A group G defines a groupoid $\mathcal{G}$, in the following manner. We set $\operatorname{Ob}\mathcal{G} = G$, and $\operatorname{Hom}_{\mathcal{G}}(g, h)$ is a set reduced to one element denoted hg^{-1} (1 if $g = h$). Notice the consistency of the definition $kh^{-1} \circ hg^{-1} = kg^{-1}$.

Exercise 1.4. *A set can be seen as a discrete groupoid. Indeed any discrete groupoid $\mathcal{G}$ is equivalent to a set, namely $[\mathcal{G}]$.*

[2]Implicit in the definition is the fact that this is really a set.

We need one more definition to be able to speak about the category of modules over rings, quasi-coherent sheaves on schemes, or the category of étale covers of curves for example. Let $p : \mathcal{C} \to \mathcal{S}$ be a functor. For any $S \in \operatorname{Ob}\mathcal{S}$, let us denote $\mathcal{C}(S)$ the subcategory of $\mathcal{C}$ with objects[3] those $x \in \operatorname{Ob}\mathcal{C}$ with $p(x) = S$ (the sections of $\mathcal{C}$ over S). The morphisms $u : x \to y$ in $\mathcal{C}(S)$ are the morphisms in $\mathcal{C}$ such that $p(u) = 1_S$. The category $\mathcal{C}(S)$ is the **fiber** category over S.

Definition 1.5. Let $p : \mathcal{C} \to \mathcal{S}$ be a functor as above. We say that this data yields a **fibered category** if for any $f : T \to S$ and $x \in \mathcal{C}(S)$ there exists $y \in \mathcal{C}(T)$ and a **cartesian** arrow $u : y \to x$. This means that for any diagram

$$\begin{tikzcd}
z \arrow[dr, dotted, "\exists! v"'] \arrow[drr, "w"] \arrow[ddd, mapsto] & & \\
& y \arrow[r, "u"] \arrow[dd, mapsto] & x \arrow[dd, mapsto] \\
& & \\
U = p(z) \arrow[dr, "g"'] \arrow[drr, "h=p(w)"] & & \\
& T = p(y) \arrow[r, "p(u)=f"'] & p(x) = S
\end{tikzcd}$$

there is a unique $v : z \to y$ such that $uv = w$ and $p(v) = g$ (i.e., there is a unique way to fill in the top diagram such that its image under p is the bottom diagram).

In other words the "square" at the right with horizontal arrow (f, u) is **cartesian**. We may think y as "the" **pullback** of x under f, and for this reason it is justified to denote it $f^*(x)$, even if y is not unique but only unique up to a *unique* isomorphism. Indeed the uniqueness property in the definition yields the fact that for any other (y', u') there exists a unique morphism $v : y' \longrightarrow y$ with $uv = u'$ and $p(v) = 1$, likewise a unique $w : y \longrightarrow y'$ with $p(w) = 1$, and $u'w = u$. Uniqueness yields $vw = 1 = wv$. In particular with obvious notations we have a **canonical** isomorphism, whenever this makes sense

$$c_{f,g} : g^*f^*(x) \xrightarrow{\sim} (fg)^*(x). \tag{1.4}$$

At this stage f^* is not exactly a functor, but as explained below we will often think f^* as a functor. The uniqueness in (1.4) suggests that these canonical isomorphisms enjoy a compatibility property for any triple of morphisms (f, g, h):

$$\begin{tikzcd}
h^*(g^*f^*).f^* \arrow[r, "h^*(c_{f,g})"] \arrow[d, "c_{g,h}"'] & h^*(fg)^* \arrow[d, "c_{fg,h}"] \\
(gh)^*f^* \arrow[r, "c_{f,gh}"'] & (fgh)^* .
\end{tikzcd}$$

[3]Objects of $\mathcal{C}$ are in small letters, while objects of $\mathcal{S}$ are in capital letters.

With some care we can drop these associativity isomorphisms, and simply keep in mind that they are implicit.

We say that $S \mapsto \mathcal{C}(S)$ is a **pseudo-functor**, or a **lax functor**, or a **presheaf in groupoids**. We point out a further convention that will be used sometimes: if $f : T \to S$ is a morphism of $\mathcal{S}$, we write x_T instead of $f^*(x)$, thinking of x_T as the "restriction" of x to T. An alternative way of thinking about lax presheaves is in categorical terms. The definition goes as follows:

Definition 1.6. Let $\mathcal{C}, \mathcal{D}$ be fibered categories over $\mathcal{S}$. A **morphism of fibered categories** from $\mathcal{C}$ to $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ such that $p_{\mathcal{D}} F = p_{\mathcal{C}}$ and F sends cartesian arrows to cartesians arrows.

Such an F yields a functor $F(S) : \mathcal{C}(S) \to \mathcal{D}(S)$ for each $S \in \operatorname{Ob} \mathcal{S}$. In our last definition below we restrict somewhat the definition of a fibered category.

Definition 1.7. A fibered category in groupoids is a fibered category (see Definition 1.5) such that for each $S \in \operatorname{Ob} \mathcal{S}$ the category $\mathcal{C}(S)$ is a groupoid.

In that case any morphism u as in Definition 1.5 is cartesian. Indeed let $w : z \to x$ be a cartesian arrow over f as given by the definition. There is a morphism $v : y \to z$, with $u = wv$ and $p(v) = 1$.

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between two fibered categories in groupoids. Since F maps a cartesian square to a cartesian square, for any $f : S \to S'$, and $x' \in \mathcal{C}(S')$, there is a canonical isomorphism

$$F(f^*(x')) \xrightarrow{\sim} f^*(F(x'))$$

which means that the diagram

$$\begin{tikzcd} \mathcal{C}(S') \arrow[r, "F(S')"] \arrow[d, "f^*"'] & \mathcal{D}(S') \arrow[d, "f^*"] \\ \mathcal{C}(S) \arrow[r, "F(S)"] & \mathcal{D}(S) \end{tikzcd} \tag{1.5}$$

commutes up to a canonical isomorphism.

We shall now record the fact that fibered categories in groupoids over a fixed $\mathcal{S}$ are part of a structure a bit more complex than an ordinary category, called a (strict) 2-category. In a (strict) 2-category, one finds two levels of morphisms, the 1-morphisms and the 2-morphisms, and consequently two levels of compositions, the horizontal composition and the vertical composition. Assume given two morphisms $F, G : \mathcal{C} \to \mathcal{D}$ as in Definition 1.6.

Definition 1.8. A **2-morphism** $\theta : F \to G$ is a base-preserving natural transformation, that is, for any $x \in \mathcal{C}(S)$ the morphism $\theta(x) : F(x) \to G(x)$ projects to the identity in $\mathcal{S}$ (thus it is a morphism of $\mathcal{D}(S)$, hence an isomorphism).

Notice that in our setting, a 2-morphism is an isomorphism. The fibered categories in groupoids are the objects of a **2-category**. The morphisms, more accurately called 1-morphisms, are the base-preserving functors, and the 2-morphisms

are the base-preserving natural transformations. The notation $\mathbf{Hom}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ stands for the category of 1-morphisms; this is a groupoid. The composition in $\mathbf{Hom}_{\mathcal{S}}(\mathcal{C},\mathcal{D})$ is the **vertical composition**.

In order to work with stacks, the complete formalism of 2-categories is not necessary. A flavor of the definition is enough, and we refer to [62], Definition 003H for more details. Simply put, the datum of a 2-category includes:

i) a set (a class) of objects $\mathrm{Ob}\,\mathcal{F}$,

ii) for any pair (X,Y) of objects, a category $\mathbf{Hom}_{\mathcal{F}}(X,Y)$, and for any triple of objects (X,Y,Z) a composition rule

$$\mu_{X,Y,Z} : \mathbf{Hom}_{\mathcal{F}}(X,Y) \times \mathbf{Hom}_{\mathcal{F}}(Y,Z) \longrightarrow \mathbf{Hom}_{\mathcal{F}}(X,Z). \tag{1.6}$$

The image $\mu_{X,Y,Z}(F,G)$ is often denoted $G \circ F$ or simply GF. This rule is required to be **associative** in a strict sense, i.e., for all (X,Y,Z,T) it should satisfy $\mu_{X,X,Z}(1_X,G) = G$, $\mu_{X,Y,Y}(F,1_Y) = F$ and

$$\mu_{X,Z,T}(\mu_{X,Y,Z}(F,G),H) = \mu_{X,Y,T}(F,\mu_{Y,Z,T}(G,H)).$$

iii) two laws of composition for the morphisms of $\mathbf{Hom}_{\mathcal{F}}(X,Y)$: vertical 2-composition

$$\begin{tikzcd} X \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\theta"] & Y \end{tikzcd} \circ \begin{tikzcd} X \arrow[r, bend left, "F_2"] \arrow[r, bend right, "F_3"'] \arrow[r, phantom, "\Downarrow\eta"] & Y \end{tikzcd} = \begin{tikzcd} X \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_3"'] \arrow[r, phantom, "\Downarrow\eta.\theta"] & Y \end{tikzcd}$$

and horizontal 2-composition:

$$\mu_{X,Y,Z}\left(\begin{tikzcd} X \arrow[r, bend left, "F_1"] \arrow[r, bend right, "F_2"'] \arrow[r, phantom, "\Downarrow\theta"] & Y \end{tikzcd} , \begin{tikzcd} Y \arrow[r, bend left, "G_1"] \arrow[r, bend right, "G_2"'] \arrow[r, phantom, "\Downarrow\eta"] & Z \end{tikzcd} \right) = \begin{tikzcd} X \arrow[r, bend left, "G_1F_1"] \arrow[r, bend right, "G_2F_2"'] \arrow[r, phantom, "\Downarrow\eta\star\theta"] & Z \end{tikzcd} .$$

The objects of $\mathbf{Hom}_{\mathcal{F}}(X,Y)$ are called 1-morphisms, and the morphisms in $\mathbf{Hom}_{\mathcal{F}}(X,Y)$ are called 2-morphisms. As an example, the category of groupoids denoted **GPO** is in an obvious way a 2-category[4]. Likewise, and to summarize our discussion:

> **The categories fibered in groupoids over a base $\mathcal{S}$, are the objects of a 2-category[5] CFG, the 1-morphisms are the functors, the 2-morphisms the natural transformations.**

An obvious but still very useful example of a fibered category in (discrete) groupoids, i.e., sets, is provided by a presheaf in sets, i.e., a contravariant functor $F : \mathcal{S} \to \mathbf{Set}$. The objects of this category denoted $\mathcal{F}$ are the pairs (S,x), $x \in F(S)$. A morphism $f : (T,y) \to (S,x)$ is simply a morphism $f : T \to S$, with $y = F(f)(x)$. Finally p is the obvious projection $p(S,x) = S$. For example any $S \in \mathrm{Ob}\,\mathcal{S}$ defines a presheaf $h_S(-) = \mathrm{Hom}_{\mathcal{S}}(-,S)$. The associated fibered category is $\mathcal{S}/S$ the category of objects of $\mathcal{S}$ over S.

[4]More generally one can speak of the 2-category of categories **Cat**.

[5]A *strict* $\mathbf{(2,1)}$*-category* in the terminology of [62], definition 003H.

Useful is the following easy result, left as an exercise:

Proposition 1.9. *Let $F : \mathcal{C} \to \mathcal{D}$ be a morphism of fibered categories in groupoids. Then F is an equivalence, i.e., there exists a quasi-inverse $G : \mathcal{D} \to \mathcal{C}$, if and only if for every object $S \in \operatorname{Ob}\mathcal{S}$, the functor on fiber categories $F(S) : \mathcal{C}(S) \xrightarrow{\sim} \mathcal{D}(S)$ is an equivalence in the usual sense.*

We close this section by the following variant of the well-known Yoneda lemma (see [45] or [62], Lemma 004B):

Proposition 1.10 (2-Yoneda Lemma). *Let $p : \mathcal{C} \to \mathcal{S}$ be a fibered category in groupoids, and let $X \in \mathcal{S}$. The evaluation functor*

$$ev_X : \mathbf{Hom}_{\mathcal{S}}(\mathcal{S}/X, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}(X)$$

is an equivalence of categories (e.g., groupoids) $\mathbf{Hom}_{\mathcal{S}}(\mathcal{S}/X, \mathcal{C}) \xrightarrow{\sim} \mathcal{C}(X)$.

Proof. It suffices to exhibit a quasi-inverse. Let $x \in \mathcal{C}(X)$. We define a 1-morphism $\phi_x : \mathcal{S}_X \to \mathcal{C}$, first on objects by the choice for any $f : S \to X$ of a pullback $f^*(x) \in \mathcal{C}(S)$. Now given a diagram $f : S \xrightarrow{g} S' \xrightarrow{f'} X$, i.e., $f'g = f$, we know there is a unique isomorphism $\nu_g : f^*(x) \cong f'^*(x)$, i.e., a cartesian diagram

$$\begin{tikzcd} f^*(x) \arrow[r, "\nu(g)"] \arrow[d] & f'^*(x) \arrow[d] \\ S \arrow[r, "g"] & S' \end{tikzcd} .$$

It is readily seen this define a 1-morphism $\phi_x : \mathcal{S}/X \to \mathcal{C}$. This construction extends easily to a functor $\psi : \mathcal{C}(X) \to \mathbf{Hom}_{\mathcal{S}}(\mathcal{S}/X, \mathcal{C})$, which is the required quasi-inverse. □

Finally let us make one more remark about the two ways of thinking about fibered categories in groupoids. Taking into account the axioms of fibered categories in groupoids, it is easy to switch from the categorical viewpoint to the more intuitive "presheaf in groupoids" picture. Assume given a fibered category in groupoids. It is tempting to see the assignment $S \in \operatorname{Ob}\mathcal{S} \mapsto \mathcal{C}(S)$ as a functor

$$\mathcal{S} \longrightarrow \mathbf{GPO}\,.$$

This is however not quite a functor, because given an object $x \in \mathcal{C}(S)$ and an arrow $f : T \to S$ in $\mathcal{S}$, the arrow $y \to x$ of Definition 1.5 is not unique. But as we said before, using the axiom of choice we can select such an arrow. Denote by $f^*(x)$ the source of this selected arrow. One also assumes that this choice is made in such a way that $1^*(x) = x$. Then f^* becomes a functor $\mathcal{C}(S) \to \mathcal{C}(T)$, i.e., a 1-morphism of **GPO**. But if $g : U \to T$ is another arrow, then we cannot expect to have the equality $g^*(f^*(x)) = (fg)^*(x)$. What we have is only a canonical isomorphism, i.e., a 2-isomorphism

$$\alpha_{f,g} : g^*f^*(x) \xrightarrow{\sim} (fg)^*(x). \tag{1.7}$$

Moreover, for any triple of arrows $V \xrightarrow{h} U \xrightarrow{g} T \xrightarrow{f} S$ we have the associativity rule, which we translate as a commutative square

$$\begin{tikzcd} h^*(g^*f^*) \arrow[r, "h^*(\alpha_{f,g})"] \arrow[d, "\alpha_{g,h}\circ f^*"'] & h^*(fg)^* \arrow[d, "\alpha_{hg,h}"] \\ (gh)^*f^* \arrow[r, "\alpha_{f,gh}"'] & (fgh)^*. \end{tikzcd} \tag{1.8}$$

There is an important consequence of this 2-associativity. Let $x_1, x_2 \in \mathcal{C}(S)$. We define a contravariant functor, i.e., a presheaf[6]

$$\mathbf{Isom}_S(x_1, x_2) = \mathcal{S}/S \longrightarrow \mathbf{Set} \tag{1.9}$$

as follows. We set

$$\mathbf{Isom}(x_1, x_2)(T \xrightarrow{f} S) = \mathrm{Isom}_T(f^*(x_1), f^*(x_2)) \tag{1.10}$$

and for a morphism $g : V \xrightarrow{u} T \xrightarrow{f} S$, we define the restriction map

$$\rho_u(\xi) = \alpha_{f,u}(x_2)\, u^*(\xi)\, \alpha_{f,u}(x_1)^{-1}. \tag{1.11}$$

Proposition 1.11. $\mathbf{Isom}(x_1, x_2)$ *is a presheaf of sets.*

Proof. Let us consider the diagram

$$\begin{tikzcd} U \arrow[r, "v"] \arrow[drr, "h"'] & V \arrow[r, "u"] \arrow[dr, "g"] & T \arrow[d, "f"] \\ & & S. \end{tikzcd}$$

We must check that $\rho_v \circ \rho_u = \rho_{uv}$. Fix $\xi \in \mathrm{Isom}_T(f^*(x_1), f^*(x_2))$. For the left-hand side, the definition yields:

$$\rho_v\rho_u(\xi) = \alpha_{g,v}(x_2)\, v^*\, \alpha_{f,u}(x_2)\, v^*u^*(\xi)v^*\, \alpha_{f,u}(x_1)^{-1}\, \alpha_{g,v}(x_1)^{-1}.$$

Using the associativity constraint (1.8), this expression becomes

$$\begin{aligned} &\alpha_{f,u}(x_2)\alpha_{u,v}(x_2)v^*\, u^*(\xi)\alpha_{u,v}(x_1)^{-1}\, \alpha_{f,uv}(x_1)^{-1} \\ &\quad = \alpha_{f,uv}(x_2)(uv)^*(\xi)\alpha_{f,uv}(x_1)^{-1} \\ &\quad = \rho_{uv}(\xi) \end{aligned}$$

as expected. □

In case $x_1 = x_2 = x$, $\mathbf{Isom}(x, x)$ is a presheaf of groups. In the sequel, i.e., in the section about stacks, the presheaf $\mathbf{Isom}(x_1, x_2)$ will become a sheaf. But for this we need a topology. This will be the subject of the next section.

[6]If there is no chance of confusion the subscript S will be omitted.

Example 1.12. *Quasi-coherent modules.* Let $\mathbf{Qcoh}(X)$ be the category of quasi-coherent modules over the scheme X. Given a morphism $f : Y \to X$ we have the pullback functor[7]

$$f^* : \mathbf{Qcoh}(X) \to \mathbf{Qcoh}(Y).$$

If $h = fg : Z \xrightarrow{g} Y \xrightarrow{f} X$ is a product, we know there is a canonical functorial isomorphism $g^*f^* \xrightarrow{\sim} h^*$. This ensures that $X \mapsto \mathbf{Qcoh}(X)$ defines a lax functor, equivalently a fibered category $\mathbf{Qcoh}$. Indeed an object of $\mathbf{Qcoh}$ is a pair $(X, \mathcal{F})$ where $\mathcal{F} \in \mathbf{Qcoh}(X)$. A morphism $(Y, \mathcal{G}) \to (X, \mathcal{F})$ is a pair (f, ϕ) where $f : Y \to X$, and ϕ is a morphism $\phi : f^*(\mathcal{F}) \to \mathcal{G}$, i.e., the composition

$$(Z, \mathcal{H}) \xrightarrow{(g,\psi)} (Y, \mathcal{G}) \xrightarrow{(f,\phi)} (X, \mathcal{F})$$

is the natural one, viz.

$$(f, \phi).(g, \psi) = (f.g, \psi \circ g^*(\phi)).$$

It is not difficult to check that (f, ϕ) is cartesian if and only if ϕ is an isomorphism. One can take as morphisms only the cartesian ones, getting in this way a (sub)-fibered category which now is fibered in groupoids. There are many variations of this construction. For example one can define the fibered category in groupoids $\mathbf{Fib}_n$, if one takes as objects the locally free $\mathcal{O}_X$-modules of rank n instead all (quasi-)coherent modules.

Exercise 1.13. *Prove that an arrow $(f, \phi) : (Y, \mathcal{G}) \to (X, \mathcal{F})$ over $f : Y \to X$ is cartesian if and only if $\phi = f^*(\mathcal{F}) \xrightarrow{\sim} \mathcal{G}$ is an isomorphism.*

Example 1.14. *Étale covers.* Let us fix a scheme X over a field k. Define a category $\mathcal{E}$ together with a functor $p : \mathcal{E} \to \mathbf{Sch}_k$ as follows. The fiber category $\mathcal{E}(S)$ has for objects the finite étale covers $\pi : Y \to X \times_k S$ say of fixed degree d. A morphism of $\mathcal{E}$ is a cartesian diagram

$$\begin{tikzcd} Z \arrow[r, "\phi"] \arrow[d, "\nu"'] & Y \arrow[d, "\pi"] \\ X \times T \arrow[r, "1\times f"] & X \times S \end{tikzcd} \tag{1.12}$$

where $p(\phi, f) = (f : T \to S)$ is a morphism of $\mathbf{Sch}_k$, and $\phi : Z \xrightarrow{\sim} Y \times_S T$. Clearly if $S = T$ and $f = 1$, then standard facts about étale morphisms yield that ϕ is an isomorphism. Let us check quickly the axioms of fibered categories. Consider a

[7]The direct image $f_*(\mathcal{G})$ is not necessarily in $\mathbf{Qcoh}(X)$, unless some restrictions are put on f. Quasi-compacity and quasi-separatedness are an example, see [33], Chap. II, Proposition 5.8.

diagram

$$\begin{tikzcd}
Y'' \arrow[rr, dotted, leftarrow, "?"] \arrow[dr, "u''"] \arrow[dd, "\pi''"'] & & Y' \arrow[dl, "u'"'] \arrow[dd, "\pi'"] \\
& Y \arrow[dd, "\pi"] & \\
X \times S'' \arrow[rr, leftarrow, "1\times f"] \arrow[dr, "1\times f''"'] & & X \times S' \arrow[dl, "1\times f'"] \\
& X \times S &
\end{tikzcd}$$

with two cartesian squares and $f''f = f'$. It suffices to fill in the horizontal upper arrow in a way the upper square becomes cartesian. The answer is

$$? = ((1 \times f)\pi', u').$$

This example will be amplified in Section 4.2 about Hurwitz stacks. A very particular case is when $X = \operatorname{Spec} k$. Then an étale cover of fixed degree n takes the form $\operatorname{Spec} L \to \operatorname{Spec} k$, for L/k a separable algebra[8] of degree n.

In the next section we shall study in great detail another basic example, the classifying fibered category in groupoids associated to a group scheme, more generally to an action of a group scheme on a scheme (Section 2.2).

1.2.2. 2-fiber product. Our next and last construction is that of fiber products in a 2-category $\mathcal{C}$. Since only the category **CFG** is really of interest for us, the definition will take place in this 2-category (although it works perfectly within any 2-category). In the 2-category **CFG**, assume given a diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "F"] & \mathcal{Z} \\
& \mathcal{Y} \arrow[u, "G"]
\end{tikzcd} \tag{1.13}$$

where $\mathcal{X}, \mathcal{Y}, \mathcal{Z} \in \operatorname{Ob} \mathbf{CFG}$, and F, G are 1-morphisms. The 2-fiber product is an object $\mathcal{W}$, together with two 1-arrows P, Q, filling the previous diagram into a 2-commutative square

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "F"] & \mathcal{Z} \\
\mathcal{W} \arrow[u, "P"] \arrow[r, "Q"] & \mathcal{Y}. \arrow[u, "G"']
\end{tikzcd} \tag{1.14}$$

This means that there exists a 2-isomorphism $\theta : FP \Longrightarrow GQ$. The square is called **2-commutative**. The data (W, P, Q, θ) must enjoy a suitable uniqueness property, which ensures that it is in some sense unique. Indeed, consider another

[8] A product of separable extensions.

2-commutative square

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F"] & \mathcal{Z} \\ \mathcal{V} \arrow[u, "R"] \arrow[r, "T"] & \mathcal{Y} \arrow[u, "G"'] \end{tikzcd}$$

together with a 2-morphism $\xi : FR \Longrightarrow GT$. Then what we want is a 1-morphism $\phi : \mathcal{V} \to \mathcal{W}$, with the **strict commutativity**, $P\phi = R$, $Q\phi = T$, and the equality between 2-morphisms $\theta.\phi = \xi$. The morphism ϕ as above should be unique. Here is the answer to this problem.

Definition 1.15. The objects of $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ over S are the triples (x, y, θ) with $x \in \mathcal{X}(S)$, $y \in \mathcal{Y}(S)$, and $\theta : F(x) \xrightarrow{\sim} G(y)$ an isomorphism. The morphisms $(x, y, \theta) \to (x', y', \theta')$ over $f : S' \to S$, are the pairs of morphisms $(u : x' \to x, v : y' \to y)$ over f, such that the square

$$\begin{tikzcd} F(x') \arrow[r, "F(u)"] \arrow[d, "\theta'"'] & F(x) \arrow[d, "\theta"] \\ G(y') \arrow[r, "G(v)"] & G(y) \end{tikzcd} \tag{1.15}$$

is commutative. The composition is the obvious one.

It is readily seen that the category $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a fibered category in groupoids. The projection functor P (resp. Q) is $P(x, y, \theta) = x$ (resp. $Q(x, y, \theta) = y$). The 2-isomorphism $FP \Longrightarrow GQ$ is provided by θ, viz.

$$\theta : F(x) = FP(x, y, \theta) \xrightarrow{\sim} G(y) = GQ(x, y, \theta).$$

It is very easy to check that this construction provides the answer.

We can think of the 1-morphism $Q : \mathcal{W} \to \mathcal{Y}$ as the base change of F along $G : \mathcal{Y} \to \mathcal{Z}$. A special case leads to the fibers of a 1-morphism. Let $S \in \operatorname{Ob} \mathcal{S}$, and take for $\mathcal{Y}$ the fibered category in sets $\mathcal{S}_S$ (the presheaf of points of S). Yoneda's lemma tells us that a 1-morphism $S \to \mathcal{Y}$ is given by a section $y \in \mathcal{Y}(S)$. By base change $y : S \to \mathcal{Y}$, we get the fiber of $F : \mathcal{X} \to \mathcal{Z}$ over y:

$$\mathcal{X} \times_{\mathcal{Z}, y} S \to S. \tag{1.16}$$

A section of $\mathcal{X} \times_{\mathcal{Z}, y} S$ over T is a triple (x, f, θ) where $x \in \mathcal{X}(T)$, $f \in \operatorname{Hom}_{\mathcal{S}}(T, S)$ and $\theta : x \to y$ is a morphism over f, equivalently an isomorphism $\theta : x \cong f^*(y)$ in $\mathcal{X}(T)$. A morphism $(x, f, \theta) \to (x', f', \theta')$ over T occurs if $f = f'$, it is simply an isomorphism $u : x \cong x'$ in $\mathcal{X}(T)$ making the triangle

$$\begin{tikzcd} x \arrow[rr, "u"] \arrow[dr, "\theta"'] & & x' \arrow[dl, "\theta'"] \\ & f^*(y) & \end{tikzcd}$$

commutative.

Exercise 1.16. *Call an object F in a 2-category $\mathcal{C}$* **final** *if for any $X \in \operatorname{Ob}\mathcal{C}$ there exists a 1-morphism $X \to F$, and for two 1-morphisms $X \to F$, there is a unique 2-isomorphism between them. Check that the 2-fiber product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a final object in a suitably defined 2-category.*

Exercise 1.17. *Show that there is between the triple fiber products $(\mathcal{X} \times_{\mathcal{U}} \mathcal{Y}) \times_{\mathcal{V}} \mathcal{Z}$ and $\mathcal{X} \times_{\mathcal{U}} (\mathcal{Y} \times_{\mathcal{V}} \mathcal{Z})$ a canonical isomorphism of fibered categories. Given morphisms $\mathcal{X} \to \mathcal{Y} \to \mathcal{Z}$ and $\mathcal{V} \to \mathcal{Z}$, build an isomorphism of fibered categories in groupoids $\mathcal{X} \times_{\mathcal{Y}} (\mathcal{Y} \times_{\mathcal{Z}} \mathcal{V}) \cong \mathcal{X} \times_{\mathcal{Z}} \mathcal{V}$.*

The 2-category **CFG** has a final object (see Exercise 1.16), viz. $id : \mathcal{S} \to \mathcal{S}$. The 2-fiber product $\mathcal{C} \times_{\mathcal{S}} \mathcal{C}$ is simply the direct product $\mathcal{C} \times \mathcal{C}$. There is also a diagonal 1-morphism

$$\Delta_{\mathcal{C}} : \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C} \tag{1.17}$$

sending x to (x, x) and an arrow $u : x \to y$, to the pair $(u, u) : (x, x) \to (y, y)$. Very useful are the "fibers" of the diagonal.

Proposition 1.18. *Let $(x, y) \in \mathcal{C}(S)^2$. The fiber $\mathcal{I}_{(x,y)}$ of $\Delta_{\mathcal{C}}$ over the section $(x, y) \in (\mathcal{C} \times \mathcal{C})(S)$ is a category fibered in sets equivalent to the presheaf* $\mathbf{Isom}(x, y)$.

Proof. A section of $\mathcal{I}_{(x,y)}$ over T is a 2-commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\Delta"] & \mathcal{C} \times \mathcal{C} \\ T \arrow[u, "\xi"] \arrow[r, "f"] & S \arrow[u, "(x,y)"'] \end{tikzcd}$$

the 2-commutativity given by $\theta = (\alpha, \beta) : (\xi, \xi) \cong (f^*(x), f^*(y))$, or equivalently a diagram

$$\begin{tikzcd} f^*(x) \arrow[rr, bend left, "\beta\alpha^{-1}"] & \xi \arrow[l, "\alpha"'] \arrow[r, "\beta"] & f^*(y). \end{tikzcd}$$

The equivalence is given by $(\xi, f, \theta) \mapsto \beta\alpha^{-1}$. □

Exercise 1.19. *Write down the details of the proof of Proposition* 1.18.

The fact that fibered categories in groupoids are objects of a 2-category forces us to rewrite the definition of a monomorphism. Let $F : \mathcal{C} \to \mathcal{D}$ be a 1-morphism of fibered categories in groupoids.

Definition 1.20. The morphism F is a **monomorphism** if for all objects x, y in $\mathcal{C}(S)$, the functor $\operatorname{Hom}_{\mathcal{C}(S)}(x, y) \longrightarrow \operatorname{Hom}_{\mathcal{D}(S)}(F(x), F(y))$ is fully faithful.

This definition extends the usual definition of monomorphism in the following way: if $G, H : \mathcal{C}' \to \mathcal{C}$ are two morphisms such that there exists a 2-isomorphism $F \circ G \cong F \circ H$, then there exists a 2-isomorphism $G \cong H$.

1.2.3. Sites and Sheaves. We begin with the general notion of a *site*, i.e., a category endowed with a topology, which is the correct general setting for sheaves and stacks. We will first present the definition of a site based on sieves, as this often provides the most elegant constructions. Unless stated otherwise, it will be assumed that finite products always exist, and even more generally that finite inverse (projective) limits exist in the categories involved. The following three definitions, originally due to Grothendieck [58], [59] are taken with minor modifications from Artin [5], MacLane and Moerdijk [45]. See also the chapter "Sites and sheaves" in [62].

Definition 1.21. Given a category $\mathcal{C}$ and an object $C \in \operatorname{Ob}\mathcal{C}$, a **sieve** (French "crible") S on C is a family of arrows of $\mathcal{C}$, all with target C, such that

$$f \in S \implies fg \in S$$

whenever fg is defined. (I.e., S is a right ideal under composition.) Given a sieve S on C and an arrow $h : D \to C$, we define the **pullback** sieve $h^*(S)$ by

$$h^*(S) = \{g \mid \operatorname{target}(g) = D,\ hg \in S\}.$$

Some people prefer to see a sieve on $C \in \operatorname{Ob}\mathcal{C}$ as a subfunctor $S \subset C$ (C identified with $h_C(-)$).

Definition 1.22. A **site** $(\mathcal{C}, J)$ is a category $\mathcal{C}$ equipped with a **Grothendieck topology** J, that is, a function J which assigns to each object C of $\mathcal{C}$ a collection $J(C)$ of sieves on C, called **covering sieves**, such that

1. the maximal sieve $t_C = \{f \mid \operatorname{target}(f) = C\}$ is in $J(C)$;
2. (stability) if $S \in J(C)$, then $h^*(S) \in J(D)$ for any arrow $h : D \to C$;
3. (transitivity) if $S \in J(C)$ and R is any sieve on C such that $h^*(R) \in J(D)$ for all $h : D \to C$ in S, then $R \in J(C)$.

It is useful to note two simple consequences of these axioms. First, there is a somewhat more intuitive transitivity property:

3′. (transitivity′) If $S \in J(C)$ is a covering sieve and for each $f : D_f \to C$ in S there is a covering sieve $R_f \in J(D_f)$, then the set of all composites $f \circ g$, where $f \in S$ and $g \in R_f$, is a covering sieve of C.

Next we have the fact that any two covering sieves have a common refinement, in fact, their intersection.

4. (refinement) If $R, S \in J(C)$ then $R \cap S \in J(C)$.

It is often more intuitive to work with a *basis* for a topology (also called a *pretopology*).

Definition 1.23. A **basis** for a Grothendieck topology on a category $\mathcal{C}$ is a function Cov which assigns to every object C of $\mathcal{C}$ a collection $\operatorname{Cov}(C)$ of families of arrows $(C_i \to C)_{i \in I}$ with target C[9], called **covering families**, such that

[9]The set I will often be omitted from the notation.

1. if $f : C' \to C$ is an isomorphism, then (f) alone is a covering family;
2. (stability) if $(f_i : C_i \to C)$ is a covering family, then for any arrow $g : D \to C$, the pullbacks $C_i \times D$ exist and the family of pullbacks $\pi_2 : C_i \times D \to D$ is a covering family (of D);
3. (transitivity) if $(f_i : C_i \to C | i \in I)$ is a covering family and for each $i \in I$, one has a covering family $(g_{ij} : D_{ij} \to C_i \mid j \in I_i)$, then the family of composites $(f_i g_{ij} : D_{ij} \to C \mid i \in I, j \in I_i)$ is a covering family.

Any basis Cov generates a topology J by

$$S \in J(C) \Leftrightarrow \exists R \in \mathrm{Cov}(C) \text{ with } R \subset S.$$

In other words, the covering sieves on C are those which refine some covering family R, see Example 1.25 below. Usually we will describe sites in terms of a basis. The reader must convince himself that the two definitions are really equivalent. This means that every site has a basis, this is readily seen, and if bases are used the topology does not depend of a choice of a base.

We will often abuse notation and refer to a site $(\mathcal{C}, J)$ or $(\mathcal{C}, \mathrm{Cov})$ simply as $\mathcal{C}$. Let $(C_i \to C)_i$ and $(U_\alpha \to C)_\alpha$ be two covering families. By a **morphism** $(U_\alpha \to C) \longrightarrow (C_i \to C)$ we mean a map $\alpha \to i$, and a morphism $U_\alpha \to C_i$ in $\mathcal{C}_C$. We can think $(U_\alpha \to C)$ as a refinement of $(C_i \to C)$.

One simple way in which new sites arise is the *induced site.*

Definition 1.24. Let $(\mathcal{C}, J)$ be a site and let $u : \mathcal{A} \to \mathcal{C}$ be a functor. Assume that u preserves all pullbacks that exist in $\mathcal{A}$. The **induced topology** $J|_A$ on $\mathcal{A}$ is defined in terms of the following basis. A family $(f_i : A_i \to A)_i$ is a covering family for the induced topology if and only if the family $(u(f_i) : u(A_i) \to u(A))_i$ is a covering family for J.

Let $(\mathcal{C}, J)$ be a site and let $\mathcal{A} \subset \mathcal{C}$ be a full subcategory. Assume that the inclusion functor preserves all pullbacks that exist in $\mathcal{A}$. Then the induced topology on $\mathcal{A}$ will also be called the *restriction* of J to $\mathcal{A}$ and will be denoted $J|_A$.

We now present key examples of sites.

Example 1.25. The small site of a topological space. Let X be a topological space, for example a scheme with its Zariski topology, and let $\mathrm{Open}(X)$ be the category of open subsets of X, where arrows are given by inclusions of open sets. (Hence there is at most one arrow between any two objects.) Say that $(U_i \to U)_i$ covers U if $U = \bigcup U_i$ (the usual definition of an open cover). This is easily seen to be a basis for a Grothendieck topology on $\mathrm{Open}(X)$. The covering sieve generated by $(U_i \to U)_i$ is the family of all sets V such that $V \subset U_i$ for some i, i.e., the maximal refinement of $(U_i \to U)$.

The resulting site is called the **small site** of the space X. This is the original and motivating example for the notion of a site. However it is special in that the underlying category is just a partial order; there are no nontrivial endomorphisms.

Example 1.26. *The fpqc, fppf and étale sites.* Our goal is to introduce suitable topologies on the categories **Sch** or **Sch** $/S$. The first natural candidate is the

Zariski topology, either big or small. Let X be a scheme, then a Zariski covering of X is a family of open immersions $(f_i : U_i \to X)_{i\in I}$ such that $X = \cup_i f_i(U_i)$. This definition satisfies the axioms of sites and produces the big (or small) Zariski site $\mathbf{Sch}_{\mathrm{Zar}}$. Clearly if X is quasi-compact, for example affine, then any Zariski covering has a refinement $(V_j \to X)_{j\in J}$ with J finite.

A more refined example for the sequel of these notes is the small **étale site** (resp. fppf, fpqc) of a scheme X. For the basics about flat, faithfully flat and étale maps see [20] Chap. 6, [33], [62]. The construction goes as follows. Let $\mathcal{P}$ be one of the following properties of morphisms of **Sch**: étale, faithfully flat locally of finite presentation, faithfully flat and quasi-compact.

Definition 1.27. The **big** $\mathcal{P}$-site of $X \in \mathbf{Sch}$ is the topology on $\mathbf{Sch}/X$ generated by the pretopology with covering families of $Y \to X$

$$\left(\begin{tikzcd} U_i \arrow[r] \arrow[rr, bend right] & Y \arrow[r] & X \end{tikzcd} \right)_i$$

where each $U_i \to Y$ is in $\mathcal{P}$. We get the **small** $\mathcal{P}$-site if all three arrows are taken in $\mathcal{P}$.

Obviously one can take for $\mathcal{P}$ the open immersions, and recover the Zariski site **Zar**. The étale topology is for geometric reasons the most natural. In this case $\mathcal{P}$ is the collection of étale locally of finite presentation morphisms. Thus an étale covering of X is a family of morphisms $(f_i : U_i \to X)_{i\in I}$ such that each f_i is étale, and $X = \cup_i f_i(U_i)$. Since an étale morphism is open, the étale topology refines the Zariski topology. Also any Zariski covering is an étale covering. If X is quasi-compact (affine) then we can work with coverings $(U_j \to X)_{j\in J}$ with U_j affine, and J finite.

If $\mathcal{P}$ means faithfully flat and locally of finite presentation, the resulting topology is named **fppf**. For example the small étale site of X has for objects the étale maps $Y \to X$, and coverings of $Y \to X$ the collection of jointly surjective étale maps $f_i : U_i \to Y$, i.e., $Y = \cup_i f_i(U_i)$. Notice that the morphisms in the small étale site X_{et} turn out to be étale. When X is affine, it suffices to consider the standard open covering, namely the finite family of étale maps $(f_i : U_i \to X)_i$, with each U_i affine, and $\cup_i f_i(U_i) = X$, which in turn says that $\coprod_j U_j \to X$ is a covering, but now with a single object.

Likewise, with the fppf topology it suffices to deal with standard fppf coverings of an affine scheme X, namely the finite collections of finite presentation maps $(f_i : U_i \to X)_i$, such that $\cup_i f_i(U_i) = X$. These topologies can be compared:

$$\textbf{Zariski} \subset \textbf{étale} \subset \textbf{fppf}.$$

If $\mathcal{P}$ is faithfully flat and quasi-compact the resulting topology is not in full generality the **fpqc** topology. One must add the Zariski covers[10]. This is not

[10]The fpqc topology behaves differently; as it stands it is not a refinement of the Zariski topology, we must add the open embeddings $U \hookrightarrow X$ at least if X is not quasi-compact. We refer to [64], Section 2.3.2 for the correct definition.

important for us because we will work essentially with the étale (or sometimes fppf) topology.

The key definition involving a topology, i.e., a site, is that of a **sheaf**. Let $\mathcal{C}$ be a site.

Definition 1.28. A **presheaf** (of sets) on $\mathcal{C}$ is a contravariant functor $F : \mathcal{C} \longrightarrow \mathbf{Set}$. If $(\mathcal{C}, \mathrm{Cov})$ is a site, then F is a **sheaf** if and only if for every object $X \in \mathcal{C}$, and every covering $(U_i \to X)_i \in \mathrm{Cov}(X)$, the sequence

$$F(X) \longrightarrow \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j) \tag{1.18}$$

with obvious arrows is exact.

For a cover by a single object $X' \to X$, the sequence (1.18) reads

$$F(X) = \ker\Big(F(X') \rightrightarrows F(X' \times_X X') \Big).$$

A diagram of sets

$$A \xrightarrow{f} B \underset{h}{\overset{g}{\rightrightarrows}} C$$

is called **exact** if f identifies A with the kernel of the double arrow (g, h), i.e., with the subset $\{b \in B,\ g(b) = h(b)\}$. When we have only the injectivity of $F(X) \to \prod_i F(U_i)$, we say that the presheaf is **separated**. Morphisms of presheaves are functorial morphisms. To check that Definition 1.28 is consistent, one need to see that the sheaf property is independent of the choice of a basis, i.e., is a property of the topology, not of the basis chosen, see Exercise 1.29 below.

Exercise 1.29. *With the same notations as before, prove that a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is a sheaf if and only if for any $C \in \mathrm{Ob}\,\mathcal{C}$, and any covering sieve S of C, the natural map*

$$\mathrm{Hom}(C, F) \longrightarrow \mathrm{Hom}(S, F)$$

is bijective. Here a sieve of C is seen as a subfunctor of $C = h_C(-)$, and Hom *stands for the functorial morphisms.*

Let us denote by $\mathcal{P}Shv_{\mathcal{C}}$ (resp. $\mathcal{S}hv_{\mathcal{C}}$) the category of presheaves (resp. sheaves) on $\mathcal{C}$.

The category of sheaves injects fully faithfully into the category of presheaves. Fundamental is the following fact [5], [45]:

Proposition 1.30. *Let $\mathcal{C}$ be a site. The inclusion $\mathcal{S}hv_{\mathcal{C}} \hookrightarrow \mathcal{P}Shv_{\mathcal{C}}$ has a left adjoint $F \mapsto \tilde{F}$, where $\tilde{F}$ is a sheaf (the associated sheaf), together with a map $\imath_F : F \to \tilde{F}$ such that a map from F to an arbitrary sheaf factors uniquely through $\tilde{F}$. A presheaf is separated if the canonical map $\imath_F$ is injective.*

Proof. (sketch) Let $X \in \mathrm{Ob}\,\mathcal{C}$, and let F be a presheaf. To shortcut the proof assume F separated. For any covering $\mathcal{U} = (U_i \to X)_i$ we set

$$F(\mathcal{U}) = \{(a_i) \in \prod_i F(U_i),\ a_{i|U_{ij}} = a_{j|U_{ij}} \quad (U_{ij} = U_i \times_X U_j). \tag{1.19}$$

Likewise one can define $F(\mathcal{U})$ for any sieve $\mathcal{U}$ of C. One can think of $F(\mathcal{U})$ as the set of sections of F defined locally on $\mathcal{U}$. If $\mathcal{V}$ is a refinement of $\mathcal{U}$ then we have an obvious restriction map

$$F(\mathcal{U}) \longrightarrow F(\mathcal{V}) \tag{1.20}$$

roughly, if we set $\tilde{F}(C) = \lim_{\leftarrow \mathcal{U}} F(\mathcal{U})$, then this not only defines a presheaf but indeed a sheaf. More concretely we can set

$$\tilde{F}(C) = \coprod_{\mathcal{U}} F(\mathcal{U}) \,/\, \sim \tag{1.21}$$

where two families $(a_i) \in F(\mathcal{U})$ and $(a'_\alpha) \in F(\mathcal{U}')$ are identified if they have the same image in the covering $U_i \times_C U'_\alpha$. Since our presheaf is separated, it is easy to check this defines an equivalence relation. It is easily seen that the presheaf $\tilde{F}$ is separated. To prove the sheaf property let us take a collection of sections $a^i \in \tilde{F}(U_i)$ where $(U_i \to C)_i$ is a covering. This means that there exists a covering $\mathcal{U}_i = (U_{i\alpha} \to U_i)_\alpha$ of U_i with $a^i \in F(\mathcal{U}_\alpha)$, and for any (i,j) the gluing property $a^i = a^j$ in $\tilde{F}(U_i \times_C U_j)$. Let $a^i = (a_{i\alpha} \in F(U_{i\alpha}))$. We translate this property as

$$a_{i\alpha|U_{i\alpha}\times_C U_{j\beta}} = a_{j\beta|U_{i\alpha}\times_C U_{j\beta}}.$$

This provides a well-defined element of $\tilde{F}(C)$, viz.

$$a = (a_{i,j,\alpha,\beta} = a_{i\alpha|U_{i\alpha}\times_C U_{j\beta}} = a_{j\beta|U_{i\alpha}\times_C U_{j\beta}})$$

living on the covering $(U_{i\alpha} \times_C U_{j\beta} \to C)$ of C. It is readily seen that this section is the gluing of the local sections a^i. □

It is technically interesting that the concept of a sheaf is local. To explain this, let first $S \in \operatorname{Ob}\mathcal{C}$, then the category $\mathcal{C}/S$ of S-objects of $\mathcal{C}$ has in an obvious manner a topology induced by the topology of $\mathcal{C}$ (we assume that finite projective limits exist). Any (pre)sheaf $\mathcal{F}$ on $\mathcal{C}$ induces a (pre)sheaf on $\mathcal{C}/S$, denoted throughout $\mathcal{F}_{|S}$, viz $\mathcal{F}_{|S}(T \to S) = \mathcal{F}(T)$. Let there be given $\mathcal{F}, \mathcal{G}$ sheaves on $\mathcal{C}$, then a presheaf on $\mathcal{C}$ is defined according to the rule

$$\mathcal{H}om(\mathcal{F}, \mathcal{G})(S) = \operatorname{Hom}(\mathcal{F}_{|S}, \mathcal{G}_{|S}).$$

Proposition 1.31.

i) *The presheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is a sheaf. Equivalently a morphism of sheaves on a site can be defined locally.*

ii) *Let $(U_i \to S)_i$ be a covering family of S, and for any i, $\mathcal{F}_i$ a sheaf on $\mathcal{C}/U_i$, such that on the $U_{ij} = U_i \times_S U_j$, $\mathcal{F}_i$ and $\mathcal{F}_j$ agree compatibly (see the proof for a precise meaning), then there is a (unique) sheaf $\mathcal{F}/S$ inducing the $\mathcal{F}_i$'s.*

Proof. i) Assume given a covering $(S_i \to S)$ of S, and for all i, a morphism $f_i : \mathcal{F}_{|S_i} \to \mathcal{G}_{|S_i}$. We want to glue together the f_i's into $f : \mathcal{F}_{|S} \to \mathcal{G}_{|S}$. It suffices to define for $T \to S$ and $\xi \in \mathcal{F}(T)$ the image $f(\xi) \in \mathcal{G}(T)$.

ii) The proof is very similar of the corresponding one in the "classical case", see for example [59]. First our assumption is the existence of a collection of gluing isomorphisms $\varphi_{ji} : \mathcal{F}_{i|U_{ij}} \xrightarrow{\sim} \mathcal{F}_{j|U_{ij}}$ with the cocycle condition $\varphi_{ki} = \varphi_{kj}\varphi_{ji}$ on

the triple "intersections" U_{ijk}. Let $T \to S$ and set $V_i = T \times_S U_i$, $V_{ij} = T \times_S U_{ij}$, etc. We take as $\mathcal{F}(T)$ the set (or abelian group) of families $(x_i) \in \prod_i \mathcal{F}_i(V_i)$ such that

$$\varphi_{ji}(x_i) = x_j.$$

It is easy to define for a morphism $T' \to T$ a "restriction map" $\mathcal{F}(T) \to \mathcal{F}(T')$. Then $\mathcal{F}$ as defined is a presheaf. We leave as an exercise to check that it is indeed a sheaf. □

Exercise 1.32. *Assume given $F, G, H : \mathcal{C}^{\mathrm{op}} \to$ **Set** a triple of sheaves, together with two morphisms $f : F \to H$, $h : G \to H$. Prove the presheaf $F \times_H G$ given by $(F \times_H G)(X) = F(X) \times_{H(X)} G(X)$ (fiber product of sets) is indeed a sheaf.*

Exercise 1.33. *With the same notations as above, if $f : T \to S$ is a morphism of $\mathcal{C}$, show that one can define a functor $f^* : \mathcal{S}hv_S \longrightarrow \mathcal{S}hv_T$ by $f^*(\mathcal{F})(X \to T) = \mathcal{F}(X \to S)$ ($\mathcal{S}hv_S = \mathcal{S}hv(\mathcal{C}/S)$). In case f is a cover, show that f^* is an equivalence of categories.*

As we said before, a fibered category in groupoids generalizes in some sense the concept of presheaf. We can ask, at least when $\mathcal{S}$ is a site, what is the proper generalization of a separated presheaf and of a sheaf. That is, how to define a sheaf in groupoids? The answer will lead us directly to prestacks and stacks, as follows

presheaf	fibered category in groupoids
separated presheaf	prestack
sheaf	stack
sheafification	stackification

1.2.4. Descent in a fibered category. The next important topic we want to review briefly is **descent theory** ([57], Exposé VIII). This technology plays a key role in the theory of stacks as a substitute of the usual gluing process along an open covering. The key words are descent datum, cocycle condition, and effectiveness.

Let us start with the elementary example of gluing sheaves (see [33], Exercise 1.22). Let X be a topological space and let $\mathcal{U} = (U_i)$ be an open cover of X, or a collection of open embeddings $(U_i \to U)_i$. Suppose that we are given for each i a sheaf $\mathcal{F}_i$ on U_i, and for each i, j an isomorphism

$$\varphi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$$

such that for each i we have $\varphi_{ii} = id$, and for each (i, j, k) we have $\varphi_{ik} = \varphi_{ij} \circ \varphi_{jk}$ on $U_i \cap U_j \cap U_k$ (this is called the **cocycle condition**). Then there exists a unique sheaf $\mathcal{F}$ on X, together with isomorphisms $\psi_i : \mathcal{F}|_{U_I} \xrightarrow{\sim} \mathcal{F}_i$ such hat for each i, j, $\psi_j = \varphi_{ij} \circ \psi_i$. We say loosely that $\mathcal{F}$ is obtained by **gluing** the $\mathcal{F}_i$ along the gluing data φ_{ij}.

We can see the open cover as a continuous map $\pi : X' = \coprod_i U_i \to X$, and the collection of $\mathcal{F}_i$ as a sheaf $\mathcal{F}'$ on X'. Let us form the fiber product $X' \times_X X' = \coprod_{i,j} U_i \cap U_j$, with the obvious projections $p_i : X' \times_X X' \to X$. The isomorphisms (φ_{ij}) can be seen as an isomorphism

$$\varphi : p_1^*(\mathcal{F}') \xrightarrow{\sim} p_2^*(\mathcal{F}').$$

Let us form the triple fiber product $X' \times_X X' \times_X X' = \coprod_{i,j,k} U_i \cap U_j \cap U_k$, with the corresponding projections p_{ij} on $X' \times_X X'$. The cocycle takes the compact form

$$p_{13}^*(\varphi) = p_{12}^*(\varphi) \circ p_{23}^*(\varphi).$$

Then the answer is there exists a unique sheaf $\mathcal{F}$ on X, together with an isomorphism $\psi : \pi^*(\mathcal{F}) \cong \mathcal{F}'$, such that $\varphi = p_2^*(\psi) \circ p_1^*(\psi)$.

It is not difficult to translate this archetypal example in a more general setting. Let $\mathcal{C} \to \mathcal{S}$ be a fibered category[11] in groupoids thought as a presheaf in groupoids. In the sequel it will be implicit that finite projective limits exist. Let $f : X' \to X$ be a morphism in $\mathcal{S}$. If $x' \in \mathcal{C}(X')$ it is natural to ask if we can find $x \in \mathcal{C}(X)$ together with an isomorphism $\theta : x' \xrightarrow{\sim} f^*(x)$, i.e., if x' **descends** to x over X. It is easy to understand what additional information on x' comes from such an x, assuming it exists. Let

$$X'' = X' \times_X X' \underset{p_2}{\overset{p_1}{\rightrightarrows}} X'$$

be the fiber product with its canonical projections. Pulling back to X'' yields a diagram

$$\begin{tikzcd} p_1^*(x') \arrow[r, "p_1^*(\theta)"] \arrow[d] & p_1^* f^*(x) \arrow[d] \\ p_2^*(x') \arrow[r, "p_2^*(\theta)"] & p_2^* f^*(\theta) \end{tikzcd}$$

where the vertical arrow $p_1^* f^*(x) \xrightarrow{\sim} p_2^* f^*(x)$ is the canonical isomorphism. The result is an isomorphism $\varphi : p_1^*(x') \xrightarrow{\sim} p_2^*(x')$ making the diagram commutative. Pulling back one step further on $X''' = X' \times_X X' \times_X X'$, if $(p_{ij})_{1 \le i < j \le 3}$ denote the projections on X'', it is not difficult to check the "cocycle" identity

$$p_{32}^*(\varphi) \circ p_{21}^*(\varphi) = p_{31}^*(\varphi). \tag{1.22}$$

Indeed, this follows easily taking into account the associativity isomorphisms in the definition of a fibered category in groupoids, and we leave this as an exercise. We can extract from this a definition:

Definition 1.34. A descent datum on x' relative to f, is an isomorphism $\varphi : p_1^*(\mathcal{F}') \xrightarrow{\sim} p_2^*(\mathcal{F}')$ which satisfies the cocycle condition (1.22). The descent datum is **effective** if the descent problem can be solved strictly, i.e., if there exists $x \in \mathcal{C}(S)$ together with an isomorphism $\psi : x' \xrightarrow{\sim} f^*(x)$, making the square

$$\begin{tikzcd} p_1^*(x) \arrow[r, "\varphi", "\sim"'] \arrow[d, "p_1^*(\psi)"'] & p_2^*(x) \arrow[d, "p_2^*(\psi)"] \\ p_1^* f^*(x) \arrow[r, "can", "\sim"'] & p_2^* f^*(x) \end{tikzcd}$$

[11] Fibered category suffices to formulate a descent datum.

commutative. The pair (x, φ) should be unique, i.e., if $(\overline{x}, \overline{\varphi})$ is another solution to the descent problem, then there is a unique isomorphism $\alpha : x \xrightarrow{\sim} \overline{x}$, such that the diagram

$$\begin{tikzcd} f^*(x) \arrow[r, "\varphi"] \arrow[rr, bend right, "f^*(\alpha)"'] & x' & f^*(\overline{x}) \arrow[l, "\overline{\varphi}"'] \end{tikzcd}$$

commutes.

This definition will be used in the following framework, very close to our basic example. Suppose that we are given a finite collection of morphisms $X_i \to X$ and for each i a section $x_i \in \mathcal{C}(X_i)$. We can ask if the x_i's come from some $x \in \mathcal{C}(X)$. This is the problem to descend $\oplus_i x_i \in \mathcal{C}(\coprod_i X_i)$ to X along the morphism $X' = \coprod_i X_i \to X$, assuming the existence of X'. Clearly φ amounts to a collection of isomorphisms $\varphi_{ij} : x_{j|X_{ij}} \xrightarrow{\sim} x_{i|X_{ij}}$, where $X_{ij} = X_i \times_X X_j$. The cocycle condition (1.22) reads

$$\varphi_{ij} \circ \varphi_{jk} = \varphi_{ik} \quad \text{on} \quad X_{ijk} = X_i \times_X X_j \times_X X_k. \tag{1.23}$$

More specifically, taking into account the fact that $\mathcal{S}$ is a site, suppose $\{f_i : U_i \to U\}_{i \in I}$ is a covering of an object U. We will use the previous notations, viz. $U_{ij} = U_i \times_U U_j$ the "pairwise intersections", and the $U_{ijk} = U_i \times_U U_j \times_U U_k$. It is important to allow $i = j$ and even $i = j = k$ in the following. The projection maps from U_{ij} to U_i, resp. U_j are as before denoted p_1, resp. p_2. The projection maps from U_{ijk} to the twofold fiber products are $p_{12} : U_{ijk} \to U_{ij}$, $p_{13} : U_{ijk} \to U_{ik}$, and $p_{23} : U_{ijk} \to U_{jk}$. The projection maps from U_{ijk} to U_i, U_j, and U_k are p_1, p_2, and p_3, respectively. In a compact form we have the picture[12]

$$\coprod U_{ijk} \Rrightarrow \coprod U_{ij} \rightrightarrows \coprod U_i \longrightarrow U\ . \tag{1.24}$$

This notation is potentially ambiguous, but it is standard in the literature. Sometimes in order to shortcut the diagrams, we will drop the tag above a projection arrow. Sometimes the cover $(U_i \to U)$ will be encapsulated into a single map $U' = \coprod U_i \to U$, in which case (1.24) becomes

$$U' \times_U U' \times_U U' \Rrightarrow U' \times_U U' \rightrightarrows U' \longrightarrow U\ . \tag{1.25}$$

Let $x_i \in \mathrm{Ob}(\mathcal{S}_{U_i})$, $i \in I$ be a collection of objects, and let

$$\phi_{ij} : \mathrm{pr}_1^* x_i \xrightarrow{\sim} \mathrm{pr}_2^* x_j \qquad (i, j \in I) \tag{1.26}$$

be a collection of morphisms in the fiber categories $\mathcal{S}_{U_{ij}}$. In the definition that follows we will identify, for instance, $p_1^* x_i$ on U_{ijk} with both $p_{12}^* p_1^* x_i$ and $p_{13}^* p_1^* x_i$.

We can rephrase Definition 1.34.

[12]Truncated cosimplicial object.

Definition 1.35. The collection $\{x_i, \phi_{ij}\}$ is a **descent datum** if the following cocycle condition is satisfied: for every triple $(i,j,k) \in I^3$, in the fiber category $\mathcal{S}(U_{ijk})$ the diagram

$$\begin{tikzcd} p_1^* x_i \arrow[rr, "p_{13}^*\phi_{ik}"] \arrow[dr, "p_{12}^*\phi_{ij}"'] & & p_3^* x_k \\ & p_2^* x_j \arrow[ur, "p_{23}^*\phi_{jk}"'] & \end{tikzcd} \tag{1.27}$$

is commutative.

Remark 1.36. Two remarks are in order about the definition.

(1) There is a diagonal morphism $\Delta_i : U_i \to U_{ii}$. We can pull back ϕ_{ii} via this morphism to get an automorphism ${\Delta_i}^*(\phi_{ii}) \in \mathrm{Aut}_{U_i}(x_i)$. On pulling back the cocycle condition for the triple (i,i,i) by $\Delta_{123} : U_i \to U_{iii}$ we deduce that ${\Delta_i}^*\phi_{ii} \circ {\Delta_i}^*\phi_{ii} = {\Delta_i}^*\phi_{ii}$; thus ${\Delta_i}^*\phi_{ii} = \mathrm{id}_{x_i}$.

(2) There is a morphism $\Delta_{13} : U_{ij} \to U_{iji}$ and we can pull back the cocycle condition for the triple (i,j,i) to get the identity $(\sigma^*\phi_{ji}) \circ \phi_{ij} = \mathrm{id}_{p_{12}^* x_i}$, where $\sigma : U_{ij} \to U_{ji}$ is the switching morphism.

To complete the picture, we can even define the category of descent data.

Definition 1.37. A **morphism of descent data** $\alpha : \{x_i, \phi_{ij}\} \to \{y_i, \psi_{ij}\}$ is given by a collection of morphisms $\alpha_i : x_i \to y_i$ in the fiber category over U_i such that the following diagrams

$$\begin{tikzcd} p_1^* x_i \arrow[r, "p_1^*\alpha_i"] \arrow[d, "\phi_{ij}"'] & p_1^* y_i \arrow[d, "\psi_{ij}"] \\ p_2^* x_j \arrow[r, "p_2^*\alpha_j"'] & p_2^* y_j \end{tikzcd} \tag{1.28}$$

commute.

Note that every morphism of descent data is an isomorphism.

Therefore the descent datum associated to an object of the fiber category $\mathcal{C}(U)$ and to a covering $(U_i \to U)_i$ of U, and morphisms between them, form a groupoid denoted $\mathbf{DD}_{\mathcal{C}}((U_i \to U)_i))$.

An object $x \in \mathrm{Ob}(\mathcal{S}_U)$ gives rise to a canonical descent datum in the following manner. First we set $x_i = f_i^* x$. Then we set $\phi_{ij} = {t_j}^{-1} \circ t_i$, where $t_i : \mathrm{pr}_1^* f_i^* x \to (f_i \circ \mathrm{pr}_1)^* x$ and $t_j : \mathrm{pr}_2^* f_j^* x \to (f_j \circ \mathrm{pr}_2)^* x$ are the canonical isomorphisms guaranteed by (1.7), and called α there. The lemma below shows this is really a descent datum; we will call this the **canonical descent datum** associated to x.

Lemma 1.38. *The cocycle condition holds for the "canonical descent datum".*

Proof. First, note that $f_i \circ \mathrm{pr}_1 = f_j \circ \mathrm{pr}_2 = f_k \circ \mathrm{pr}_3$. Then note that $\mathrm{pr}_{13}^*\phi_{ik}$, $\mathrm{pr}_{23}^*\phi_{jk}$ and $\mathrm{pr}_{12}^*\phi_{ij}$ factor uniquely through $(f_i \circ \mathrm{pr}_1)^* x = (f_j \circ \mathrm{pr}_2)^* x = (f_k \circ \mathrm{pr}_3)^* x$ by Lemma 3.1.3. $\square$

Definition 1.39. A descent datum $\{x_i, \phi_{ij}\}$ is called effective if there exists an $x \in \mathrm{Ob}(\mathcal{S}_U)$ such that $\{x_i, \phi_{ij}\}$ is isomorphic to the canonical descent datum associated to x.

The lesson of these definitions is that we have defined a functor

$$\mathcal{C}(U) \longrightarrow \mathbf{DD}_{\mathcal{C}}((U_i \to U)_i) \tag{1.29}$$

which sends an object to its canonical descent datum. Obviously descent data can be formulated similarly for a cover encapsulated in a single map $p : U' \to U$. A descent datum on x' which lies over U' is an isomorphism $\phi : p_1^*(x') \xrightarrow{\sim} p_2^*(x')$ constrained by the cocycle condition when pulled back to $U' \times_U U' \times_U U'$ and taking into account the associativity isomorphisms

$$p_{31}^*(\phi) = p_{32}^*(\phi) \circ p_{21}^*(\phi). \tag{1.30}$$

It will be useful to collect here the main results about fpqc descent, with only sketchy proofs, referring to the literature for the details. These results will help us to motivate the future definition of stacks. Let X be a scheme, $h_X \in \mathbf{Fun}(\mathbf{Sch}^{\mathrm{op}}, \mathbf{Set})$ its functor of points. The presheaf h_X is clearly a Zariski sheaf, but it enjoys a stronger property, as we shall see.

Recall that a ring homomorphism $f : A \to B$ is **faithfully flat** if B is a flat A-module and $\operatorname{Spec} B \to \operatorname{Spec} A$ is onto. The main property of a faithfully flat morphisms is the following fact: let $u : M \to N$ be a morphism of A-modules, then u is injective (resp. surjective, bijective) if and only if $u \otimes 1$ is injective (resp. surjective, bijective), see [62], Definition 00HB or [20]. A faithfully flat morphism enjoys a nice exactness property:

Lemma 1.40. *If $f : A \to B$ is faithfully flat then the sequence*

$$0 \longrightarrow A \xrightarrow{f} B \underset{i_2}{\overset{i_1}{\rightrightarrows}} B \otimes_A B \tag{1.31}$$

is exact, i.e., f is injective and $\ker(i_1, i_2) = f(A)$. *More generally for any A-module M, the sequence*

$$0 \longrightarrow M \longrightarrow M \otimes_A B \rightrightarrows M \otimes_A B \otimes_A B$$

is exact.

Proof. The well-known trick is to perform the base change $- \otimes_A B$. Indeed from the faithfully flat assumption it suffices to check the expected property after this base change. The sequence (1.31) becomes

$$0 \longrightarrow B \xrightarrow{f \otimes 1} B \otimes_A B \underset{i_2 \otimes 1}{\overset{i_1 \otimes 1}{\rightrightarrows}} B \otimes_A B \otimes_A B\ .$$

Let us define a ring homomorphism $\rho : B \otimes_A B \to B$ by $\rho(\alpha \otimes \beta) = \alpha\beta$. Clearly

$$\rho(f \otimes 1) = id, \quad (1 \otimes \rho)(i_1 \otimes 1) = id \quad \text{and} \quad (1 \otimes \rho)(i_2 \otimes 1) = (f \otimes 1)\rho.$$

Assume that $(i_1 \otimes 1)(\xi) = (i_2 \otimes 1)(\xi)$, then

$$\xi = (1 \otimes \rho)(i_1 \otimes 1)(\xi) = (1 \otimes \rho)(i_2 \otimes 1)(\xi) = (f \otimes 1)(\rho(\xi)).$$

This proves our claim for $M = A$. For an arbitrary M the argument is exactly the same. □

To go further, we need a topological property of faithfully flat morphisms, but this requires a finiteness condition:

Lemma 1.41.

i) *If $f : X \to Y$ is faithfully flat and quasi-compact*[13]*, then a subset $V \subset Y$ is open if and only if $f^{-1}(V)$ is open in X, therefore the topology of Y is the quotient topology of X by the equivalence relation defined by f.*
ii) *Let $f : X \to Y$ be a locally of finite presentation (of finite type, assuming the schemes locally noetherian), and faithfully flat. Then f is open.*

Proof. See [33], Exercise 9.1 or [62], Lemmas 02JY and 01UA. □

Then we can state the main property of the functor of points of a scheme. See Mumford's Red book [46], Chap. II, § 6 for a nice discussion about Grothendieck's existence problem. By an fpqc morphism, we mean a faithfully flat and quasi-compact morphism (see Lemma 1.41).

Theorem 1.42. *The functor of points h_S of a scheme S is an fppf sheaf (even an fpqc sheaf).*

Proof. Let $\rho : X' \to X$ be an fpqc morphism. The proof amounts to checking that given $f' : X' \to S$ such that $f'p_1 = f'p_2$, then f' factors uniquely through X, as in the diagram:

$$\begin{tikzcd} X'' = X' \times_X X' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X' \arrow[r, "\rho"] \arrow[d, "f'"'] & X \arrow[dl, dotted, "f"] \\ & S. & \end{tikzcd} \tag{1.32}$$

One says that $\rho : X' \to X$ is an effective epimorphism. In case all schemes are affine, the exact sequence (1.31) gives immediately the answer. Let us consider the general case. The functor h_S is a Zariski sheaf, thus we can easily reduce the checking to the case where X is affine. Assuming this, since ρ is quasi-compact, X' is covered by finitely many affine open sets, let $X' = \cup_{i=1}^n X'_i$, $X'_i = \mathrm{Spec}(A'_i)$ be such a covering. This Zariski covering yields a faithfully flat morphism $\tilde{X} =$

[13] This means that the inverse image of a quasi-compact subset is quasi-compact.

$\coprod_{i=1}^n X'_i \to X'$ (see Exercise 1.47). We have the following commutative diagram

$$\begin{tikzcd}
X' \times_X X' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & X' \arrow[r, "\rho"] \arrow[d, bend left, "f'"] & X \arrow[d, equal] \\
\tilde{X} \times_X \tilde{X} \arrow[u, "v"] \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & \tilde{X} \arrow[u, "u"] \arrow[r, "\tilde{\rho}"] \arrow[d, "\tilde{f}'"'] & X \arrow[dl, dotted, "\tilde{f}"] \\
& S &
\end{tikzcd}$$

the arrows being the obvious ones. Clearly $\tilde{f}'q_1 = \tilde{f}'q_2$, thus assuming the result true for $\tilde{\rho} : \tilde{X} \to X$, we get a unique morphism $\tilde{f}$ such that $\tilde{f}\tilde{\rho} = \tilde{f}'$, equivalently $\tilde{f}\rho u = f'u$. But $u : \tilde{X} \to X$ is a Zariski cover and h_S a Zariski sheaf, this in turn yields the equality $\tilde{f}\rho = f'$. Clearly $\tilde{f}$ is the expected unique solution. Notice that $\tilde{X}' = \operatorname{Spec}(\prod_{i=1}^n A'_i)$, thus the problem is reduced to the purely affine case $X' = \operatorname{Spec} A' \to \operatorname{Spec} A$. Let $S = \cup_\alpha S_\alpha$ be a covering by affine open subsets. Under these assumptions, let us show first the solution f is unique. Indeed if $f_1\rho = f_2\rho$, since ρ is surjective, we have $f_1 = f_2$ as maps between sets. Then $f'^{-1}(S_\alpha) = \rho^{-1}(f_i^{-1}(S_\alpha))$. What has just been proved yields the result if S is affine, which in turn yields $f_1 = f_2$. The last claim is the existence of f. We first build f as a map between sets. It suffices to check that $\rho(x'_1) = \rho(x'_2)$ implies $f'(x'_1) = f'(x'_2)$. We know that under this assumption

$$p_1^{-1}(x'_1) \cap p_2^{-1}(x'_2) = \operatorname{Spec} k(x'_1) \otimes_{k(y)} k(x'_2) \neq \emptyset \quad (\rho(x'_1) = \rho(x'_2) = y)$$

taking $z \in p_1^{-1}(x'_1) \cap p_2^{-1}(x'_2)$, we get $f'(x'_1) = f'p_1(z) = f'p_2(z) = f'(x'_2)$. In this way we can fill in the dotted arrow in diagram (1.32) by a continuous map h, for this use Lemma 1.41. Now let $S = \cup_\alpha S_\alpha$ be a covering by affine open sets. Then $f'^{-1}(S_\alpha) = \rho^{-1}(h^{-1}(S_\alpha))$ is a saturated open set in X'. Our previous reduction to X and X' affine works identically, and in this case the result comes from (1.31). This shows that h is a morphism of schemes, and this completes the proof. □

The foundational result Theorem 1.42 says that in order to get a scheme from a Zariski sheaf you need first to check this sheaf in indeed an fppf (even fpqc) sheaf. We will see in the next section how to enlarge the category **Sch** from this point of view.

A second remarkable descent result deals with the fibered category of quasi-coherent sheaves **Qcoh** (Example 1.12). It states that quasi-coherent sheaves satisfy the descent property with respect to fpqc coverings, and as a consequence for an fppf or an étale covering. The easy part concerns the morphisms of quasi-coherent sheaves:

Theorem 1.43. *Let $\rho : X' \to X$ be a quasi-compact faithfully flat morphism. For any pair of quasi-coherent modules over X, the diagram*

$$\mathrm{Hom}_Y(\mathcal{F},\mathcal{G}) \xrightarrow{f^*} \mathrm{Hom}_X(f^*(\mathcal{F}), f^*(\mathcal{G})) \underset{p_2^*}{\overset{p_1^*}{\rightrightarrows}} \mathrm{Hom}_{X'}(q^*(\mathcal{F}, q^*(\mathcal{G}))$$

deriving from (1.32) *is exact.*

Proof. The proof is local on X, and as in the preceding proof, we can easily reduce it to X and X' affine. Then the conclusion follows from the exact sequence (1.31). See also [9], Chapter 6 or [64]. □

The descent result for quasi-coherent modules is:

Theorem 1.44. *Let $f : X' \to X$ be a quasi-compact faithfully flat morphism of schemes. Then the pullback functor $\mathcal{F} \mapsto f^*(\mathcal{F})$ is an equivalence of categories between* $\mathbf{Qcoh}(X)$ *on one hand and the category of quasi-coherent $\mathcal{O}_{X'}$-modules with descent data on the other hand.*

Proof. See for example [9], Theorem 6.1/4., or [62], Proposition 023T. □

Ultimately the objects we want to descend are the schemes themselves. The setting for this problem is the fibered category of schemes over schemes, i.e., the fibered category $\mathcal{C}$ over $\mathcal{S} = \mathbf{Sch}$, the objects of $\mathcal{C}$ being the morphisms $f : X \to S$, with the obvious morphisms. Then $p(X \to S) = S$. The fiber category is $\mathbf{Sch}/S$ the category of S-schemes. We already saw in Theorem 1.42 that we are able to descend the morphisms of schemes along an fpqc descent datum. This combined with the descent of quasi-coherent sheaves suggest that we are able to descend quasi-coherent sheaves of algebras, and then schemes affine over a base scheme. We slightly different notations, let $\rho : S' \to S$ be a quasi-compact faithfully flat morphism. Let us consider the diagram

$$S'' = S' \times_S S' \underset{p_2}{\overset{p_1}{\rightrightarrows}} S' \xrightarrow{\rho} S$$

Let $X' \in \mathrm{Ob}(\mathbf{Sch}/S')$ be an S'-scheme, equipped with a descent datum $\phi : p_1^*(X') \xrightarrow{\sim} p_2^*(X')$ relative to the fpqc covering $S' \to S$ (1.30). An open set $U' \subset X'$ is called **stable** under ϕ, if ϕ maps $p_1^*(U')$ onto $p_2^*(U')$, equivalently ϕ restricts to a descent datum on U'. Even if it is not true that descent data on schemes are always effective, with some restrictions on S and S' a positive result can be achieved. For details see for example [9], 6.1/6.

Theorem 1.45. *Let $\rho : S' \to S$ be faithfully flat and quasi-compact. Then*

i) *The functor $X \mapsto \rho^*(X) = X \times_S S'$ from S-schemes to S'-schemes equipped with descent data is fully faithful.*
ii) *Assume that S and S' affine schemes. Then a descent datum on an S'-scheme is effective if and only if X' can be covered by quasi-affine*[14] *open sets which are stable under ϕ.*

[14] Open subscheme of an affine scheme.

Proof. (sketch) Assertion i) is a direct consequence of Theorem 1.42. In order to verify the if part of ii), we can use i) to reduce the general case to the case X' quasi-affine. Thus $X' \hookrightarrow \operatorname{Spec}\Gamma(X', \mathcal{O}_{X'})$ is a quasi-compact open immersion. Observe that there is a descent datum on $Y' = \operatorname{Spec}\Gamma(X', \mathcal{O}_{X'})$ coming from ϕ. Then use Theorem 1.43 to solve the descent of Y'. Then conclude. The only-if part is easy. □

Exercise 1.46. *Let $p : \mathcal{C} \to \mathcal{S}$ be a fibered category (in groupoids), and let $f : S' \to S$ be a morphism of $\mathcal{S}$. Assume that f admits a section, i.e., $\sigma : S \to S'$, with $f\sigma = 1_S$. Prove that any descent datum $\phi : p_1^*(x') \xrightarrow{\sim} p_2^*(x')$ on an object $x' \in \mathcal{C}(S')$ is effective.*

Exercise 1.47. *Let A be a commutative ring, and $(f_1, \ldots, f_n)$ a family of elements of A such that $A = \sum_{i=1}^n Af_i$. Prove that the canonical morphism $A \to \prod_{i=1}^n A_{f_i}$ is faithfully flat.*

Exercise 1.48. *With the notations of Theorem 1.45, let (S_i) be an open covering of S. Denote S_i', X_i' the corresponding open subschemes of S', X'. Then a descent datum $\phi : p_1^*(X') \xrightarrow{\sim} p_2^*(X')$ is effective if and only if for each i, the induced descent datum ϕ_i on X_i' relatively to $\rho_i : S_i' \to S_i$ is effective.*

Exercise 1.49. *Let $f : X \to S$ be a morphism of schemes, and $S' \to S$ a faithfully flat morphism. If after base-change $f \times 1 : X \times_S S' \to S'$ is an isomorphism, then f is an isomorphism.*

1.2.5. Descent: examples. Our aim now is to illustrate the descent formalism in two very special cases. The first one, elementary, is the descent of schemes along a Zariski cover, and the second one is the so-called Galois descent of schemes.

These two examples can help to understand the definition of stacks (coming soon). Let $X \in \mathbf{Sch}$ a scheme. Its functor of points $h_X : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$, $S \mapsto h_X(S) = \operatorname{Hom}_{\mathbf{Sch}}(S, X)$, is clearly a Zariski sheaf, indeed an fppf sheaf. In other words if $S = \cup_i U_i$ is an open cover of $S \in \mathbf{Aff}$, then the following diagram with obvious arrows is exact

$$h_X(S) \longrightarrow \prod_i h_X(U_i) \rightrightarrows \prod_{i,j} h_X(U_i \cap U_j). \tag{1.33}$$

To recover X from the covering $X' = \coprod_i U_i$ is typically a descent problem. As explained before, this problem is essentially equivalent to checking that the Zariski sheaf $F : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ is representable. This is the most elementary result among the *techniques de construction*. Before doing this we need a definition, giving an answer to the question: how to translate in categorical language, some properties of morphisms of schemes, see [4] or [37] Chap. 4.

Definition 1.50. Let $\phi : F \to G$ be a morphism between two presheaves $\mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$. We say ϕ is **relatively representable**, or **schematic**, if for each scheme $S \in \mathbf{Sch}$, and $y \in G(S)$ the fiber product $F \times_G S$, a presheaf over $\mathbf{Sch}/S$, is represented by an S-scheme.

Notice that the fiber product (Exercise 1.32) means precisely $F \times_{G,y} S$, where $y : S \to G$ is the morphism given by the lemma of Yoneda (see Subsection 1.2.1).

We can be a bit more precise, for example we can say ϕ is an open (closed) immersion if for each pair (S, y) the induced morphism $T \to S$ is an open (closed) immersion. We will see that we may even extend the definition to a various class of properties of morphisms of schemes, étale, or étale surjective, for example.

The following well-known result is the Grothendieck **representability theorem** (see [4]) in its most elementary form:

Theorem 1.51. *A Zariski sheaf $\mathcal{X}$ is representable by scheme X if and only if there exists a family of open immersions $u_i : U_i \to \mathcal{X}$ (Definition 1.50) such that the following two conditions hold:*

i) *$u : \coprod_i U_i \to \mathcal{X}$ is surjective,*
ii) *u is representable.*

Furthermore,

iii) *X is separated if and only if for all pairs (i, j), the morphism of schemes $U_i \times_{\mathcal{X}} U_j \to U_i \times U_j$ is a closed immersion.*

Proof. This is the standard construction of a scheme by gluing schemes along open subsets [18]. As we will see, it is also a typical descent problem along a Zariski cover, or equivalently the problem to perform a quotient under an equivalence relation. For these reasons it is instructive to write down the details. Let us write the standard diagram

$$\begin{tikzcd} U_i \arrow[r, "u_i"] & \mathcal{X} \\ U_i \times_{\mathcal{X}} U_j \arrow[u, "v_i"] \arrow[r, "v_j"] & U_j \arrow[u, "u_j"'] \end{tikzcd} \tag{1.34}$$

so that condition ii) says $U_i \times_{\mathcal{X}} U_j$ is a scheme. Furthermore the arrows v_i, v_j are both open immersion. Let us denote by $U_{i,j} \subset U_i$ and $U_{j,i} \subset U_j$ the corresponding open sets. The isomorphism $U_i \times_{\mathcal{X}} U_j \to U_{i,j}$ together with the similar one to $U_{j,i}$ yields an isomorphism $\theta_{j,i}$, viz.

$$\begin{tikzcd} & U_i \times_{\mathcal{X}} U_j \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ U_{i,j} \arrow[rr, dotted, "\theta_{j,i}", "\sim"'] & & U_{j,i}. \end{tikzcd} \tag{1.35}$$

To get triple intersection one has to perform one more fiber product. Taking the fiber product of the diagram (1.35) with U_k, we get

$$\begin{tikzcd} & (U_i \times_{\mathcal{X}} U_j) \times_{\mathcal{X}} U_k \arrow[dl, "\sim"'] \arrow[dr, "\sim"] & \\ U_{i,j} \cap U_{i,k} \arrow[rr, dotted, "\theta_{j,i}", "\sim"'] & & U_{j,i} \cap U_{j,k}. \end{tikzcd}$$

Then, writing down the associativity of the fiber product

$$\begin{tikzcd}
(U_i \times_{\mathcal{X}} U_j) \times_{\mathcal{X}} U_k \arrow[r, "\sim"] \arrow[d, equal] & U_{i,j} \cap U_{i,k} \subset U_i \arrow[d, dotted, "\theta_{j,i}"] \\
(U_i \times_{\mathcal{X}} U_j) \times_{\mathcal{X}} U_k \arrow[r, "\sim"] \arrow[d, "\wr"'] & U_{j,i} \cap U_{j,k} \subset U_j \arrow[d, dotted, "\theta_{k,j}"] \\
U_i \times_{\mathcal{X}} (U_j \times_{\mathcal{X}} U_k) \arrow[r, "\sim"] & U_{k,j} \cap U_{k,i} \subset U_j
\end{tikzcd} \tag{1.36}$$

we get the cocycle condition

$$\theta_{k,j\,|U_{j,i}\cap U_{j,k}}\theta_{j,i\,|U_{i,j}\cap U_{i,k}} = \theta_{k,i\,|U_{k,j}\cap U_{k,i}}, \quad \theta_{i,j}\theta_{j,i} = 1_{U_{i,j}}. \tag{1.37}$$

We can rephrase this as follows. Let $U = \coprod_i U_i$. Then

$$R := U \times_{\mathcal{X}} U = \coprod_{i,j} U_i \times_{\mathcal{X}} U_j$$

embeds in $U \times U$. It is equipped with two canonical projections $d_0, d_1 : R \to U$. Notice that the restrictions of d_0 and d_1 to $U_i \times_{\mathcal{X}} U_j$ are v_i and v_j. There are more maps. Indeed denote $s : R \xrightarrow{\sim} R$ the map that switches the two factors, and let $e : U \to R$ be the diagonal. Finally the $\theta_{i,j}{}'^s$ glue together to yield a morphism, the gluing morphism

$$\theta : R \times_{d_1,U,d_0} R \longrightarrow R. \tag{1.38}$$

Let us denote by p_0, p_1 the projections $R \to U$. The cocycle condition (1.37) can be implemented in a commutative diagram

$$\begin{tikzcd}
R \times_{d_1,U,d_0} \times R \times_{d_1,U,d_0} R \arrow[r, "\theta\times id"] \arrow[d, "1\times m"'] & R \times_{d_1,U,d_0} R \arrow[d, "m"] \\
R \times_{d_1,U,d_0} R \arrow[r, "m"] & R
\end{tikzcd} \tag{1.39}$$

together with

$$\begin{tikzcd}
R \times_{d_1,U,d_0} R \arrow[r, "\theta"] \arrow[d, "p_1"'] & R \arrow[d, "d_1"] \\
R \arrow[r, "d_1"] & U
\end{tikzcd}$$

with the corresponding one. We can rephrase this by saying the datum

$$(U, R, d_0, d_1, \theta, s, e)$$

defines an **equivalence relation** on the scheme U, meaning that $R \subset U \times U$ is a locally closed immersion making the image of the set of points $|R|$ in $|X| \times |X|$ the graph of a set-theoretical equivalence relation. This can be seen as follows. We have $x \sim y$ $(x, y \in X)$ if and only if there is a point $\alpha \in R$, with $d_0(\alpha) = x$, $d_1(\alpha) = y$. We can think of α as a "path" joining x to y. The opposite path joins

y to x. If $y \sim z$, and $\beta \in R$ is a path joining y to z, the "product" $\alpha\beta$ which makes sense in R, yields $x \sim z$.

We can build a quotient scheme $X = U/R$ as follows. The topological space X is simply the quotient topological space U/R. Let us denote $\pi : U \to X$ the quotient map. It is easy to see that the open sets of the form $p^{-1}(\Omega)$, $\Omega \subset X$ open, are the R-stable open sets $V \subset U$, i.e., $d_0^{-1}(V) = d_1^{-1}(V)$. Notice π embeds U_i as an open set in X and $\pi^{-1}(U_i) = \coprod_j U_{j,i}$. Now we can use the isomorphisms $\theta_{i,j}$ to glue together the structural sheaves $\mathcal{O}_i$ and finally get a sheaf of ring $\mathcal{O}_X$ on X. This means that a section of $\mathcal{O}_X$ over $U_i \subset X$ can be seen as a family of sections $a_j \in \Gamma(U_{j,i}, \mathcal{O}_{U_j})$ such that $\theta_{j,i}^\sharp(a_j) = a_{i|U_{i,j}}$. Equivalently $\mathcal{O}_X$ is equal to the subsheaf of R-invariant sections $p_*(\mathcal{O}_U)^R$.

The claim is that $(X, \mathcal{O}_X)$ is a scheme that represents $\mathcal{X}$. Viewing $\{U_i\}$ as an open cover of X, the construction shows that the sections $u_i \in \Gamma(U_i, \mathcal{X})$ satisfy

$$u_{i|U_i \cap U_j} = u_{j\,|U_i \cap U_j} \tag{1.40}$$

for all (i, j). Since $\mathcal{X}$ is a Zariski sheaf, we get a section $\eta \in \mathcal{X}(X)$ that restricts to u_i on U_i. This shows that the morphism $\eta : U \to \mathcal{X}$ factors through X. It is not difficult to show that the morphism $X \to \mathcal{X}$ is indeed an isomorphism of sheaves, i.e., $h_X \xrightarrow{\sim} \mathcal{X}$. Let $\varphi : S \to \mathcal{X}$ be a section over S. We can perform the fiber product

$$\begin{tikzcd} U_i \arrow[r] & \mathcal{X} \\ U_i \times_{\mathcal{X}} S \arrow[u, "f_i"] \arrow[r] & S \arrow[u] \end{tikzcd} \tag{1.41}$$

The hypothesis shows that $S_i = U_i \times_{\mathcal{X}} S \hookrightarrow S$ is an open immersion, furthermore $(S_i)_i$ is an open cover of S. We must check that the morphism

$$U \times_{\mathcal{X}} S = \coprod_i S_i \xrightarrow{\coprod_i f_i} U \longrightarrow X \tag{1.42}$$

factors through S. This amounts to seeing the equality on each intersection

$$f_{i,j} = f_{i|S_i \cap S_j} = f_{j\,S_i \cap S_j} = f_{j,i}. \tag{1.43}$$

This is a simple consequence of the functoriality of the gluing morphism θ, leading to the commutative diagram (setting $S_{i,j} = S_{j,i} = S_i \cap S_j$):

$$\begin{tikzcd} U_{i,j} \arrow[r, "\theta_{j,i}"] & U_{j,i} \\ S_{i,j} \arrow[u, "f_{i,j}"] \arrow[r, equal] & S_{j,i}. \arrow[u, "f_{j,i}"'] \end{tikzcd}$$

Thus the sheaf $\mathcal{X}$ is isomorphic to X. Finally notice that ii) implies the morphisms in iii) are morphisms of schemes. The condition amounts to saying the diagonal of X is closed in $X \times X$. We leave the details as an exercise. $\square$

Exercise 1.52. *Let R be a complete local noetherian ring with maximal ideal $\mathcal{M}$. If F : $\mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ is representable, show that the natural map*

$$F(R) \longrightarrow \varprojlim F(R/\mathcal{M}^n)$$

is bijective, where $F(A)$ stands for $F(\operatorname{Spec} A)$.

A basic example where (1.51) works is the functorial definition of projective spaces. Recall that the functor of points $\mathcal{P}_n$ of the projective space $\mathbb{P}^n$ ([33], chap. II, section 7) is defined as follows. Let $\mathcal{P}_n(S)$ be the set of equivalence classes of morphisms $\varphi : \mathcal{O}_S^n \to \mathcal{L}$, where $\mathcal{L}$ is a line bundle over S, φ is a surjective morphism, and the equivalence is the relation $\sim$ defined by saying that

$$(\mathcal{O}_S^n \xrightarrow{\varphi} \mathcal{L}) \sim (\mathcal{O}_S^n \xrightarrow{\varphi'} \mathcal{L}')$$

if there is an isomorphism $\psi : \mathcal{L} \xrightarrow{\sim} \mathcal{L}'$ with $\psi\varphi = \varphi'$. Clearly this defines a contravariant functor $\mathcal{P}_n$ over the category **Sch**, which fulfills the hypothesis of Theorem 1.51. It is represented by the n-dimensional projective space $\mathbb{P}^n = \operatorname{Proj} k[X_0, \ldots, X_n]$. The construction extends over an arbitrary base scheme X. Let $\mathcal{E}$ be a locally free sheaf of rank $n \geq 1$ over X. Then the presheaf $(\mathbf{Sch}\,/X)^{\mathrm{op}} \to \mathbf{Set}$ given by

$$\mathbb{P}(\mathcal{E})(S \xrightarrow{f} X) = \{f^*(\mathcal{E}) \to \mathcal{L}\,/\mathrm{i}so\} \tag{1.44}$$

with $\mathcal{L}$ a line bundle on S and $f^*(\mathcal{E}) \to \mathcal{L}$ is onto, is a Zariski sheaf. It is not difficult to see that Theorem 1.51 applies verbatim, which in turn yields an X-scheme $p : \mathbb{P}(\mathcal{E}) \to X$, the **projective bundle** of rank 1 quotients of $\mathcal{E}$. The identity 1 of $\mathbb{P}(\mathcal{E})$ corresponds to a rank 1 quotient $p^*(\mathcal{E}) \to \mathcal{O}(1)$, the so-called tautological line bundle.

Exercise 1.53. *If $\mathcal{E}, \mathcal{E}'$ are two locally free sheaves of rank n over X, then $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}')$ are isomorphic over X if and only if there is an invertible sheaf $\mathcal{L}$ on X such that $\mathcal{E}' \cong \mathcal{E} \otimes \mathcal{L}$).*

Exercise 1.54. *Let $p : \mathbb{P}(\mathcal{E}) \to X$ be a projective bundle over X, and relative dimension two* ($\operatorname{rank} \mathcal{E} = 2$). *Show that the sections $\sigma : X \to \mathbb{P}(\mathcal{E})$ are in one-to-one correspondence with the rank 1 subbundles $\mathcal{K} \subset \mathcal{E}$. Under this correspondence, if $\mathcal{N} = \ker(\mathcal{E} \to \mathcal{L})$, and if $D = \sigma(X)$, then show D is a Cartier divisor with invertible sheaf $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(D) \cong \mathcal{O}(1) \otimes p^*(\mathcal{N})^{-1}$.*

Our next example is a glimpse about the **Galois descent**, a special but important case of étale descent. Let as usual a morphism (covering) $\rho : S' \to S$, which in this example is an étale Galois cover with Galois group G (a finite group). This means that G acts (on the left) on S', by mean of S-automorphisms, viz.

$$\sigma : G \times_S S' \longrightarrow S'$$

and that this action is **free**. At the level of points the action is denoted $\sigma(g, x') = gx'$. That the action is free simply means that the diagram

$$\begin{tikzcd} G \times S' \arrow[r, shift left, "\sigma"] \arrow[r, shift right, "p_2"'] \arrow[d, "\psi"', "\wr"] & S' \arrow[d, equal] \\ S'' \arrow[r, shift left, "p_1"] \arrow[r, shift right, "p_2"'] & S' \end{tikzcd} \tag{1.45}$$

is cartesian, where $\psi = (\sigma, p_2)$, and $\psi(g, x') = (gx', x')$, equivalently the canonical morphism

$$G \times S' \to S' \times_S S',\ (g, x') \mapsto (gx', x')$$

is an isomorphism. It will be convenient to identify the two horizontal rows of this diagram.

We can ask how to write a descent datum on a scheme X' over S', in terms of the upper row of the diagram. The first projection becomes σ, and the S''-isomorphism $\phi : p_1^*(X') \xrightarrow{\sim} p_2^*(X')$ becomes

$$\phi : \sigma^*(X') = (G \times S') \times_{\sigma, S'} X' \xrightarrow{\sim} p_2^*(X') = (G \times S') \times_{p_2, S'} X',$$

or

$$\phi : G \times X' \xrightarrow{\sim} G \times X'. \tag{1.46}$$

With these identifications in mind, let us consider the morphism

$$G \times X' \longrightarrow (G \times S') \times_{p_2, S', f} X' = G \times X'$$

given by ϕ twisted by $g \mapsto g^{-1}$, that is, on the level of points:

$$(g, x') \longrightarrow \phi(g^{-1}, x') = (g^{-1}, gf(x'), \varphi(g, x')) \rightsquigarrow (g^{-1}, \varphi(g, x')).$$

This defines a morphism $\varphi : G \times X' \to X'$, which given the previous identifications has the property $f(\varphi(g, x')) = gf(x')$. We must find how the cocycle condition (1.37) reads after these identifications. We have

$$S''' = S'' \times_{p_2, S', p_1} S'' = (G \times S') \times_{p_2, S', \sigma} (G \times S') = G \times G \times S'.$$

At the level of points, this identification is $((g, x'), (h, y'))_{x' = hy'} \mapsto (g, h, y')$. To go further we need to identify the projections p_{ij}

$$S''' = G \times G \times X' \underset{p_{23}}{\overset{p_{12}}{\underset{\longrightarrow}{\overset{\longrightarrow}{\overset{p_{13}}{\longrightarrow}}}}} S'' = G \times X'.$$

They are given on the level of points by

$$(g, h, y') \mapsto \begin{cases} p_{12}(g, h, y') = (g, hy') \\ p_{23}(g, h, y') = (h, y') \\ p_{13}(g, h, y') = (gh, y'). \end{cases}$$

It is not difficult to check that the cocycle condition reads $\varphi(g, \varphi(h, y')) = \varphi(gh, y')$. This together with $\varphi(1, y') = y'$ says that the descent data on X' are in one-to-one correspondence with the actions of G on X' that lift the action on S', i.e.,

making f G-equivariant. Theorem 1.45 applies, and gives us the following descent result:

Proposition 1.55. *The notations being as before, assume X' has a covering by affine G-stable open subschemes. Then X' together with its G-action comes from a uniquely defined S-scheme X, viz. there is a G-isomorphism $X' \xrightarrow{\sim} S' \times_S X$, where the G-action on the right comes from S'.*

Obviously there are many variations on the Galois descent problem. We can work with (quasi-)coherent sheaves instead of schemes. In a case the descent datum on the quasi-coherent $\mathcal{O}_{S'}$-module $\mathcal{F}$ is identical to a lifting of the action of G on S' to $\mathcal{F}$, this consists of an isomorphism $\varphi : \sigma^*(\mathcal{F}) \xrightarrow{\sim} p_2^*(\mathcal{F})$ on $G \times S'$, satisfying the cocycle condition

$$(m \times 1)^*(\varphi) = p_{23}^*(\varphi) \circ (1 \times \sigma)^*(\varphi) \tag{1.47}$$

where $m : G \times G$ denotes the multiplication of G, the notations being the obvious ones. Since G is a finite discrete group, this amounts to a collection of isomorphisms for all $g \in G$, $\varphi_g : g^*(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}$ with

$$\varphi_h \circ h^*(\varphi_g) = \varphi_{hg}.$$

This is what is often called a **G-linearization** of $\mathcal{F}$. Clearly this definition applies verbatim to more general groups than finite discrete groups, for example the multiplicative group $\mathbf{G}_m$.

The effectiveness of descent data on quasi-coherent sheaves says that in this setting $\mathcal{F}$ together with its G-linearization comes from, or descent to, an $\mathcal{O}_X$-module. The most elementary manifestation of this result is the classical Galois descent, viz. $S = \operatorname{Spec} k$, $S' = \operatorname{Spec} k'$, where k'/k is a Galois extension of fields with group G. Then any k'-vector space endowed with an action G, compatible with the action on k', comes from a k-vector space by the extension of scalars $k \to k'$.

1.3. Stacks

1.3.1. Algebraic spaces. As we have seen, in order to define a scheme, the categorical viewpoint invites us to start with a Zariski sheaf $F : \mathbf{Sch}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, and then try to prove it is representable by a scheme X, i.e., $F(-) = \operatorname{Hom}_{\mathbf{Sch}}(-, X)$. Even if F is an fppf sheaf, a necessary condition, it is not obvious to achieve such a result. A foundational problem falling into this perspective is the **Picard functor** (see [4], [9], Chap. 8, [39]). For any scheme X, recall that the Picard group is the group of isomorphism classes of invertible sheaves on X, equivalently $\operatorname{Pic}(X) = \mathrm{H}^1(X, \mathcal{O}_X^*)$ ([33], Chap. III, Ex. 4.5). We can extend this group to a presheaf of abelian groups $\mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$ by $S \mapsto \operatorname{Pic}(X \times S)$, but for trivial reasons (*cf* introduction) this is not a Zariski sheaf, not even a separated presheaf. Indeed the invertible sheaves coming from S by pullback must be trivial. If $p_2 : X \times S \to S$ has a section this forces a line bundle on S to be trivial. This is obviously not the case if $S = \mathbb{P}^1_X$ with the sheaf $\mathcal{O}(1)$.

This suggests to take as a plausible definition of the Picard functor of a k-scheme:

$$\mathbf{Pic}_X(S) = S \mapsto \mathrm{Pic}(X \times S)/p^*(\mathrm{Pic}(S)) \quad (S \in \mathbf{Sch}_k). \tag{1.48}$$

An element of $\mathbf{Pic}_X(S)$ can be seen roughly as a family of invertible sheaves on X parameterized by S. Assuming $X(k) \neq \emptyset$, it is not difficult to check this really defines a Zariski sheaf, indeed an fppf sheaf. More generally we can define a relative Picard functor for a morphism $f : X \to S$ (loc. cit.), it is defined over $\mathbf{Sch}/S$ as in (1.48) by $(T \to S) \mapsto \mathbf{Pic}_{X/S}(T) = \mathrm{Pic}(X \times_S T)/p_2^*(\mathrm{Pic}(T)$. In general, i.e., unless f has a section, one need to sheafify this presheaf for the fppf topology to get a sheaf[15] (Proposition 1.30). The resulting functor $\mathcal{P}_{X/S_{(fppf)}}$ is then substantially different from the original one. The sheafification process says that a section of this sheaf over $T \to S$, can be understood as an invertible sheaf not on $X \times_S T$, but on $(X \times_S T'_i)_i$ for some fppf covering $T'_i \to T$. These sheaves must agree on an fppf covering of the overlaps $T'_i \times_T T'_j$.

Even at this stage, some strong assumptions are necessary to get the representability of this functor. The first main result along these lines is the following ([9], Theorem 8.2/3):

Theorem 1.56 (Murre, Oort). *Let X/k be a proper scheme over a field k, then the functor $\mathbf{Pic}_X$ is representable by a scheme locally of finite type over k.*

This result was extended by Grothendieck to the relative case, i.e., $f : X \to S$ is projective and flat, with the additional requirement that the geometric fibers are integral. Despite this important result, the Picard functor in the relative case $f : X \to S$, even if f is flat, proper (even projective), and finitely presented, is not necessarily a scheme. We refer to the beginning of Section 8.2 of [9], or to [39] for a discussion of an example due to Mumford. A major advance was the approach of Artin and Raynaud to this problem. With these hypotheses, and a mild additional technical assumption, their results state that the functor $\mathbf{Pic}_{X/S}$ is (represented by) an algebraic space (see [9], Section 8.3). We refer to the exposition [39] for a complete discussion and proofs.

But what kind of object is an algebraic space? The answer comes roughly by changing in the representability Theorem 1.51 a Zariski cover by an étale cover and keep only the functorial way of thinking. Algebraic spaces were introduced by Artin, precisely for that purpose. Our presentation will be very sketchy since algebraic spaces are indeed algebraic stacks, precisely Deligne-Mumford stacks.

The definition of algebraic spaces modelled on the categorical definition of (separated) schemes is as follows[16]. Recall (Definition 1.50) that a morphism of sheaves $\varphi : F \to G$ is called schematic (relatively representable) if for any scheme S and $y \in G(S)$ the sheaf $F_y = F \times_{S,y} G$ over $\mathbf{Sch}/S$ is a scheme (over S). Thus

[15] The Zariski topology is in general too weak [39], Exercise 2.4.

[16] In [62] Definition 025Y, a more flexible definition is proposed: the condition iii), separability is ignored, and F is taken as an fppf sheaf. A fundamental theorem of Artin then provides an étale atlas.

it makes sense to declare φ étale (or an open or closed immersion, or smooth), if for all (S, y), $F_y \to S$ is étale (or an open or closed immersion, or smooth).

Definition 1.57. A presheaf $F : \mathbf{Sch}^{\mathrm{op}} \longrightarrow \mathbf{Set}$ is a (separated) **algebraic space** if

i) F is an étale sheaf (see Definition 1.28).
ii) There exists a family of affine schemes U_i, and for each i an étale morphism $U_i \to F$, such that the morphism $\coprod_i U_i \to F$ is étale (in particular representable), surjective. It is called a **presentation** (or **atlas**) of F.
iii) (separability) the canonical morphism $U_i \times_F U_j \to U_i \times U_j$ is a closed immersion.

In ii) surjective means that for any pair (S, x), $x \in F(S)$ the collection of morphisms $U_i \times_F S \to U_i$ is an étale covering of S. The difference with (1.51) is the replacement of a Zariski cover by an étale cover. Finally it is instructive to understand the Definition 1.57, i.e., the resulting object, as the result of a quotient of the scheme U, but now by an étale equivalence relation. Indeed let $U = \coprod_i U_i$. Then by ii) $R = U \times_F U$ is a scheme equipped with two projections

$$R \underset{d_1}{\overset{d_0}{\rightrightarrows}} U \ . \tag{1.49}$$

As in Theorem 1.51 $R \subset U \times U$ is the graph of a closed equivalence relation. But now the projections are not local immersions, they are only étale. This is readily seen from ii), indeed the restriction of d_1 to the open set $d_1^{-1}(U_i) = \coprod_j U_i \times_F U_j$ is étale over U_i. One can conclude that the sheaf F is the quotient, in a categorical sense, $F = U/R$, meaning that the étale sheaf F is a coequalizer

$$R \rightrightarrows U \longrightarrow F.$$

This must be compared with the beautiful existence result ([58], Exp. 5, Theorem 4.1) which we recall (almost) without proof[17]:

Theorem 1.58. *Assume given an affine scheme U and an equivalence relation $R \subset U \times U$. Assume that $d_0 = pr_1 : R \to U$ (then also $d_1 = pr_2 : R \to U$) is finite and locally free. The coequalizer fppf sheaf is an affine scheme.*

Proof. The proof follows closely the classical proof when the equivalence relation comes from the action of a finite group on an algebra of finite type over a field. Suppose that $U = \operatorname{Spec} A$, then let $A^R = \{a \in A \mid d_0^*(a) = d_1^*(a)\}$ be the subalgebra of invariant elements. Then one can check that $\operatorname{Spec}(A^R)$ satisfies the property of the coequalizer. □

The separability condition as in the case of schemes is a property of the diagonal $\Delta : F \to F \times F$. Since the diagonal plays a key role in the case of stacks, it will be useful to compare this case with the case of algebraic spaces.

[17] This result is used to prove the existence of a quotient G/H of an affine group scheme, by a closed subgroup. The result of loc. cit. is more general in the sense that we may assume that R is a groupoid (Section 2.1).

Proposition 1.59. *Let F be an algebraic space. Then the diagonal $\Delta : F \hookrightarrow F \times F$ is a closed immersion. In particular it is schematic.*

Proof. The result amounts to proving that given a cartesian diagram of sheaves of sets

$$\begin{tikzcd} F \arrow[r, "\Delta"] & F \times F \\ \Delta_{\alpha,\beta} \arrow[u] \arrow[r] & S, \arrow[u, "(\alpha,\beta)"'] \end{tikzcd}$$

the étale sheaf $\Delta_{\alpha,\beta}$ is a closed subscheme of S. Notice that the diagram below, Δ_S being the diagonal of S, is cartesian

$$\begin{tikzcd} \Delta_{\alpha,\beta} \arrow[r] \arrow[d] & S \times_{\alpha,F,\beta} S \arrow[d] \\ S \arrow[r] & S \times S. \end{tikzcd}$$

It suffices to check that the natural morphism $S \times_{\alpha,F,\beta} S \to S \times S$ is a closed immersion, i.e., $S \times_{\alpha,F,\beta} S$ is a scheme, and the resulting morphism is a closed immersion. This is clear if β factors through some étale open $U_i \to F$ (check this). In the general case one can check this after the étale cover $\rho : S' = \coprod_i U_i \to S$, and then use a descent argument to get the result. This means that we have the diagram

$$\begin{tikzcd} S' \arrow[r, "\rho"] & S \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] & F \\ \Delta_{\alpha',\beta'} \arrow[u] \arrow[r] & \Delta_{\alpha,\beta} \arrow[u] & \end{tikzcd}$$

with $\alpha' = \alpha\rho, \beta' = \beta\rho$. Then we know from the previous step that $\Delta_{\alpha',\beta'}$ is a closed subscheme of S', which in turn yields that the étale sheaf $\Delta_{\alpha,\beta}$ is isomorphic to a closed subscheme of S, as expected. Indeed scheme $\Delta_{\alpha',\beta'}$ has a natural descent datum coming from the fact that as a sheaf it comes from $\Delta_{\alpha,\beta}$. Then one can descend this closed subscheme to a scheme over S, with sheaf of points $\Delta_{\alpha,\beta}$, showing it is a scheme. Finally another descent argument shows $\Delta_{\alpha,\beta} \to S$ is a closed immersion. □

It is clear from Definition 1.57 together with Theorem 1.51 that we get the inclusions:

Schemes $\subset$ **Algebraic spaces** $\subset$ **Étale sheaves**.

Finally a last comment. For a field k, call k**-point**[18] of an algebraic space F a point of $F(k)$. An étale neighborhood of a k-point x is an étale map $(U, \star) \to F$ of a pointed affine scheme U, $\star \in U(k)$, and $\star \mapsto x$. Now taking the inductive limit of the local rings $\mathcal{O}_{U,\star}$, we can define the local ring of F at x (see [4] or [62] Definition

[18]For simplicity, we write $F(A)$ instead of $F(\operatorname{Spec} A)$.

04KG). A large part of the theory of schemes (open and closed subschemes, (quasi)-coherent modules, etc.) has a natural counterpart in the setting of algebraic spaces. This will be emphasized in a next section dealing with algebraic stacks 3. To get a feeling of this assertion, be a closed (open) sub-algebraic space $Z \hookrightarrow F$ of the algebraic space F, we mean an étale subsheaf such that the inclusion is a closed (open) immersion, i.e., representable by a closed (open) immersion.

The difference between schemes and algebraic spaces is "small"; one can prove for example that any algebraic space of finite presentation has an open dense subspace which is a scheme ([4], Proposition 4.5). It would be instructive to describe an example of an algebraic space which is not a scheme, but these examples tend to be somewhat exotic. In some cases the Picard functor (1.48) leads to such an example, but not a simple one. Simpler are the examples given by contracting a configuration of curves with definite negative matrix of self-intersection numbers [4]. One can also find in [4] a discussion of Hironaka's celebrated 3-dimensional example.

Exercise 1.60. *Show that the sheaf product $F \times G$ of two algebraic spaces is an algebraic space. Extend this to the fiber product $F \times_H G$.*

1.3.2. Prestacks. In this section we add one more level to the preceding tower of categories:

Schemes $\subset$ **Algebraic spaces** $\subset$ **Algebraic stacks**.

We start by defining a prestack, an object which looks like a presheaf. Then we will give the definition a stack, roughly an object which "represents" a sheaf in groupoids. In the next subsection the stackification functor which allows to get a stack from a prestack, generalizing the sheafification functor, will be described. Throughout we work over a site $\mathcal{S}$, and a fibered category in groupoids $p : \mathcal{C} \to \mathcal{S}$ will be seen without further notification as a **lax functor** $\mathcal{S} \to$ **GPO**.

Definition 1.61. A category fibered in groupoids $p : \mathcal{C} \to \mathcal{S}$ is a **prestack** if for any $S \in \operatorname{Ob}\mathcal{S}$, and any pair $x_1, x_2 \in \mathcal{S}(S)$, the contravariant functor $\mathbf{Isom}(x_1, x_2) : \mathcal{S}/S \longrightarrow$ **Set** defined in (1.9) is a sheaf of sets.

The meaning of this definition is as follows. Let $f : T \to S$ be a morphism, $(T_i \to T)_i$ a covering family of T, and $T_{ij} = T_i \times_T T_j$. Then we have an exact diagram of sets:

$$\operatorname{Isom}_T(f^*(x_1), f^*(x_2)) \longrightarrow \prod_i \operatorname{Isom}_{T_i}(f_i^*(x_1), f_i^*(x_2)) \rightrightarrows \prod_{i,j} \operatorname{Isom}_{T_{ij}}(f_{ij}^*(x_1), f_{ij}^*(x_2)) \tag{1.50}$$

where f_i, f_{ij} stand for the "restriction" of f to T_i, T_{ij} and the arrows are the obvious restriction maps. In other words, we can glue the isomorphisms. It will be convenient to use sometimes the convenient notation $x_{|V}$ instead of $f^*(x)$, for any $f : T \to S$, and $x \in \mathcal{C}(S)$. Likewise the restriction from T to V of an isomorphism $\xi : x_1 \xrightarrow{\sim} x_2$ will be denoted for short $\xi_{|V} : x_{1|V} \xrightarrow{\sim} x_{2|V}$.

We can think of a prestack as a *separated* presheaf in groupoids, and then a stack will be a **sheaf in groupoids**.

At this point we are ready to give the definition of a stack. Recall that we work over a fix site $\mathcal{S}$.

Definition 1.62. A **stack (in groupoids) over a site** $\mathcal{S}$ is a fibered category in groupoids $p : \mathcal{C} \to \mathcal{S}$ such that:

1. $\mathcal{C} \to \mathcal{S}$ is a prestack, and
2. for all coverings $\mathcal{U} = (U_i \to U)_i$ of $U \in \mathrm{Ob}\,\mathcal{S}$ in the site $\mathcal{S}$, all descent data $\{x_i, \phi_{ij}\}$ for $\mathcal{U}$ are effective. In other words, for any U and any covering $\mathcal{U} = (U_i \to U)_i$ of U, the functor (1.29) is an equivalence of categories.

Usually the hardest part to check is the second condition. Define the 2-category $\mathbf{PStack}_{\mathcal{S}}$ (resp. $\mathbf{Stack}_{\mathcal{S}}$) as a full 2-subcategory of $\mathbf{CFG}$. This means that the objects of $\mathbf{PStack}_{\mathcal{S}}$ (resp. $\mathbf{Stack}_{\mathcal{S}}$) are the (pre)stacks based over $\mathcal{S}$, the 1-morphisms $\mathcal{X} \to \mathcal{Y}$ are the base preserving functors, and the 2-morphisms $F \cong G$ are the functorial (iso)morphisms. An important observation is the fact that the 2-subcategory of prestacks, respectively stacks is preserved by the fibered product which takes place in the 2-category $\mathbf{CFG}$, see Definition 1.15.

Proposition 1.63. *Assume given a diagram of prestacks* (*resp. stacks*)

$$\begin{tikzcd} \mathcal{X} \arrow[dr, "F"'] & & \mathcal{Y} \arrow[dl, "G"] \\ & \mathcal{Z} & \end{tikzcd}$$

over the site $\mathcal{S}$*. Then* $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ *is a prestack* (*resp. a stack*)*.*

Proof. Let us check the claim is the case of stacks. Let there be given two objects $\xi_i = (x_i, y_i, \theta_i)$ $(i = 1, 2)$ of the fiber product, see Definition 1.15. Let us describe the presheaf $\mathbf{Isom}_S(\xi_1, \xi_2)$. A section of this presheaf over $f : T \to S$ is a pair of isomorphisms $u : f^*(x_1) \to f^*(x_2)$, $v : f^*(y_1) \to f^*(y_2)$ making the diagram

$$\begin{tikzcd} F(x_1)_{|T} \arrow[r, "F(u)"] \arrow[d, "\theta_{1|T}"'] & F(x_2)_{|T} \arrow[d, "\theta_{2|T}"] \\ F(y_1)_{|T} \arrow[r, "F(v)"] & F(y_2)_{|T} \end{tikzcd}$$

commutative. This shows that $\mathbf{Isom}_S(\xi_1, \xi_2)$ is the equalizer of

$$\mathbf{Isom}_S(x_1, x_2) \times \mathbf{Isom}_S(y_1, y_2) \underset{\theta_2}{\overset{\theta_1}{\rightrightarrows}} \mathbf{Isom}_S(x_1, y_2)\ ,$$

thus a sheaf. Let $(U_i \to U)_i$ be a covering of $U \in \mathcal{S}$. Assume that (x_i, y_i, θ_i) is an object of the fiber product fibered category $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$, equipped with a descent data. In view of Definition 1.15 this means we have a pair of descent data (x_i, φ_{ij}), (y_i, ψ_{ij}), and a morphism

$$(\theta_i) : (F(x_i), F(\varphi_{ij})) \longrightarrow (G(y_i), G(\psi_{ij})).$$

We see readily that $(F(x_i), F(\varphi_{ij}))$ and $(G(y_i), G(\psi_{ij}))$ are descent data in $\mathcal{Z}$. Since $\mathcal{X}, \mathcal{Y}$ are stacks, these descent data are effective, thus they produce two ob-

jects $x \in \mathcal{X}(U)$ and $y \in \mathcal{Y}(U)$. Now since $\mathcal{Z}$ is also a stack, the morphism $(\theta_i) = F(x) \to G(y)$, given locally, comes from a unique morphism $\theta : F(x) \to G(y)$. Thus initial descent datum is effective, and this shows that the fiber product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ is a stack. □

An important example of a fiber product is what is called the **inertia stack** of the stack $p : \mathcal{C} \to \mathcal{S}$. This is the 2-fiber product, i.e., the stack

$$\mathcal{I}_{\mathcal{C}} := \mathcal{C} \times_{\Delta, \mathcal{C}\times\mathcal{C}, \Delta} \mathcal{C} \tag{1.51}$$

where $\Delta : \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the diagonal morphism (1.17). An object of the inertia stack over $S \in \operatorname{Ob}\mathcal{S}$ is a pair $(x, y) \in \mathcal{C}(S)^2$ together with a pair of isomorphisms $u, v : x \xrightarrow{\sim} y$. Morphisms are defined likewise. There is a simpler stack equivalent to $\mathcal{I}_{\mathcal{C}}$. Its objects over S are the datum (x, σ), with $x \in \mathcal{C}(S)$, and $\sigma \in \operatorname{Isom}_S(x, x)$. A morphism $(x, \sigma) \to (y, \tau)$ is a morphism $u : x \to y$ such that $\tau u = u\sigma$. It is easy to see this defines a fibered category in groupoids, equivalent to the inertia stack. The equivalence is $(x, \sigma) \mapsto (x, x; \sigma, 1_x)$. We leave the details to an interested reader.

Finally one can define a substack of a stack:

Definition 1.64. A **substack** $\mathcal{Y} \subset \mathcal{X}$ of a stack $\mathcal{X}$ is a stack which is a (strictly) full subcategory of $\mathcal{X}$.

In other words, a substack is a "subsheaf in groupoids". Thus for each $f : T \to S$, and $x \in \mathcal{Y}(S)$, then $f^*(x)$ computed in $\mathcal{X}$ is in $\mathcal{Y}(T)$. Furthermore if $(S_i \to S)_i$ is an étale covering of $S \in \operatorname{Ob}\mathcal{S}$, and if $x \in \mathcal{X}(S)$, then $x_{|S_i} \in \mathcal{Y}(S_i)$, if and only if $x \in \mathcal{Y}(S)$.

1.3.3. Sheafification versus Stackification. It is well known that to any presheaf F (separated or not) over a site $\mathcal{C}$, one can associate a sheaf $\tilde{F}$, together with a canonical morphism $F \to \tilde{F}$ (which is a monomorphism if F is separated) such that every morphism $F \to G$ with G a sheaf factors uniquely through $\tilde{F}$. In other words, the map $\operatorname{Hom}(\tilde{F}, G) \to \operatorname{Hom}(F, G)$ is bijective. This can also be formulated by the assertion that the inclusion functor $\mathcal{P}Shv \hookrightarrow \mathcal{S}hv$ has a left adjoint

$$\begin{tikzcd} \mathcal{P}Shv \arrow[r, hook] & \mathcal{S}hv \arrow[l, bend right] \end{tikzcd} .$$

The same result holds true for prestacks and stacks, i.e., the inclusion

$$\mathbf{PStack} \hookrightarrow \mathbf{Stack}$$

taking into account that we are dealing with 2-categories. A precise formulation involves the category $\mathbf{Hom}(-, -)$ instead of the ordinary Hom, giving a 2-category generalization of the previous construction.

Proposition 1.65. *Let $\mathcal{F}$ be a prestack. There exists a stack $\tilde{\mathcal{F}}$ together with a monomorphism $\imath : \mathcal{F} \to \tilde{\mathcal{F}}$, such that for any stack $\mathcal{G}$, the functor*

$$\mathbf{Hom}(\tilde{\mathcal{F}}, \mathcal{G}) \longrightarrow \mathbf{Hom}(\mathcal{F}, \mathcal{G}), \quad \phi \mapsto \phi.\imath \tag{1.52}$$

is an equivalence of categories.

Proof. (sketch) The construction is formally equivalent to the case of a presheaf, taking into account the facts that the objects live in a groupoid. Thus, we will ignore some details. Let $S \in \mathcal{S}$. An object of $\tilde{\mathcal{F}}(S)$ is a collection

$$\xi = (\mathcal{U} = (U_i \to U), (x_i), (\alpha_{ij}))$$

where $\mathcal{U} = (U_i \to U)$ is a covering of U, $x_i \in \operatorname{Ob}\mathcal{F}(U_i)$, and (α_{ij}) is a descent datum associated to this covering. If $\mathcal{V} = (V_\alpha \to S)$ is a covering finer than $(U_i \to U)$, one can define the restriction $\xi_{\mathcal{V}}$ in an obvious way. Let $\xi' = ((U'_j \to U), (x'_j), (\alpha'_{jk}))$ be another object of $\tilde{\mathcal{F}}(S)$ with associated covering $\mathcal{U}'$. Let $(W_k \to S)$ the covering $\mathcal{W} = \mathcal{U} \times_U \mathcal{U}'$ refining both $\mathcal{U}$ and $\mathcal{U}'$. This poor notation means we are taking the covering $W_{i,j} = U_i \times_U U'_j$. Then a morphism $\xi \to \xi'$ is a collection of morphisms $(\xi_{\mathcal{W}} \to \xi'_{\mathcal{W}})$, i.e., $(\xi_i)_{|W_{ij}} \to (\xi'_j)_{|W_{ij}}$ which is compatible with the induced descent data. Finally $\imath(x) = (S \to S, x, 1_x)$.

It is not very difficult to check that $\tilde{\mathcal{F}}$ is a stack, and that $\imath$ is a monomorphism. If $\mathcal{F}$ is already a stack, then clearly $\mathcal{F} \cong \tilde{\mathcal{F}}$. Here is a very sketchy proof. First $\tilde{\mathcal{F}}$ is a prestack. This amounts to checking that a morphism $\xi \to \xi'$ given "locally" with respect to a covering $(V_k \to U)_k$ comes from a unique true morphism. With the same notations as above, we have for each k a morphism $f^k : x_{|V_k} \longrightarrow x'_{|V_k}$ in $\tilde{F}$, i.e., a collection of morphisms

$$f^k_{ji} : x_{i|U_{ijk}} \longrightarrow x'_{j|U_{ijk}}$$

where $U_{ijk} = U_i \times_U U'_j \times_U V_k$. For a fixed k, the f^k_{ij} can be glued together in a morphism $f_{ji} : x_{i|U_{ij}} \longrightarrow x'_{j|U_{ij}}$, using the fact that $\mathcal{F}$ is a prestack. The uniqueness in the descent ensures that the f_{ji} agree with the descent data. This provides the expected morphism.

Likewise we can check the descent property for the objects. Let $(U^\alpha \to U)$ be a covering family, and $(x^\alpha, U^\alpha_i \to U^\alpha) \in \tilde{\mathcal{F}}(U^\alpha)$ a collection of objects which satisfies the descent condition with respect to the covering $(U^\alpha \to U)$. We can see x^α as given by $(x^\alpha_i, (U^\alpha_i \to U^\alpha))$ with respect to a covering $(U^\alpha_i \to U^\alpha)$. We know from Definition 1.22 that the family of maps $(U^\alpha_i \to U)_{\alpha,i}$ defines a covering family of U. It is not difficult to check that the datum $(x^\alpha_i, (U^\alpha_i \to U))$ yields an object $x \in \tilde{\mathcal{F}}(U)$ which is the result of the descent of the locally defined object x^α.

This concludes the proof that $\tilde{\mathcal{F}}$ is indeed a stack. We omit the verification of the universal property. The interested reader will find much more details in the online treatise [62]. □

We need one more definition, that of **epimorphism** in the 2-category of stacks.

Definition 1.66. A 1-morphism of stacks $F : \mathcal{F} \to \mathcal{G}$ is an **epimorphism** if given a section $y \in \mathcal{G}(S)$, there is a covering family $(S_i \to S)_i$ of S, such that $y_{|S_i}$ lifts to $\mathcal{F}(S_i)$, i.e., $y_{|S_i} = F(x_i)$, $x_i \in \mathcal{F}(S_i)$.

It is not difficult to prove the following result:

Proposition 1.67. *Let $F : \mathcal{F} \longrightarrow \mathcal{G}$ be a morphism of stacks over the site $\mathcal{S}$. Then F is an equivalence if and only if F is a monomorphism and an epimorphism.*

Proof. (sketch) This amounts to checking that F induces fiberwise an equivalence of categories, i.e., for $S \in \operatorname{Ob}\mathcal{S}$, the functor $F_S : \mathcal{F}(S) \longrightarrow \mathcal{G}(S)$ is fully faithful, and essentially surjective. The first point comes directly from the fact that F is a monomorphism (Definition 1.20). Now let $y \in \mathcal{G}(S)$. We can find a covering family $(U_i \to S)_i$ together with $x_i \in \mathcal{F}(U_i)$ and $\theta_i : F(x_i) \cong y_{|U_i}$. Notice y is endowed with a canonical descent datum with respect to the covering $(U_i \to S)$. It is readily seen that F being a monomorphism, this descent datum lifts to a descent datum on the family (x_i). Since $\mathcal{F}$ is a stack, we know there exists $x \in \mathcal{F}(S)$ inducing the x_i's. Then $F(x) \cong y$. □

Exercise 1.68. *Let $(\mathcal{F}_i)_{i\in I}$ be a family of stacks. Show that one can define the sum $\coprod_i \mathcal{F}_i$. (Hint: define naively the sum as a prestack, then stackify it to get the correct answer.)*

Exercise 1.69. *Prove that the category fibered in groupoids of Example* 1.14 *is a stack.*

Exercise 1.70. *Show $\mathcal{C} \to \mathcal{S}$ is a prestack if and only if the functor* (1.29) *is fully faithful.*

Exercise 1.71. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks. Prove there is "canonical" factorization $F : \mathcal{Y} \xrightarrow{\overline{F}} \mathcal{Z} \xrightarrow{I} \mathcal{X}$, where $\overline{F}$ is an epimorphism, and I a monomorphism.*

1.3.4. Gerbes. Gerbes are stacks of a very particular type. Roughly, gerbes are 2-principal bundles, and it is not surprising if the definition mimics that of a principal bundle. Let us start by recalling the definition of a torsor, i.e., a principal bundle as synonym, in a rather general form. The definition will be specialized later. We refer to Section 2.2 for a review of some basic facts about torsors. As usual $\mathcal{S}$ is a site. Assume given a sheaf of group $\mathcal{G}$ over the site $\mathcal{S}$. We say $\mathcal{G}$ acts on the left (resp. right) on the sheaf of sets $\mathcal{F}$ if we are given a morphism of sheaves $\mathcal{G} \times \mathcal{F} \longrightarrow \mathcal{F}$ (resp. $\mathcal{F} \times \mathcal{G} \longrightarrow \mathcal{F}$) which satisfies the usual condition for an action. Equivalently for any $X \in \mathcal{G}$ the group $\mathcal{G}(X)$ acts on the set $\mathcal{F}(X)$ making for any arrow $X \to Y$ the restriction map $\mathcal{F}(Y) \to \mathcal{F}(X)$ equivariant. If the sheaves $\mathcal{F}_i$ $(i = 1, 2)$ are endowed with an action of $\mathcal{G}$, a morphism $\phi : \mathcal{F}_1 \to \mathcal{F}_2$ is $\mathcal{G}$-equivariant if for any object X, the map $\mathcal{F}_1(X) \to \mathcal{F}_2(X)$ is.

Definition 1.72. A sheaf $\mathcal{P}$ over X is a $\mathcal{G}$-torsor if there is a (right) action $\mathcal{P} \times \mathcal{G} \to \mathcal{P}$ of $\mathcal{G}$ on $\mathcal{P}$, which is locally trivial in the sense that one can find a covering family $(U_i \to X)_i$ such that the induced torsor $\mathcal{P}_{|U_i}$ is trivial, i.e., equivariantly isomorphic to $\mathcal{G}_{|U_i}$:

$$\mathcal{P}_{|U_i} \xrightarrow{\sim} \mathcal{G}_{|U_i}.$$

A sheaf of groups is viewed as a **trivial** torsor for its action on itself by means of right translations.

Notice that the definition of a principal bundle depends on the topology, i.e., the site chosen. When $\mathcal{S} = \mathbf{Sch}$ we can speak on a Zariski torsor, or an étale (resp. fppf) torsor. The definition of a gerbe goes as follows:

Definition 1.73. Let $p : \mathcal{G} \to \mathcal{S}$ be a stack. Suppose that $X \in \operatorname{Ob}\mathcal{S}$ and that we are given a morphism $\phi : \mathcal{G} \to X$ (we write X instead of $\mathcal{S}/X$). Then we say that $(\mathcal{G}, \phi)$ is a **gerbe over** X if:

i) ϕ is an epimorphism,
ii) the diagonal $\mathcal{G} \to \mathcal{G} \times_X \mathcal{G}$ is an epimorphism.

In other words, $\mathcal{G}$ viewed as a fibered category on groupoids over $\mathcal{S}/X$, is locally non empty, and locally connected.

In more concrete terms i) says that there is a cover $(U_i \to X)$ such that $\mathcal{G}(U_i) \neq \emptyset$, and ii) says that if two sections $x, y \in \mathcal{G}(S)$ are such $F(x) = F(y)$ (same S-point of X) then they are locally isomorphic.

There is an alternative view about gerbes showing the theory is a "higher" version of that of torsors. If $x, y \in \mathcal{G}(S)$ are isomorphic, say $h : x \xrightarrow{\sim} y$, then $\sigma \mapsto h\sigma h^{-1}$ defines an isomorphism

$$\operatorname{Aut}_S(x) \xrightarrow{\sim} \operatorname{Aut}_S(y).$$

If we assume that $\operatorname{Aut}_S(x)$ is a sheaf of abelian groups, then this isomorphism becomes canonical, i.e., independent of h. Assume that this commutativity condition holds. Let us choose a cover $\{U_i \to X\}$ such that $\mathcal{G}(U_i) \neq \emptyset$. Pick $x_i \in \mathcal{G}(U_i)$. Then as seen above there is a canonical isomorphism

$$\operatorname{Aut}_{U_{ij}}(x_{i|U_{ij}}) \xrightarrow{\sim} \operatorname{Aut}_{U_{ij}}(x_{j\,|U_{ij}}). \tag{1.53}$$

Clearly these isomorphisms verify the cocycle condition which in turn allows us to glue these sheaves into a sheaf of abelian groups $\mathcal{A}$ on X (Proposition 1.31). We say $\mathcal{A}$ is the **band**, or **lien** of $\mathcal{G}$.

The gerbe $p : \mathcal{G} \to X$ is called **trivial** or **neutral** if $\mathcal{G}(X) \neq \emptyset$. If this is the case we can describe more accurately $\mathcal{G}$. Let us choose $x \in \mathcal{G}(X)$, then $\mathcal{G}$ being a stack, $\operatorname{Aut}(x) : (T \to X) \mapsto \operatorname{Aut}_T(x_{|T})$ is a sheaf of groups over $\mathcal{S}/X$. Let now $y \in \mathcal{G}(T)$ be a section over T, with $p(y) = f : T \to X$. Property ii) of gerbes says $\operatorname{Isom}_T(x_{|T}, y)$ is a $\operatorname{Aut}(x)$-torsor (= principal bundle). This means that we have a right action of $\operatorname{Aut}(x)$ on $\operatorname{Isom}_T(x_{|T}, y)$, the composition), and that locally this torsor has a section, this is ii).

We leave as an exercise to prove that this description can be inverted, i.e., that given a sheaf of groups G over $\mathcal{S}/X$, then the category whose objects over S are the pairs (P, f) where $f : T \to X$ and P is a torsor for the group $G \times_X T$, with obvious morphisms in a gerbe over X. It is a trivial gerbe since the group G acting on itself on the right is a section over X. The notation for this construction is $\mathbf{B}(G/X)$. More will be said in Section 2.2.

Keeping the same assumptions as above, locally, say over a cover $(U_i \to X)$, a gerbe can be identified with the stack $\mathbf{B}(\mathcal{A}/U_i)$. The gluing process is encoded in a second cohomology class lying in $\mathrm{H}^2_{\mathrm{top}}(X, \mathcal{A})$ (Giraud's theorem), a higher analogue of the fact that principal bundles are classified by $\mathrm{H}^1_{\mathrm{top}}(X, \mathcal{A})$, top $\in \{$étale, fppf$\}$. Gerbes will be useful for example in a future discussion about Galois versus non Galois covers.

2. Group actions versus groupoids

We saw previously that an algebraic space F can be understood as a "quotient" of a scheme U by an étale equivalence relation (Subsection 1.3.1), this construction being summarized by a diagram

$$R \underset{d_1}{\overset{d_0}{\rightrightarrows}} U \xrightarrow{p} F.$$

This alternative way of thinking about geometric objects extends to the larger category of stacks. To do this we must replace an equivalence relation by an action of a groupoid on scheme, i.e., a **scheme in groupoids**. This key insight of Grothendieck is already present in ([58], Exposé V).

2.1. Schemes in groupoids

In this section we introduce groupoid schemes, objects with a geometric flavor. We restrict ourselves to groupoid schemes, to limit the size of the definitions. It is not difficult to extend the scope of this section to *groupoids in algebraic spaces*, see the chapter "Groupoids in Algebraic Spaces" in [62], or [38]. Throughout we will work with the category $\mathcal{S} = \mathbf{Sch}$ of schemes, unless otherwise stated.

Definition 2.1. A **scheme in groupoids**, or **groupoid in the category of schemes**, is a diagram in **Sch** denoted

$$R \underset{d_1}{\overset{d_0}{\rightrightarrows}} U \qquad \text{or} \qquad d_0, d_1 : R \rightrightarrows U \tag{2.1}$$

where d_0 is called the **source** morphism and d_1 is the **target** morphism, together with i) the **unit** $e : U \to R$, ii) the **inverse** $s : R \to R$, and iii) the **product** $m : R \times_{d_1,U,d_0} R$. These arrows must satisfy the following axioms (compare with Theorem 1.51):

- (unit and inverses) $d_0 e = d_1 e = 1_U$, $d_0 s = d_1$ and $d_1 s = d_0$,
- (associativity) the diagram

$$\begin{tikzcd} R \times_{d_1,U,d_0} R \times_{d_1,U,d_0} R \arrow[r, "1\times m"] \arrow[d, "m\times 1"'] & R \times_{d_1,U,d_0} R \arrow[d, "m"] \\ R \times_{d_1,U,d_0} R \arrow[r, "m"] & R \end{tikzcd} \tag{2.2}$$

  commutes.
- (inverse) the diagram

$$\begin{tikzcd} R \arrow[r, "(1,s)"] \arrow[rr, bend right, "ed_0"'] & R \times_{d_1,U,d_0} R \arrow[r, "m"] & R \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "(s,1)"] \arrow[rr, bend right, "ed_1"'] & R \times_{d_1,U,d_0} R \arrow[r, "m"] & R \end{tikzcd}$$

  commutes.

These axioms are most obvious when depicted as diagrams:

$$\begin{tikzcd} U \arrow[r, "e"] \arrow[dr, "id"'] & R \arrow[d, "d_0"] \\ & U \end{tikzcd} \qquad \begin{tikzcd} U \arrow[r, "e"] \arrow[dr, "id"'] & R \arrow[d, "d_1"] \\ & U. \end{tikzcd} \tag{2.3}$$

If p_1 and p_2 are the two projections from $R \times_{d_1,U,d_0} R$ to R, then we have two commutative squares:

$$\begin{tikzcd} R \times_{d_1,U,d_0} R \arrow[r, "m"] \arrow[d, "p_1"'] & R \arrow[d, "d_0"] \\ R \arrow[r, "d_0"'] & U \end{tikzcd} \qquad \begin{tikzcd} R \times_{d_1,U,d_0} R \arrow[r, "m"] \arrow[d, "p_2"'] & R \arrow[d, "d_1"] \\ R \arrow[r, "d_1"'] & U. \end{tikzcd} \tag{2.4}$$

Exercise 2.2. *Show that the diagrams in* (2.4) *are cartesian.*

An important example of groupoid scheme arises whenever an algebraic group[19] G acts (on the left) on a scheme U, with action $\mu : G \times U \to U$, $\mu(g,u) = gu$. Set $R = G \times U$, and let $d_0 : G \times U \to U$ be the projection, and $d_1 = \mu : G \times U \to U$ the action. We set $e = (1_G, id_U)$, i.e., $u \mapsto (1_G, u)$. We identify $R_{d_1} \times_{d_0} R$ with $G \times G \times U$ by the map $((g,u),(g',u' = gu)) \mapsto (g,g',u)$. Then we set $m(g,h,u) = (hg,u)$, and $s(g,u) = (g^{-1}, gu)$. This defines a groupoid scheme.

$$\begin{tikzcd} u \arrow[r, "g"] \arrow[rr, bend right, "g'g"'] & u' \arrow[r, "g'"] & u'' \end{tikzcd} \qquad s(\ u \xrightarrow{g} u'\ ) = \ u \xleftarrow{g^{-1}} u'\ . \tag{2.5}$$

Let $d_0, d_1 : R \rightrightarrows U$ be a groupoid scheme. Instructed by the previous example, we can understand this datum as R **acting** on U. Let $\xi \in R(S)$ be an S-point of R, then we can picture ξ as an **arrow**, or **isomorphism**

$$x - \overset{\xi}{-} \to y$$

with source $x = d_0(\xi)$ and target $y = d_1(\xi)$. The isomorphism $s(\xi)$ is the inverse of ξ. If $d_0(\eta) = d_1(\xi)$, then $m(\xi,\eta)$ can be imagined as the product $\eta.\xi : x = d_0(\xi) \overset{\sim}{d_1} (\eta) = z$.

Let $d_0, d_1 : R \rightrightarrows U$ and $d'_0, d'_1 : R' \rightrightarrows U$ be two groupoid actions on U. A **morphism** between these actions is a morphism $\phi : R \to R'$ inducing for any S a morphism between the corresponding presheaves in groupoids, i.e., a morphism satisfying $d'_i\phi = d_i$, $\phi e = e'$, $s'\phi = \phi s$ and $m'(\phi \times \phi) = \phi m$.

On the other hand, there is an important operation on groupoids: the pullback of a groupoid scheme $d_0, d_1 : R \rightrightarrows U$ by a map $f : U' \to U$. We want a kind of

[19] For us, an algebraic group is affine, i.e., a closed subgroup scheme of a matrix group over some ground field.

cartesian diagram

$$\begin{tikzcd} R \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & U \\ R' \arrow[r, shift left, "d_0'"] \arrow[r, shift right, "d_1'"'] & U'. \arrow[u, "f"'] \end{tikzcd} \tag{2.6}$$

The answer is $R' = R \times_{U \times U} (U' \times U')$, where the fiber product is understood as

$$\begin{tikzcd} R \arrow[r, "(d_0,d_1)"] & U \times U \\ R' \arrow[u, "g"] \arrow[r, "h"] & U' \times U'. \arrow[u, "(f,f)"'] \end{tikzcd}$$

The arrows in $d_0', d_1' : R' \rightrightarrows U'$ are the obvious ones. Thus an arrow in R' with source x' and target y' corresponds bijectively to an arrow in R from $f(x')$ to $f(y')$. This construction applies in particular to a subscheme $U' \subset U$, thereby defining the induced groupoid scheme.

Definition 2.3. Let $d_0, d_1 : R \rightrightarrows U$ be a groupoid scheme. A subscheme $V \subset U$ is R-**stable** if $d_0^{-1}(V) = d_1^{-1}(V)$.

Given a stable subscheme $V \subset U$ if an arrow has one of its end points in V, then both are in V. Therefore the induced groupoid reduces to

$$\Delta = d_0^{-1}(V) = d_1^{-1}(V) \rightrightarrows V$$

with the induced maps. If the groupoid comes from a group action, then R-stable means G-stable, furthermore the induced groupoid is given by the induced group action.

Remark 2.4. Between two points x, y of R we may have one or more arrows, or no arrow at all if $x \neq y$, meaning that the "diagonal" map $(d_0, d_1) : R \longrightarrow U \times U$ is not necessarily an embedding. If this is the case, then R is essentially an equivalence relation.

Our goal in this section is to develop a dictionary between groupoid schemes and an interesting class of (pre)stacks. Throughout, the base category is $\mathcal{S} = \mathbf{Sch}$, viewed as a site with the étale topology, i.e., the big étale site $\mathbf{Sch}_{et}$. Let us start with category fibered in groupoid, i.e., a presheaf in groupoids, or a restack $p : \mathcal{F} \to \mathcal{S}$, and let $f : U \to \mathcal{F}$ be a (1-)morphism. Assume that the fiber product $U \times_{\mathcal{F}} U$ in the 2-category **CFG** is representable by a scheme R. The projection maps define a diagram $d_0, d_1 : R \rightrightarrows U$. We set $e : U \to R$ equal to the diagonal Δ, that is $e(x) = (x, x, 1_x)$, and $s(x, y, \theta) = (y, x, \theta^{-1})$. We define the composition map $R \times_{d_1, U, d_0} R \to R$ on the level of points of U:

$$m((x, y, \theta), (x', y', \theta')) = (x, y', \theta'\theta) \quad (y = x') \tag{2.7}$$

where $\theta : f(x) \cong f(y)$, $\theta' : f(x') \cong f(y')$ are isomorphisms on $\mathcal{F}$.

Lemma 2.5. *If $U \times_{\mathcal{F}} U$ is isomorphic (as a category fibered in groupoids) to a scheme R, then the diagram $d_0, d_1 : R \rightrightarrows U$ endowed with the extra morphisms (e, s, m) is a groupoid scheme.*

Proof. This is easy and left to the reader. □

Important is the opposite direction. Let there be given a groupoid scheme

$$d_0, d_1 : R \rightrightarrows U, \quad (e, s, m).$$

Define a fibered category in groupoids $p : \mathcal{F} \to \mathcal{S}$ as follows. The category of sections over $S \in \mathbf{Sch}$ has the set of objects $\mathrm{Hom}(S, U)$. An arrow (= an isomorphism) between two objects $x, y \in \mathcal{F}(S)$ is given by a S-point $\xi \in \mathrm{Hom}(S, R)$, such that $d_0(\xi) = x$ and $d_1(\xi) = y$. The composition rule is given by the map m, viz. $\eta \circ \xi = m(\xi, \eta)$. To define a presheaf in groupoids, we must define the pullback functor $f^* : \mathcal{F}(S) \to \mathcal{F}(S')$ for $f : S' \to S$ a morphism. This comes from the functoriality of the presheaf $\mathrm{Hom}(-, U)$, i.e., $f^*(x) = x \circ f$. To summarize:

Proposition 2.6. *The preceding construction yields a prestack denoted U/R and called the* **quotient prestack** *of U by the action of R.*

Proof. Everything is almost clear, excepted the fact that U/R is a prestack. Let $x, y \in \mathrm{Hom}(S, U)$. Then the construction yields readily that $\mathbf{Isom}_S(x, y)$ is the functor of points of the scheme given by the cartesian diagram

$$\begin{tikzcd} R \arrow[r, "(d_0,d_1)"] & U \times U \\ \mathbf{Isom}_S(x,y) \arrow[u] \arrow[r] & S. \arrow[u, "(x,y)"'] \end{tikzcd} \tag{2.8}$$

Since the presheaf of points of a scheme is a sheaf for the étale topology, the conclusion follows. □

The prestack U/R needs not be a stack: the problem is that descent data are not necessarily effective. Thus to get a stack we need to stackify the prestack U/R (see Subsection 1.3.3), that add to the prestack the solutions of all descent data:

Definition 2.7. The **quotient stack** denoted $[U/R]$ defined by a groupoid scheme $d_0, d_1 : R \rightrightarrows U$ is the stack associated to the prestack U/R. It comes together with an epimorphism $p : U \to [U/R]$.

A section over S of the stack $[U/R]$ has a rather "concrete" description. It is an S'-point of $x' : S' \to U$, for a suitable cover $S' \to S$, together with a descent datum relative to S'/S, the isomorphism $\phi : p_1^*(x') \xrightarrow{\sim} p_2^*(x')$ given by a $S' \times_S S'$-point of R. As previously explained this means that ϕ must satisfy the relation

$$p_{31}^*(\phi) = m(p_{21}^*((\phi), p_{32}^*(\phi)).$$

It is instructive to compare this construction with various standard constructions in algebraic geometry. One starts with a presheaf, then sheafifies it with respect

to a given topology (étale, fppf...), and then tries to prove that the sheaf thus obtained is representable. The philosophy here is to work with the quotient stack, representable or not, as a true geometric object.

Let $p : U \to \mathcal{F} = [U/R]$ be the quotient stack of U by the action of R on U. The construction of $[U/R]$ says that the two sections $pd_0, pd_1 \in [U/R](R)$ are isomorphic, the canonical isomorphism θ being given by 1_R. The isomorphism θ satisfies the following identity

$$m^*(\theta) = p_2^*(\theta) \circ p_1^*(\theta). \tag{2.9}$$

This says that the groupoid acts trivially on $[U/R]$. It is not difficult to check that $([U/R], p, \theta)$ is universal in the 2-categorical sense (see Exercise 2.9 below).

Exercise 2.8. *Let $d_0, d_1 : R \rightrightarrows U$ be a groupoid scheme with $U = \operatorname{Spec} L$, L a field. Assume that d_i is finite and étale. Let $K = L^R = \{x \in L,\ d_0^*(x) = d_1^*(x)\}$ be the subset of invariant elements. Show the Galois type result that K is a subfield and that L/K is a separable finite extension* ([29], *Theorem* 2.10).

Exercise 2.9. *Let $\phi : U \longrightarrow \mathcal{M}$ be a morphism from U to a stack $\mathcal{M}$. Assume that there is a 2-isomorphism $\varphi : d_0^*(\phi) \cong d_1^*(\phi)$ such that* (2.9) *holds true. Prove that ϕ factors through $[U/R]$ in the 2-categorical sense, i.e., there exists $\overline{\phi} : [U/R] \to \mathcal{M}$ together with an isomorphism $\overline{\phi} \cong \phi$.*

Exercise 2.10. *Let $p : \mathcal{C} \to \mathcal{S}$ be a stack over the site $\mathcal{S}$. Let $S \in \operatorname{Ob}\mathcal{S}$. Show $\mathcal{C} \times S := \mathcal{C} \times_{\mathcal{S}} \mathcal{S}/S \to \mathcal{S}/S$ is a stack over the site $\mathcal{S}/S$. If $\mathcal{C} = [U/R]$, show $\mathcal{C} \times S = [U \times S/R \times S]$.*

2.2. Classifying stack, quotient stack

In this section, we are going to specialize a groupoid scheme to a down-to-earth group action on a scheme. Throughout, we will work over $\mathbf{Sch}/S$, $S = \operatorname{Spec} A$ the spectrum of a noetherian ring, to simplify we may even assume that $A = k$ a field. To begin with, let G/S be an S **affine group scheme** (smooth or not), i.e., a closed subgroup scheme of some $\mathrm{GL}_n \times S$ flat over $\operatorname{Spec} A$, and let $A[G]$ be its algebra of functions, so that $G = \operatorname{Spec} A[G]$. The multiplication $m : G \times_S G \to G$ yields at the level of functions a coproduct

$$m^* : A[G] \to A[G] \otimes_A A[G]$$

together with a counit $e^* : A[G] \to A$, and a coinverse $s^* : A[G] \to A[G]$ with obvious dualized group axioms. Our preferred examples are the multiplicative group $G_{S,m} = \operatorname{Spec} A[T, T^{-1}]$, with coproduct $T \mapsto T \otimes T$, counit $T \mapsto 1$, the subgroups $\mu_n = \operatorname{Spec} A[T]/(T^n - 1) \subset \mathbf{G}_m$, or a constant finite group G. In this last case $A[G] = A^G = \oplus_{g \in G} A e_g$ (product algebra) with coproduct given by the convolution $m^*(e_g) = \sum_{st=g} e_s \otimes e_t$. Notice μ_n is étale if $n \in A^*$, and even constant if $A = k = \overline{k}$.

Assume G acts on a S-scheme X, with action $\mu : G \times X \to X$, or to simplify $\mu(g, x) = gx$, at the level of points. If $X = \operatorname{Spec} R$, the action can be read as a

coaction $\mu^* : R \to A[G] \otimes_A R$. An equivariant morphism $f : X \to Y$, assuming G acts on X and Y, is characterized at the points level by $f(gx) = gf(x)$. When $G = \mathbf{G}_m$, an action of G on $\operatorname{Spec} R$ give rises to $R = \oplus_{n\in\mathbb{Z}} R_n$, i.e., R is a graded ring, nd the coaction reads $f = \sum_n f_n \mapsto \sum_n T^n \otimes f_n$.

Before we begin our description of quotient stacks $[X/G]$, I record here some basic facts about G-principal bundles (or G-torsors). To be able to speak about torsors, the category $\mathbf{Sch}/S$ must be enriched by a topology taken as usual in the list: Zariski, étale, fppf, fpqc, the fppf topology being the natural choice to be able to work with sufficient generality. It is important to note that the definition of a G-bundle is sensitive to the chosen topology, i.e., site. In most cases, working with affine group schemes and quasi-compact objects, there will be no loss of generality in working with a cover $f : T' \to T$ given by a single object.

We have already encountered the definition of a G-torsor for a sheaf of groups on a site (Definition 1.72). Let us specialize this definition to the case when our sheaf of groups G is an affine group scheme over $\operatorname{Spec} A$. This contains the case of a constant finite group, by this we mean a finite ordinary group G, but viewed as a group scheme, say over $\mathbb{Z}$, or as $G \times \operatorname{Spec} A$ over A. In the rest of this subsection G is an affine, flat, group scheme over A.

Definition 2.11. A **G-principal bundle** with base $T \in \mathbf{Sch}/S$ (or $G \times_S T$-torsor, or even G-torsor) is the datum of a sheaf $\mathcal{P}$ (for a given topology on $\mathbf{Sch}_S$) endowed with a right action of G, together with a morphism $p : \mathcal{P} \to T$ which is G-equivariant with respect to the trivial action of G on T. This datum must be locally trivial on T in the sense there exists a covering (often given by a single object) $f : T' \to T$ for a chosen topology, and an equivariant isomorphism

$$\varphi : T' \times_S G \xrightarrow{\sim} T' \times_T \mathcal{P}.$$

It is understood that G acts on $T' \times_S G$ by right translations.

Said differently, a G-bundle over T is an fppf-sheaf (resp. étale sheaf) on $\mathbf{Sch}/S$ which is locally isomorphic to $T \times_S G$. It is a result that under our assumptions, viz. G affine, the sheaf $\mathcal{P}$ is a scheme (see Exercise 2.14). The scheme P is called the **total space**, and T is the **base**.

The G-principal bundles are the objects of a category. A **morphism of torsors** $F : (p' : P' \to T') \longrightarrow (p : P \to T)$ is a cartesian diagram

$$\begin{tikzcd} P' \arrow[r, "\varphi"] \arrow[d, "p'"'] & P \arrow[d, "p"] \\ T' \arrow[r, "f"] & T. \end{tikzcd} \tag{2.10}$$

A principal bundle over T is trivial if it is isomorphic to $p_1 : T \times_S G \to T$, with the obvious right action of G on $T \times_S G$.

Indeed the knowledge of φ suffices, as shown in the proposition below. Details can be found in [17] or [64], §§ 4.3 and 4.4.

Proposition 2.12. *Let $p : P \to T$ be a G-principal bundle.*

i) *The morphism p is affine, of finite type, and faithfully flat. If G is smooth, then p is smooth.*

ii) *The diagram*

$$\begin{tikzcd} P \times G \arrow[r, "\mu"] \arrow[d, "p_2"'] & P \arrow[d, "p"] \\ P \arrow[r, "\pi"] & T \end{tikzcd} \tag{2.11}$$

is cartesian. Conversely, if $p : P \to T$ is a scheme over T and if G acts on P over T making the above diagram cartesian, then $P \to T$ is a G-torsor.

iii) *Let $p' : P' \to T$ be another G-principal bundle with the same base T. A morphism of bundles $\varphi : P' \to P$ is an isomorphism.*

iv) *A principal bundle $\pi : P \to T$ is trivial if and only if it has a section $T \to P$.*

Proof. (sketch) i) Let $f : T' \to T$ be a morphism which trivializes $p : P \to T$. Then p is obtained from $p_1 : T' \times_S G \to T'$ which is affine, of finite type and faithfully flat. Then by faithfully flat descent p shares these properties. Likewise if G is smooth, then p_1 is smooth, and so is p.

ii) We must check that $\varphi : P \times G \longrightarrow P \times_S P$, $(x, g) \mapsto (x, xg)$ is an isomorphism. This is clear for the trivial bundle (check this), so if $S' \to S$ trivializes π, then $\varphi \times 1_{S'}$ is an isomorphism, and again by a descent argument φ is an isomorphism.

iii) First the result is true when both bundles are trivial. Indeed if $(1, e) : T \to T \times G$ is the obvious unit section of $T \times G \to T$, then $\varphi(1, e) = (1, g)$ for some point $g : T \to G$. Then φ is the left translation by g.

iv) Suppose that $s : T \to P$ is a section, which in turn yields an equivariant map $T \times G \to P$, $(t, g) \mapsto (s(t)g)$. here is a covering family which trivializes the bundle, then making the pullback of φ an isomorphism, therefore φ must be an isomorphism. □

Remark 2.13. Here are some remarks. From iii) it follows that given a commutative square

$$\begin{tikzcd} P' \arrow[r, "\varphi"] \arrow[d, "p'"'] & P \arrow[d, "p"] \\ T' \arrow[r, "f"] & T \end{tikzcd}$$

where the vertical arrows are G-principal bundles, and φ is equivariant, then the square is cartesian. Notice also that if G is smooth, then so is p. In this case p is locally trivial in the étale topology, since p has a section locally for the étale topology, due to the smoothness property[20].

[20]This follows from the fact that a smooth morphism locally in the étale sense is an open subscheme of relative affine space. Think of uniformizing parameters, see [46] for example.

Therefore, if G is a smooth affine group scheme, we have the following equivalence concerning triviality of bundles:

locally trivial for the étale topology

$$\Updownarrow$$

locally trivial for the fppf topology.

This is not true in general. One can see that the morphism $\mathbf{G}_m \to \mathbf{G}_m$, $t \mapsto t^m$ is a μ_m-principal bundle for the fppf topology, but not for the étale topology if the characteristic $p > 0$ divides m.

To fix ideas, in the sequel of this section the topology will be the fppf topology, unless otherwise specified. When working with smooth group schemes there will be no loss of generality in switching to the étale topology. This will be implicit.

The case $G = \mathbf{G}_m$ will be of great interest for us. It is not difficult to check that a $\mathbf{G}_m$-fppf bundle, equivalently an étale bundle, is of the form

$$P = \mathrm{Spec}(\oplus_{n\in\mathbb{Z}}\mathcal{L}^{\otimes n})$$

for some invertible sheaf $\mathcal{L}$ on the base T. This comes from the "eigenspace" decomposition $\pi_*(\mathcal{O}_P) = \oplus_{n\in\mathbb{Z}}\mathcal{L}^{\otimes n}$. Indeed Proposition 2.12, i) says that $p : P \to T$ is affine, so we can assume that all schemes are affine. Therefore an action f $\mathbf{G}_m$ on $P = \mathrm{Spec}\, R$ can be translated through the coaction as the fact that R is a graded ring $R = \oplus_{n\in\mathbb{Z}}R_n$. In general this says that the direct image $p_*(\mathcal{O}_P)$ splits as a direct sum of eigensheaves $p_*(\mathcal{O}_P) = \oplus_{n\in\mathbb{Z}}\mathcal{L}_n$. The proof amounts to checking that $\mathcal{L} := \mathcal{L}_1$ is a line bundle and that $\mathcal{L}_n \cong \mathcal{L}^{\otimes n}$. This is clear when the bundle is trivial, then $p_*(\mathcal{O}_P) = \mathcal{O}_T[X^{\pm 1}]$. The result follows in general by descent. Since p is affine (Proposition 2.12,i)), we get $P = \mathrm{Spec}(p_*(\mathcal{O}_P))$.

A convenient way to think about principal bundles is as follows. Let $\pi : P \to T$ be a principal G-bundle, and assume $(U_i \to T)_i$ is a covering (for the chosen topology) which trivializes π. Let us choose an isomorphism of trivial bundles

$$\varphi_i : U_i \times_S G \xrightarrow{\sim} U_i \times_T P$$

then $\varphi_i^{-1}\varphi_j$ is an automorphism of the trivial G-bundle $U_{ij} \times G$ ($U_{ij} = U_i \times_T U_j$). There is a morphism $\theta_{ij} : U_{ij} \to G$ such that at the level of points

$$\varphi_i^{-1}\varphi_j(x, g) = (x, \theta_{ij}(x)g).$$

Clearly the following **cocycle** condition holds:

$$\theta_{ij}(x)\theta_{jk}(x) = \theta_{ik}(x) \quad (x \in U_i \times_T U_j \times_T U_k). \tag{2.12}$$

This cocycle condition is a descent datum on the trivialized bundle over $T' = \coprod_i U_i$. Descent theory gives the converse (Exercise 2.14).

Exercise 2.14. *Suppose that you have a G-cocycle (descent datum) relative to a covering $(U_i \to S)$ as in (2.12). Using a descent argument prove that there is a unique (up to a unique isomorphism) G-bundle $P \to T$ giving rises this cocycle when trivialized on $(U_i \to S)$.*

Exercise 2.15. *Let $\pi : P \to T$ be a $\mathbf{G}_m$-bundle. Prove that such a bundle can be trivialized for the Zariski topology. Precisely, prove that there is an invertible sheaf $\mathcal{L}$ on S such that $P = \mathrm{Spec}(\oplus_{n\in\mathbb{Z}}\mathcal{L}^{\otimes n})$. In particular the $\mathbf{G}_m$-bundles over T are "classified" by $\mathrm{Pic}(T)$, precisely the fibered category $\mathbf{B}\,\mathbf{G}_m$ is equivalent to the fibered category of invertible sheaves on $\mathbf{Sch}\,/S$.*

Exercise 2.16. *Explain why when $G = \mathrm{GL}_n$, a principal G-bundle (in the fppf topology) is Zariski locally trivial.*

Keeping the previous assumptions, consider the groupoid $d_0, d_1 : G \rightrightarrows \mathrm{Spec}\, k$ coming from the trivial action of G on the point $pt = \mathrm{Spec}\, k$. Recall that the ground ring is now $A = k$, therefore we can use freely the letter S. The prestack pt/G has for sections over $S \in \mathbf{Sch}\,/k$, only one object viz. the trivial G-bundle $p_1 : S \times G \to S$. Recall that we are working implicitly on the fppf-site $\mathbf{Sch}\,/k$, but when working with smooth group schemes it will be convenient to switch to the étale site.

The question is: what does the associated stack look like? The answer is:

Proposition 2.17. *The quotient stack $[pt/G]$ is the stack $\mathbf{BG} \to \mathbf{Sch}$ whose objects are the G-principal bundles in the étale (resp. fppf) sense.*

Proof. The definition says that a section over S of the associated stack $[pt/G]$ is given by an étale (resp. fppf) covering $U_i \to S$, together with a family of isomorphisms of G-bundles

$$\varphi_{ij} : U_{ji} \times G = (U_i \times_S U_j) \times G \xrightarrow{\sim} U_{ij} \times G = (U_i \times_S U_j) \times G \tag{2.13}$$

such that over the triple intersections $U_i \times_S U_j \times_S U_k$, the cocycle condition

$$\varphi_{ij}\varphi_{jk} = \varphi_{ik} \tag{2.14}$$

holds true. As explained in the previous proposition, the $\varphi_{ij}'^{s}$ define a descent datum for the trivial G-bundle $\coprod_i U_i \times G$ relatively to the cover $(U_i \to S)$. Thus the theory of descent says that the U_i's come from a unique scheme $P \to S$. Since the descent datum is compatible with the G action, the action descents to S. Thus we get $\pi : P \to S$ together with a right G-action. Since the covering $U_i \to S$ trivializes $\pi : P \to S$, Proposition 2.12 shows that $\pi : P \to S$ is a G-principal bundle. It is not difficult to check that the morphisms of $[pt/G]$ are exactly the bundle morphisms. This proves the proposition. □

Exercise 2.18. *Show that the category $\mathbf{Fib}_n$ whose objects over $S \in \mathbf{Sch}$ are the vector bundles of rank n over S (i.e., locally free coherent sheaves of rank n), and with morphisms $\mathcal{F}' \to \mathcal{F}$ over $f : S' \to S$ the isomorphisms $f^*(\mathcal{F}) \xrightarrow{\sim} \mathcal{F}'$ is a stack. Prove that $\mathbf{Fib}_n \cong \mathbf{B}(\mathrm{GL}_n(k))$ (this generalizes Exercise 2.15).*

The stack $\mathbf{BG}$ is an interesting toy model, so let us study some of its basic and useful properties in case G is a smooth affine group scheme over $A = k$. Recall that the 1-morphism $pt \to \mathbf{BG}$ given by the trivial bundle $G \to pt$ is a smooth atlas. We want to say something more. Let $\pi : P \to S$ be a principal bundle viewed as a 1-morphism $p : S \to \mathbf{BG}$. Then:

Lemma 2.19. *The square*

$$\begin{tikzcd} P \arrow[r, "\pi"] \arrow[d] & S \arrow[d, "p"] \\ pt \arrow[r, "Q"] & \mathbf{BG} \end{tikzcd} \tag{2.15}$$

is 2*-cartesian.*

Proof. An object of the 2-fiber product $pt \times_{\mathbf{BG}} S$ above T is a pair $(f : T \to S, \varphi)$ where $\varphi : T \times G \xrightarrow{\sim} P \times_S T$ is an isomorphism of principal bundles with base T. This provides a morphism $T \to P$. Conversely let $T \to P$ be a morphism. By base change, the isomorphism $P \times G \xrightarrow{\sim} P \times_S P$ yields an isomorphism $\varphi : T \times G \xrightarrow{\sim} P \times_S T$. It is readily seen that we can extend this to an equivalence of stacks $P \xrightarrow{\sim} pt \times_{\mathbf{BG}} S$. In particular, if $P \to S$ is the trivial bundle $G \to pt$, we get a 2-cartesian square

$$\begin{tikzcd} G \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r, "Q"] & \mathbf{BG}\,. \end{tikzcd} \tag{2.16}$$

This shows that we can see the morphism $pt \to \mathbf{BG}$ as a G-principal bundle, indeed the "universal" G-bundle. □

There are many important operations on principal bundles, all of them having a stacky flavor. Let us recall two of these operations. The first one is the **induction**, or extension of the structural group. Let $G \to H$ be a group morphism. If $P \to X$ is a G-bundle, we can build a H-bundle as follows. More generally, let F be a scheme acted on by G (on the left). The group G acts[21] on $P \times F$ by

$$s(g, x) = (gs^{-1}, sx). \tag{2.17}$$

It is tempting to perform the quotient $P \times F/G$ in the category of schemes. This quotient is usually denoted $P \times^G F$, and called the **associated fiber space** to P/S with fiber F. Obviously if $P \to X$ is trivial (i.e., $P(X) \neq \emptyset$) then $P \times^G F = X \times F$. This suggests that in order to give a sense to this quotient in general, we have to trivialize $P \to X$ over an étale (or more generally fppf) covering $X' \to X$. This trivialization put on $X' \times G$ a descent datum, indeed pulling back to $X' \times_X X' \rightrightarrows X'$ we get two copies of the trivial bundle together with an automorphism given by an element of G. This descent datum propagates on $X' \times F$, which in turn by the descent machinery produces canonically a scheme over X. The result is the expected $P \times^G F$. The descent will be possible only if some technical assumptions on F are fulfilled, for example F affine. With the help of this construction we are able to describe "explicitly" the scheme $\mathbf{Isom}_S(P_1, P_2)$ (Exercise 2.21).

Exercise 2.20. *With the same notations as above, let $H \subset G$ be a closed subgroup of G. Check that $P \times^G (G/H) \cong P/H$ (see* [58] *for the existence of the quotient G/H).*

[21] Schemes are identified with their functor of points.

Exercise 2.21. *Let $P_i \to S$ $(i = 1, 2)$ be two principal G-bundles with base S. Show that* $\mathbf{Isom}_S(P_1, P_2)$ *is represented by the scheme* $(P_1 \times_S P_2) \times^{G \times G} G$, *where $G \times G$ acts on G according to* $(g_1, g_2).g = g_1 g g_2^{-1}$, *and $P_1 \times_S P_2$ is viewed as a $G \times G$-bundle.*

A typical situation where the previous construction applies is when you have a group morphism $\rho : G_1 \to G_2$, then extending the structure group from G_1 to G_2 by means of $P_1 \mapsto P_1 \times^{G_1} G_2$ yields a morphism of stacks

$$F_\rho : \mathbf{BG}_1 \longrightarrow \mathbf{BG}_2 \,. \tag{2.18}$$

The trivial subgroup $\{1\}$ of G yields a morphism $\mathbf{B}\{1\} = \operatorname{Spec} k \to \mathbf{BG}$. It is readily seen that this morphism corresponds from the Yoneda lemma to the section $(G \to \operatorname{Spec} k) \in \mathbf{BG}(\operatorname{Spec} k)$. In the example below, this remark will be emphasized.

Exercise 2.22. *Write down the details of the proof that a group morphism $\rho : G_1 \to G_2$ determines a 1-morphism of stacks $F : \mathbf{BG}_1 \to \mathbf{BG}_2$.*

Exercise 2.23. *The notations being as before, suppose F is a k-group scheme, and that $\rho : G \to \operatorname{Aut}(F)$ is a group morphism defining a group action of G on F. Let $P \to T$ be a G-principal bundle. Prove that the associated fiber space $P \times^G F$ has a well-defined structure of T-group scheme, according to the composition law (at the level of points)*

$$[x, f].[x', f'] = [x, f.(\rho(g)(f'))] \quad (x' = xg)$$

where $[x, f]$ means the image of $(x, f) \in P \times F$ in $P \times^G F$. Compare with Exercise 2.20.

Example 2.24. *The groupoid* $\mathbf{Hom}(\mathbf{BG}_1, \mathbf{BG}_2)$. Let there be given two finite constant (smooth suffices) group schemes G_1, G_2 over $A = k$. We would like to describe the 1-morphisms $\mathbf{BG}_1 \to \mathbf{BG}_2$, i.e., the groupoid $\mathbf{Hom}(\mathbf{BG}_1, \mathbf{BG}_2)$. Although this is not essential, to simplify the answer we will assume that $k = \overline{k}$.

Then let $P_1 = \operatorname{Spec} k \times G_1$ be the trivial principal bundle over k. We have $\operatorname{Aut}(P_1) = G_1$ where G_1 acts on itself by means of left translations. Given a morphism $F : \mathbf{BG}_1 \to \mathbf{BG}_2$, the image $F(P_1)$ is a G_2-bundle over $\operatorname{Spec} k$, thus trivial, due to our assumption $k = \overline{k}$, which in turn yields $\operatorname{Aut}(F(P_1)) = G_2$. Therefore the functor F yields a morphism of groups

$$\rho : G_1 = \operatorname{Aut}(P_1) \longrightarrow G_2 = \operatorname{Aut}(P_2).$$

It is not difficult to check that F is completely determined by ρ, indeed $F = F_\rho$ (see Exercise 2.22). A sketch of this is as follows. Let $p_1 : S \times G_1 \to S$ the trivial bundle over S. The diagram

$$\begin{tikzcd} S \times G_1 \arrow[r, "p_2"] \arrow[d, "p_1"'] & G_1 \arrow[d] \\ S \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

is cartesian and the same is true for the image by F. This in turn says that the G_2-bundle $F(S \times G_1) \to S$ is canonically isomorphic to $S \times G_2 \to S$. In the general case the G_1-bundle $P_1 \to S$ is only "locally" trivial, the gluing datum encoded in a cocycle $\theta_{ij} : U_{ij} \to G_1$. Clearly the G_2-bundle $F(P_1 \to S)$ is described on the

same covering by the cocycle $\rho \circ \theta_{ij}$, as expected. To complete our answer, we must identify the 2-isomorphisms

$$\begin{tikzcd} \mathbf{BG}_1 \arrow[r, bend left, "F_{\rho_1}"] \arrow[r, bend right, "F_{\rho_2}"'] \arrow[r, phantom, "\Downarrow \theta"] & \mathbf{BG}_2 \,. \end{tikzcd}$$

Our previous discussion proves that θ is determined if it is known at the level of trivial bundles, i.e., $\theta : G_2 = F_{\rho_1}(G_1/k) \cong F_{\rho_2}(G_1/k) = G_2$. This isomorphism is a left translation say by $\xi \in G_2$. We must have for all $s \in G_1$, $\theta\rho_1(s) = \rho_2(s)\theta$, that is $\rho_2(s) = \xi\rho_1(s)\xi^{-1}$. Therefore a 2-isomorphism is the conjugation by an element of G_2. If we drop the assumption that $k = \overline{k}$, the result does not hold exactly in such a simple form.

Example 2.25. *The exact sequence of cohomology revisited.* Let H be a subgroup of the finite constant group G, or more generally a smooth subgroup of an affine smooth group. Consider the 2-fiber product $\mathbf{B}\mathrm{H} \times_{\mathbf{BG}} \operatorname{Spec} k$. A section over $S \in \mathbf{Sch}_k$ of this fiber product is given by a pair $(\pi : P \to H, \phi)$ where $P \to S$ is an H-bundle, and $\phi : P \times^H G \xrightarrow{\sim} S \times G$ is an isomorphism of G-bundles, that is a trivialization of $\sigma : S \to P \times^H G$ of the G-bundle $P \times^H G \to S$. There are obvious morphisms

$$P \xrightarrow{\imath} P \times^H G \xrightarrow{q} G/H, \quad \imath(x) = [x, 1],\ q([x, g]) = Hg^{-1}.$$

Let $f = \phi\sigma : S \to G/H$. Notice that the definitions show that $f(\pi(x)) = H\theta(x)$. There is also a well-defined morphism $\theta : P \times^H G \longrightarrow G$ such that $\xi = \sigma\pi(\xi)\theta(\xi)$, for all $\xi \in P \times^H G$. Clearly $\theta(\xi g) = \theta(\xi)g$, which in turn show that θ is characterized by its restriction $\theta : P \to G$. It is not difficult to check that $\psi(x) = (\pi(x), H\theta(x))$ is a well-defined H-equivariant morphism, which in turn yields an isomorphism of H-bundles over S

$$P \xrightarrow{\psi} S \times_{f, G/H} G \,.$$

It is by now clear that a section of the fiber product $\mathbf{B}\mathrm{H} \times_{\mathbf{BG}} \operatorname{Spec} k$ can be identified with an H-equivariant morphism $\theta : P \longrightarrow S \times G$. An obvious inspection of the morphisms permits us to conclude that we have an isomorphism $\mathbf{B}\mathrm{H} \times_{\mathbf{BG}} \operatorname{Spec} k \cong G/H$, that is a 2-cartesian diagram

$$\begin{tikzcd} \mathbf{B}\mathrm{H} \arrow[r] & \mathbf{BG} \\ G/H \arrow[u] \arrow[r] & pt. \arrow[u] \end{tikzcd} \tag{2.19}$$

Let now $H = G/A$ be a quotient group, and $\pi : P \to X$ a H-bundle over X. We can ask if wether or not P/X comes from a G-bundle via (2.18). Locally for the étale topology the bundle is trivial, so locally admits a lift to $\mathbf{BG}$. Assume given $Q \to S$, $Q' \to S$ two G-bundles with isomorphic images in $\mathbf{B}\mathrm{H}$. Let

$$\theta : Q \times^G H \xrightarrow{\sim} Q' \times^G H$$

be such an isomorphism of H-bundles. Locally all things are trivial, so θ is produced by a section $\theta \in H(S)$. Since all groups are assumed to be smooth, localizing further, we can lift θ to G, which in turn yields an isomorphism $Q \xrightarrow{\sim} Q'$. This can be interpreted in terms of stacks. Let us define $\mathcal{G}$, the fibered category in groupoids over $\mathbf{Sch}/X$ of G-lifts to P/X. The objects are the data $(Q \to S, \xi)$ where Q/S is a G-torsor, and $\xi : Q \times^G H \xrightarrow{\sim} P$ is an isomorphism of H-torsors. That $\mathcal{G}$ is such a category is clear. The definition shows that $\mathcal{G} = \mathbf{BG} \times_{\mathbf{BH}} X$, in particular $\mathcal{G}$ is a stack. The result obtained before is more precise:

Proposition 2.26. *Let the notations be as above, with G and A smooth. Then $\mathcal{G}$ is a gerbe over X.*

Proof. If P/X becomes trivial over the cover $X' \to X$, then clearly we can lift as a G-bundle over X', this show $\mathcal{G}(X') \neq \emptyset$. Assume now that $Q/S, Q'/S$ are two lifts of $P \times_X S/S$ for some X-scheme $S \to X$. Then we get an isomorphism

$$\theta : Q \times^G H \xrightarrow{\sim} Q' \times^G H.$$

Localizing further on S, we may assume that both Q, Q' are trivial. In that case the isomorphism comes from a point $S \to H = G/A$ of H. Since the morphism $G \to H$ is smooth, this morphism has a section étale-locally. This shows that étale-locally we can lift θ to an isomorphism $Q \cong Q'$. □

The previous construction of the classifying stack $\mathbf{BG}$ can be generalized. Let us consider an action of an (affine) group scheme G on a scheme X. To keep things simple enough, we work over a ground field k, but everything works verbatim over an arbitrary base. We change for a moment our convention about the action (on the right) of G on the total space of a principal bundle. Throughout an action of a group on a scheme is written on the left, except for torsors, where the action is on the right.

Definition 2.27. Let $[X/G]$ be the category whose objects are the diagrams

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\pi"'] & X \\ S & \end{tikzcd} \tag{2.20}$$

where $\pi : P \to S$ is a principal bundle, and φ is an equivariant[22] morphism. A morphism from $S \leftarrow P \to X$ to $S' \leftarrow P' \to X$ is given by a commutative diagram

$$\begin{tikzcd} S \arrow[d, "f"'] & P \arrow[l, "\pi"'] \arrow[r, "\varphi"] \arrow[d, "\psi"] & X \arrow[d, equal] \\ S' & P' \arrow[l, "\pi'"'] \arrow[r, "\varphi'"] & X \end{tikzcd} \tag{2.21}$$

[22]On the level of points $\varphi(\xi g) = g^{-1}\varphi(\xi)$.

where the morphism ψ is G-equivariant. The category $[X/G]$ is called the **quotient stack** of X by the action of G.

The terminology of quotient stack will be justified soon.

Notice that the square on the left is cartesian (see Proposition 2.12). The objects like (2.20) are the S-sections of a fibered category in groupoids, which derives from the presheaf in groupoids where f^* is defined on the objects by

$$\begin{array}{ccc} P' & \xrightarrow{\varphi'} & X \\ {\scriptstyle \pi'}\downarrow & & \\ S' & & \end{array} \quad \mapsto \quad \begin{array}{ccc} P' \times_{S'} S & \xrightarrow{\varphi'.p_1} & X \\ {\scriptstyle \pi' \times 1}\downarrow & & \\ S. & & \end{array} \tag{2.22}$$

We have two naturally defined 1-morphisms

$$X \xrightarrow{Q} [X/G] \xrightarrow{F} pt. \tag{2.23}$$

The functor F maps a diagram (2.20) onto the base S, and Q maps an X-point $u : S \to X$ onto the diagram

$$\begin{array}{ccc} P = G \times S & \xrightarrow{\varphi} & X \\ {\scriptstyle \pi}\downarrow & & \\ S & & \end{array} \tag{2.24}$$

where φ is the morphism which on points is given by $(g, s) \mapsto g.u(s)$. If we argue as for **BG**, then we see readily the following important fact:

Proposition 2.28. *Let the notations be as above.*

i) $[X/G]$ *is a stack, and* Q *is an epimorphism* (*Definition* 1.66).

ii) *Let* $P : S \to [X/G]$ *be given by a diagram* (2.20), *then the diagram*

$$\begin{array}{ccc} P & \longrightarrow & S \\ \downarrow & & \downarrow{\scriptstyle P} \\ X & \xrightarrow{Q} & [X/G] \end{array} \tag{2.25}$$

is 2*-cartesian.*

As suggested by the notation, the stack $[X/G]$ can be identified with the quotient stack of the groupoid scheme

$$R = G \times X \underset{d_1}{\overset{d_0}{\rightrightarrows}} U = X$$

where d_0 is the first projection, and d_1 the action, i.e., $[X/G] = [X/R]$. This remark shows that the stack $[X/G]$ is really a natural substitute to the ordinary quotient of X by the G-action. To prove our claim in elementary terms, it suffices to see that our definition of $[X/G]$ identifies it with the stackification of the prestack

associated to the groupoid scheme (R, X). An object over S of this prestack is a map $f : S \to X$. We can see f as an equivariant map $F : S \times G \to X$, with $F(s, g) = g^{-1}f(s)$ at the points level. A morphism from f to f' is given by an element $\sigma \in G$ such that $f' = \sigma f$. We can translate this into a commutative diagram

$$\begin{tikzcd}
 & X & \\
S \times G \arrow[ur, "F"] \arrow[rr, "(1,\sigma)"] \arrow[dr] & & S \times G \arrow[ul, "F'"'] \arrow[dl] \\
 & S. &
\end{tikzcd}$$

Now consider an object of the associated stack, i.e., the stackification of the prestack. It is given by a covering family $(S_i \to S)$ (étale or fppf), an object of the prestack over S_i, i.e., a map $f_i : S_i \to X$, and a descent datum with respect to this covering. It is readily seen that we can use the descent datum to glue together the trivial bundles $S_i \times G$ to get a G-principal bundle $P \to S$ (Exercise 2.14). Finally since the functor $\mathrm{Hom}(-, X)$ is a sheaf, the F_i's glue together in an equivariant map $P \to X$. We have produced a functor $[X/R] \longrightarrow [X/G]$, which is easily seen an isomorphism. Notice this functor follows more quickly from the universal property of the stack $[X/R]$.

The quotient stack operation is transitive in an appropriate sense. A particular case of this fact is the following:

Proposition 2.29. *Let $G \to G'$ be a surjective morphism of affine smooth algebraic groups. Let H be the kernel (not necessarily smooth). The group G acting on G' in an obvious way, then $\mathbf{B}\mathrm{H} \cong [G'/G]$.*

Proof. Denote $G' = G/H$ the quotient group. Our claim amounts to finding an equivalence between these two sheaves in groupoids. This follows quickly from standard facts about reduction of the structural group in a fiber bundle. Let us start with a H-bundle $P \to S$ with base S. If one extends the group from G to G', the G-bundle $P' = P \times^H G$ becomes trivial, i.e., $P' \times^G G'$ has a section. The converse is a well-known result giving a characterization of G-bundles coming from an H-bundle. This is a manifestation of the exact sequence $\mathrm{H}^1(S, H) \to \mathrm{H}^1(S, G) \to \mathrm{H}^1(S, G')$ the cohomology being the flat cohomology. If H is smooth, étale cohomology suffices. It turns out that a trivialization of $P' \times^G G'$ is the same datum that a G-equivariant morphism $P' \to G'$. Things put together yield the result. $\square$

Example 2.30. Let k be a field of arbitrary characteristic, and work over $\mathbf{Sch}_k$. Let $\mu_n = \ker(\mathbf{G}_m \to \mathbf{G}_m)$, $\lambda \mapsto \lambda^n$ the group of n-roots of the unity. Prove that $\mathbf{B}\,\mu_n \cong [\mathbf{G}_m \,/\, \mathbf{G}_m]$, where $\mathbf{G}_m$ acts on itself through $\lambda \mapsto \lambda^n$.

Exercise 2.31. *Let $H \subset G$ be a subgroup of G. Prove that the morphism $\mathbf{B}\mathrm{H} \longrightarrow \mathbf{BG}$ is representable (schematic); this extends Example 2.25.*

Exercise 2.32. *Let X be a scheme acted on by the affine smooth group scheme G. Show that the natural morphism $[X/G] \longrightarrow \mathbf{BG}$ is representable (schematic). More precisely, prove that for any map $S \to \mathbf{BG}$ given by an object $P \in \mathbf{BG}(S)$ we have an isomorphism $S \times_{\mathbf{BG}} [X/G] \xrightarrow{\sim} P \times^G X$.*

Exercise 2.33. *Prove that $\mathcal{G} \cong [P/G]$, and that if A is central in G, then $\mathcal{G} \xrightarrow{\sim} X \times \mathbf{B}\mathrm{A}$.*

3. Algebraic stacks

3.1. Algebraic stacks

Let us recall the notations. Sheaves, stacks, etc. will be denoted by calligraphic, or bold letters, while schemes or algebraic spaces are represented by capital letters. We are going to define the **algebraic stacks**, the ultimate objects of these lectures. Without further notification a stack is a stack in groupoids over the site $\mathcal{S} = \mathbf{Sch}_{et}$, i.e., with the **étale** topology[23]. We begin by recalling a previous Definition 1.50:

Definition 3.1.

i) A stack $\mathcal{X}$ is **representable (resp. schematic)** if it is 1-isomorphic to an algebraic space (resp. a scheme).
ii) Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of stacks. It is **representable**[24] (resp. **schematic**) if for any morphism $\mathcal{S} \to \mathcal{Y}$ where $\mathcal{S}$ is an algebraic space (resp. scheme), the 2-fiber product $\mathcal{X} \times_{\mathcal{Y}} \mathcal{S}$ is an algebraic space (resp. a scheme).

   Thus in a cartesian diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F"] & \mathcal{Y} \\ \mathcal{T} \arrow[u, "g"] \arrow[r, "f"] & \mathcal{S}, \arrow[u, "h"'] \end{tikzcd} \tag{3.1}$$

if $\mathcal{S}$ is an algebraic space (resp. scheme), then $\mathcal{T}$ is an algebraic space (resp. scheme). Let $\mathcal{P}$ be a property of morphisms of schemes of local nature on the target (étale, smooth, flat). One says that F has the property $\mathcal{P}$ if in all such diagrams f has the property $\mathcal{P}$. Notice this mimics a similar definition for morphisms of algebraic spaces. A more precise definition will follows.

Definition 3.2. A stack $p : \mathcal{X} \to \mathbf{Sch}$ is algebraic (or an Artin stack) if the following two properties are satisfied:

i) The diagonal morphism $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable, of finite type, separated and quasi-compact (Definition 3.1),
ii) There is an algebraic space U, and a morphism $P : U \to \mathcal{X}$, which is smooth and surjective. We say (U, P) is a presentation, or an atlas, of $\mathcal{X}$.

[23] Another natural choice would be to start with the fppf topology as in [62]. It can be shown that this choice leads ultimately to the same objects.
[24] Another terminology is **relatively representable**, see, e.g., [37].

If smooth is replaced by étale in ii), the stack is a **Deligne-Mumford stack**, DM stack for short.

The quasi-compactness of the diagonal in i) is a weak form a being separated. In ii) there is no loss of generality in assuming that U is a scheme, since an algebraic space has an étale atlas (Subsection 1.3.1). The definition need some comments. First the representability assumption i) amounts to saying that given any pair (x, y) of sections of $\mathcal{X}$ over S, then the sheaf $\mathbf{Isom}_S(x, y)$ is representable, i.e., an algebraic space (resp. a scheme). To check this, notice that we have a cartesian diagram:

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\Delta"] & \mathcal{X} \times \mathcal{X} \\ \mathbf{Isom}_S(x, y) \arrow[u] \arrow[r] & S. \arrow[u, "(x,y)"'] \end{tikzcd} \tag{3.2}$$

Then ii) means that $\mathbf{Isom}_S(x, y) \to S$ is a morphism of algebraic spaces, separated, quasi-compact (see Subsection 1.3.1).

As a consequence a 1-morphism $S \to \mathcal{X}$ from an algebraic space is always representable. Notice a schematic morphism is representable.

The remark that follows is important.

Proposition 3.3. *Let $\mathcal{X}$ be an algebraic stack. Then, its diagonal is of finite type. If $\mathcal{X}$ is DM (Deligne-Mumford), then its diagonal is unramified, hence quasi-finite, and schematic.*

Proof. (sketch) See for instance [62] for explanations about unramified morphisms. Let $P : U \to \mathcal{X}$ be a presentation. It is easy to check that we have an isomorphism

$$\mathcal{X} \times_{\Delta, \mathcal{X} \times \mathcal{X}, (P,P)} U \times U \xrightarrow{\sim} U \times_{\mathcal{X}} U.$$

Therefore we must check the required property for the morphism of algebraic spaces $\delta : U \times_{\mathcal{X}} U \longrightarrow U \times U$. The composition $pr_1 \circ \delta : U \times_{\mathcal{X}} U \to U$ comes from the cartesian square

$$\begin{tikzcd} U \arrow[r, "P"] & \mathcal{X} \\ U \times_{\mathcal{X}} U \arrow[u] \arrow[r] & U \arrow[u, "P"'] \end{tikzcd}$$

therefore is of finite type. In turn δ is of finite type. If $\mathcal{X}$ is DM, then $pr_1.\Delta$ is étale, thus δ is unramified, in particular quasi-finite. Since an algebraic space which is separated, quasi-finite over a scheme, is also a scheme [4] for a proof of this fact, the last claim follows. □

Thus if $\mathcal{X}$ is DM, a morphism $S \to \mathcal{X}$, with S a scheme, is schematic. Importantly, the fact that a DM stack has an unramified diagonal indeed characterizes the Deligne-Mumford stacks, namely:

Proposition 3.4. *An algebraic stack is DM if and only if its diagonal is unramified.*

The proof consists in finding an étale atlas starting from a smooth one, by a slice argument, see, e.g., [44], Theorem 8.1. □

As a corollary, one gets a characterization of algebraic spaces within the category of stacks (loc. cit.):

Proposition 3.5. *Let $\mathcal{X}$ be an algebraic stack. Then $\mathcal{X}$ is an algebraic space if and only if for all $S \in \mathbf{Sch}$, $x \in \mathcal{X}(S)$, we have $\mathbf{Isom}_S(x,x) = \{1\}$.*

Exercise 3.6. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a representable morphism from a stack to an algebraic stack $\mathcal{Y}$. Then show that $\mathcal{X}$ is an algebraic stack.*

Exercise 3.7. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic stacks. Prove that the relative diagonal $\Delta_{\mathcal{X}/\mathcal{Y}} : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$, $\Delta(x) = (x, x; 1)$ is representable.*

The most prolific examples of algebraic stacks are the quotient stacks $[X/G]$, where X is a scheme acted on by an affine smooth group scheme[25]. To simplify, when a group scheme enters, we will assume schemes are defined over a field k.

Theorem 3.8. *If G is a smooth k-group scheme, then* **BG** *is an algebraic stack (a DM stack if G is étale). Likewise, the stack $[X/G]$ is algebraic, it is Deligne-Mumford if and only if the stabilizers of geometric points are étale.*

Proof. i) The main point is to check the representability of the diagonal, that is for two sections $x = (P/S, f), x' = (P'/S, f')$ of $[X/G]$ over S, the representability of the sheaf $\mathbf{Isom}_S(x, x')$. Let us check this. In view of the previous definition of morphisms of a quotient stack, a section over $T \to S$ of $\mathbf{Isom}_S(, x, x')$ is an equivariant isomorphism ψ such that the diagram

$$\begin{tikzcd}
 & X & \\
P \arrow[ur, "\varphi"] \arrow[rr, bend left, "\psi"] \arrow[rr, "p"'] & S & P' \arrow[ul, "\varphi'"'] \arrow[ll, "p'"] \\
P \times_S S' \arrow[u] \arrow[r] & S' \arrow[u] & P' \times_S S' \arrow[u] \arrow[l]
\end{tikzcd} \tag{3.3}$$

commutes. It is clear that it suffices to prove the result when $X = pt$, i.e., for the principal bundles $P \to S$ and $P' \to S$. Indeed once this partial result known, the sheaf $\mathbf{Isom}_S(x, x')$ will be represented by a closed sub-algebraic space of $\mathrm{Isom}_S(P/S, P'/S)$. In the case of principal bundles, one first trivializes both bundles over an étale cover, then the sheaf $\mathbf{Isom}(G \times S', G \times S')$ is $G \times S'$. The sheaf $\mathbf{Isom}_S(P/S, P'/S)$ is the quotient of $G \times S'$ by an étale equivalence relation. See Exercise 2.20 for a more precise answer.

ii) The definition shows that $Q : X \to [X/G]$ is a presentation. This is due to the étale-local triviality of a principal bundle with a smooth group. Finally if G is

[25] If the affine group is replaced by a non affine one, an elliptic curve for example, the construction may no longer lead to an algebraic stack.

étale over the ground field, for example finite constant, then **BG**, more generally $[X/G]$, is a DM stack. □

The next proposition shows that $[X/G]$ is in a weak sense a categorical quotient:

Proposition 3.9. *Let $f : X \to Y$ be an equivariant morphism of schemes, the action of G on Y being trivial. Then f factors canonically through the quotient stack $[X/G]$.*

Proof. Assume given a section of $[X/G]$, that is a diagram

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "\pi"'] & X \arrow[d] \\ S & Y. \end{tikzcd}$$

We must check that $f\varphi$ factors through S. If $P \to S$ is the trivial bundle, this is obvious. In the general case, one trivializes the bundle over an étale covering $S' \to S$. This yields a morphism $S' \to Y$ which by an étale descent argument factors through S. □

In a stronger sense Y would be a stack with trivial action of G, then the conclusion is similar, but the diagram is only 2-commutative (see [55] for the details).

Exercise 3.10. *Let the ground field k of characteristic $p > 0$. Prove that $\mathbf{B}\,\mu_p$ is an algebraic stack although not a DM stack* (*use Example* 2.30).

Exercise 3.11. *Show that the* 2*-fiber product $\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y}$ of algebraic stacks is an algebraic stack* (*Subsection* 1.2.2).

The quotient stack $[X/G]$ is a particular case of a more general and very useful result. It corresponds to the groupoid $G \times X \rightrightarrows X$ which underlies the action of G on X.

Theorem 3.12. *Let $d_0, d_1 : R \rightrightarrows U$ be a scheme in groupoids. Assume that d_0 (and then also d_1) is smooth (resp. étale), and the "diagonal" $R \to U \times U$ separated and quasi-compact. Then the quotient stack $[U/R]$ is algebraic, and $U \to [U/R]$ is a smooth presentation, as a consequence the groupoid R can be identified with $R = U \times_{[U/R]} U$.*

Proof. (sketch) This follows the same lines as the proof of Theorem 3.8. We first check the representability of the étale sheaf $\mathbf{Isom}_S(x, y)$, if $x, y \in [U/R](S)$. This is clear if the sections come from the prestack. Indeed it suffices to note that the

diagram below is cartesian

$$\begin{tikzcd} R \arrow[r, "(d_0,d_1)"] & U \times U \\ \mathrm{Isom}_S(x,y) \arrow[u] \arrow[r] & S. \arrow[u, "(x,y)"'] \end{tikzcd}$$

In the general case, we reduce the problem to this case via a suitable étale covering, as in the previous example. The second step is to prove that $P : U \to [U/R]$ is smooth and surjective. The surjectivity follows from the definition of the quotient stack (Definition 2.7). Let $\alpha : S \to [U/R]$, and assume first that α lifts to U. We have a cartesian diagram

$$\begin{tikzcd} U \arrow[r, "P"] & {[U/R]} \\ U \times_{[U/R]} S \arrow[u] \arrow[r] & S. \arrow[u, "\alpha"'] \arrow[ul, "u"'] \end{tikzcd}$$

Then a T-section of $U \times_{[U/R]} S$ is the data of $f : T \to S$, together with $g : T \to U$ and an isomorphism $\theta : Pg \cong Puf$, i.e., $d_0(\theta) = g, d_1(\theta) = uf$. Thus this is the same as a point of the fiber of $d_1 : R \to U$ over uf. The smoothness follows. □

One can think of the 2-category of algebraic stacks as "equivalent" to the category of smooth groupoid schemes. The problem is that it may occur that different groupoid schemes lead to isomorphic stacks. This comes from the fact that there is no preferred atlas for a given stack. To clarify this, we need to organize the groupoid schemes in something like a 2-category.

Definition 3.13. An **elementary equivalence** between two groupoid schemes $R \rightrightarrows U$ and $R' \rightrightarrows U'$ is a diagram

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "g"'] & U \arrow[d, "f"] \\ R' \arrow[r, shift left] \arrow[r, shift right] & U' \end{tikzcd} \tag{3.4}$$

which is cartesian, that is $R \cong R' \times_{U' \times U'} (U \times U)$ (see (2.6)) and where $f : U \to U'$ is smooth and surjective. Two groupoid schemes are **Morita equivalent** if they are joined by a chain of elementary equivalences.

Then we can easily check that:

Lemma 3.14. *Morita equivalent groupoid schemes yield equivalent stacks.*

Proof. It is readily seen that (f, g) induces a morphism of quotient prestacks, and then between stacks $F : [U/R] \to [U'/R']$. The fact that the diagram above is cartesian says that F is a monomorphism. It suffices to check that F is an epimorphism (Proposition 1.67). But f being smooth, we can find étale-locally a section, and our claim follows. □

For the converse, see Exercise 3.16. Therefore we can conclude that:

Equivalent algebraic stacks

$$\Updownarrow$$

Morita equivalent smooth groupoid schemes.

In this vaguely defined correspondence, the 2-morphisms of stacks correspond to **homotopies** between morphisms of groupoid schemes. The definition goes as follows. Let there be given two (Morita) morphisms (f, g), (f', g') as in (3.4). A **homotopy**, or **2-morphism** $(f, g) \Longrightarrow (f', g')$ is a morphism $h : U \to R'$ such that $d_0' h = f$, $d_1' h = f'$, i.e., if $x \in U(S)$, $h(x)$ is an arrow with source $f(x)$ and target $f'(x)$.

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "g,g'"'] & U \arrow[d, "f,f'"] \\ R' \arrow[r, shift left] \arrow[r, shift right] & U'. \end{tikzcd}$$

For all $x, y \in U(S)$, $\xi \in R(U)$ such that $d_0(\xi) = x$, $d_1(\xi) = y$, the commutativity of the following diagram is required:

$$\begin{tikzcd} f(x) \arrow[r, "g(\xi)"] \arrow[d, "h(x)"'] & f(y) \arrow[d, "h(y)"] \\ f'(x) \arrow[r, "g'(\xi)"] & f'(y). \end{tikzcd}$$

Exercise 3.15. *Prove that a homotopy $h : (f, g) \Longrightarrow (f', g')$ yields a 2-isomorphism $F \Longrightarrow F'$ between the induced morphisms.*

Exercise 3.16. *Prove that two different presentations of a given algebraic stack are Morita equivalent.*

3.1.1. Weighted projective line. It is instructive to work out in detail an example to enlighten the preceding definitions. Let us recall the classical definition of $\mathbb{P}^n$. Let $\mathbb{A}_*^{n+1} = \operatorname{Spec} k[X_0, \ldots, X_n] - \{(0, \ldots, 0)\}$ be the punctured affine space. The group $\mathbf{G}_m$ acts in an obvious way on $\mathbb{A}_*^{n+1}$, a geometric counterpart of the fact that $R = k[X_0, \ldots, X_n]$ is a graded ring. The action reads (on points) $\lambda(x_1, \ldots, x_n) = (\lambda x_1, \ldots, \lambda x_n)$. Then:

Proposition 3.17. $\mathbb{P}^n = [\mathbb{A}_*^{n+1} / \mathbf{G}_m]$.

Proof. The classical construction of projective space gives us the fact that the natural morphism $\mathbb{A}_*^{n+1} \longrightarrow \mathbb{P}^n$ is a $\mathbf{G}_m$-principal bundle[26]. In turn this produces

[26] Zariski locally trivial.

a morphism $F : \mathbb{P}^n \to [\mathbb{A}_*^{n+1}/\,\mathbf{G}_m]$ which fits into a diagram

$$\begin{tikzcd} & \mathbb{P}^n \arrow[dr, "F"] & \\ \mathbb{A}_*^{n+1} \arrow[ur] \arrow[rr] & & {[\mathbb{A}_*^{n+1}/\,\mathbf{G}_m]}. \end{tikzcd} \tag{3.5}$$

Since the morphisms other than F are $\mathbf{G}_m$-principal bundle, this in turn forces F to be an isomorphism. Another way to see this, is to identify the sections of the corresponding fibered categories. A section over S of the quotient stack $[\mathbb{A}_*^{n+1}/\,\mathbf{G}_m]$, is a datum $(P \to S,\, f : P \to \mathbb{A}_*^{n+1})$ of a $\mathbf{G}_m$-principal bundle together with an equivariant morphism f. But it is known that such a bundle is given as $P = \operatorname{Spec}(\oplus_{n\in\mathbb{Z}}\mathcal{L}^n)$ for a line bundle $\mathcal{L}$ over S. The equivariant morphism $f = P \to \mathbb{A}_*^{n+1}$ amounts to the choice of $n+1$ sections $s_i \in \Gamma(S, \mathcal{L})$ without common zero, equivalently a surjective morphism $\mathcal{O}_S^{n+1} \to \mathcal{L}$, that is an S-point of $\mathbb{P}^n$. $\square$

Let us focus now on the case $n = 1$, i.e., the projective line. We are free to change the usual grading of $k[X, Y]$ to a new one, viz. $\deg(X) = p$, $\deg(Y) = q$, with $(p, q) \in \mathbb{N}^2$, $(p, q) \neq (1, 1)$. The ring $R = k[X, Y]$ is a graded ring with $\deg(X^iY^j) = pi + qj$ which in turn yields a $\mathbf{G}_m$-action on $\operatorname{Spec} R = \mathbb{A}^2$. The action is very clear at the level of points

$$\lambda(x, y) = (\lambda^p x, \lambda^q y) \quad (\lambda \in \Gamma(S, \mathcal{O}_S^*), x, y \in \Gamma(S, \mathcal{O}_S)). \tag{3.6}$$

The induced action on the punctured plane is no longer free, even if $\gcd(p, q) = 1$. Indeed the X-axis $\mathbf{G}_m(1, 0)$ and the Y-axis $\mathbf{G}_m(0, 1)$, are two orbits of fixed points with respective inertia groups μ_p and μ_q, where $\mu_n = \operatorname{Spec} k[T]/(T^n - 1)$ is the group of n roots of 1, a subgroup of $\mathbf{G}_m$. On the open set $XY \neq 0$, the stabilizer of a point reduces to μ_d, if $d = \gcd(p, q)$. Let us assume from now that $d = 1$. First we can perform the Proj construction $\operatorname{Proj} k[X, Y]$, which in turn yields the same result as the geometric quotient $\mathbb{A}^2 - \{(0, 0)\}/\,\mathbf{G}_m$, i.e., the projective line. We can describe this scheme by two charts $U = \operatorname{Spec} k[X, Y]_{(X)}$ and $V = \operatorname{Spec} k[X, Y]_{(Y)}$ with usual notations. It is readily seen that $k[X, Y]_{(X)} = k[t = \frac{Y^p}{X^q}]$, and $k[X, Y]_{(Y)} = k[t^{-1}]$. The presence of fixed points makes the quotient stack $[\mathbb{A}^2 - \{(0, 0)\}/\,\mathbf{G}_m]$ fundamentally different from $\mathbb{P}^1$. On the invariant open subset $\Omega = \mathbb{A}^2$ minus the two axis, the action is free, therefore we have

$$[\Omega/\,\mathbf{G}_m] = \Omega/\,\mathbf{G}_m = \mathbb{P}^1 - \{0, \infty\}.$$

Most interesting for us is to understand the local structure of the stack $[\mathbb{A}^2 - \{(0, 0)\}/\,\mathbf{G}_m]$ near the two bad (stacky) points $0, \infty$. We wish to exhibit a local chart around these two points.

Let $0 \neq f \in R_n\,(n \geq 1)$ be a homogeneous polynomial. The open subset $U_f = D(f) = \{f \neq 0\} \subset \mathbb{A}^2$ is $\mathbf{G}_m$-invariant therefore defines an open substack $[U_f/\,\mathbf{G}_m]$ of $[\mathbb{A}^2 - \{(0, 0)\}/\,\mathbf{G}_m]$. Let us consider the curve $Y_f = \{f = 1\} \subset U_f$. For any point $\xi \in U_f$ the transporter subscheme of ξ into Y_f, that is the subscheme

of $\mathbf{G}_m$ whose S-points are the $\lambda \in \Gamma(S, \mathcal{O}_S^*)$ such that $f(\lambda\xi) = \lambda^n = 1$ is a μ_n torsor. In particular the induced groupoid which acts on Y_f is $\mu_n \times Y_f \to Y_f$, i.e., the action of the subgroup μ_n. Suppose furthermore that the curve $Y_f \subset \mathbb{A}^2_*$ is smooth, for example $f = X$ or $f = Y$. Then Y_f is an "étale slice":

Lemma 3.18. *Suppose that the curve $\{f = 1\}$ in $\mathbb{A}^2$ is smooth, and p, q, n are $\neq 0$ in the ground field k, then $[Y_f/\mu_n] \cong [U_f/\mathbf{G}_m]$ yields a local chart for $\mathbb{P}^1(p, q)$.*

Proof. We start by checking that the action morphism $\mathbf{G}_m \times Y_f \longrightarrow U_f$ is étale. Since both members are smooth, it suffices to check the morphism induces an isomorphism on the tangent spaces, i.e., injective. We can restrict ourselves to the point $(1, (x_0, y_0)) \in \mathbf{G}_m \times Y_f$. The tangent line along the orbit of (x_0, y_0) is generated by (px_0, qy_0). On the other hand differentiating the equation $f(\lambda^p x_0, \lambda^q y_0) = \lambda^d$ at $\lambda = 1$ yields

$$px_0 \frac{\partial f}{\partial x}(x_0, y_0) + qy_0 \frac{\partial f}{\partial y}(x_0, y_0) = d$$

which in turn shows that the orbit of (x_0, y_0) is transverse to Y_f at (x_0, y_0). The claim follows. Let us consider the induced groupoid acting on Y_f. As observed previously this groupoid is given by the action of μ_n on Y_f. It fits into a diagram

$$\begin{tikzcd} \mathbf{G}_m \times U_f \arrow[r, shift left] \arrow[r, shift right] & U_f \\ \mu_d \times Y_f \arrow[u] \arrow[r, shift left] \arrow[r, shift right] & Y_f \arrow[u] \end{tikzcd} \tag{3.7}$$

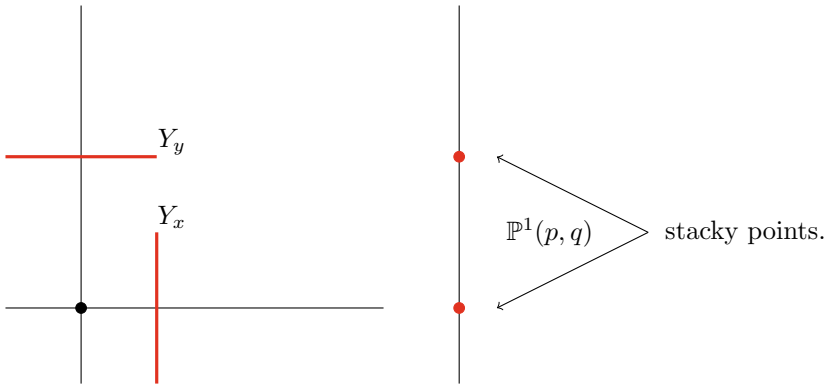
from which we get a morphism $\phi : [Y_f/\mu_d] \longrightarrow [U_f/\mathbf{G}_m]$. This morphism is an isomorphism. Firstly it is a monomorphism due to the previous remark. The action morphism $\mathbf{G}_m \times Y_f \to U_f$ is étale surjective, as a consequence every point of U_f étale-locally lifts to $\mathbf{G}_m \times Y_f$ which in turn means that ϕ is an epimorphism, thus an isomorphism by Proposition 1.67. In our example we can take $f = x$ (resp. y). Thus the stacky weighted projective line $\mathbb{P}^1(p, q)$ has two well-understood stacky points as illustrated by the picture on page 68. Since p, q are coprime, the "complementary" open substack $[\mathbb{A}^2 - \{(x, y),\, xy = 0\}/\mathbf{G}_m]$ is a scheme.

It will be useful to notice that there exists a canonical morphism

$$\mathbb{P}^1(p, q) \longrightarrow \mathbb{P}^1 \tag{3.8}$$

onto the ordinary projective line. From the presentation of $\mathbb{P}^1(p, q)$ as a quotient, this amounts to finding a natural morphism $\mathbb{A}^2 - \{(0, 0)\} \to \mathbb{P}^1$ invariant by the $\mathbf{G}_m$-action. This morphism is easy to see, indeed on the open set $x \neq 0$, this is $(x, y) \mapsto t = \frac{y^p}{x^q}$, and on the open set $y \neq 0$, we take $(x, y) \mapsto \frac{x^q}{y^p} = t^{-1}$. These morphisms glue together to give rise to the expected morphism.

There is another way to define a stacky projective line, but with a different flavor than the previous one. Let $G \subset \mathrm{PGL}(2)$ be a finite subgroup, then G acts on the ordinary projective line $\mathbb{P}^1$, which in turn yields the quotient stack $[\mathbb{P}^1/G]$, it is assumed that $|G|$ is prime to the characteristic of k. The coarse moduli space



is $\mathbb{P}^1/G \cong \mathbb{P}^1$, use for this Lüroth's theorem, i.e., the Riemann-Hurwitz formula. Obviously there are points with non-trivial (cyclic) stabilizers, indeed two or three orbits of such points, therefore $[\mathbb{P}^1/G]$ is really a stack with two or three stacky points. Take $G = \mu_n$ the group of n-roots, with the obvious action $(\zeta, z) \mapsto \zeta z$. Then it is readily seen that there are two stacky points $z = 0, \infty$ with local charts $[\mathbb{A}^1/\mu_n]$. This stack if however different from $\mathbb{P}^1(n,n)$.

Exercise 3.19. *Let $\mathbb{A}^1 = \operatorname{Spec} k[x]$, $k[x]$ graded according to $\deg(x) = d \geq 1$. The corresponding action of $\mathbf{G}_m$ at the points level is $\lambda.x = \lambda^d x$. Assuming $d \neq 0$ in k, prove the isomorphism $[\mathbb{A}^1 - \{0\}/\mathbf{G}_m] = \mathbf{B}(\mu_d)$ (μ_d = group of d-roots).*

Exercise 3.20. *Show that there is an "obvious" morphism $\mathbb{P}^1 \to \mathbb{P}^1(p,q)$.*

Exercise 3.21. *In this exercise we describe by different arguments the local structure of the stacky projective line, assuming now the pair of weights (p,q) arbitrary, i.e., $d = \gcd(p,q) \geq 1$. We keep the same notations as in the text.*

i) *Let $f \in k[X,Y]$ be homogeneous of degree $n \geq 1$. Define $V_f = \{(x,y,t) \in \mathbb{A}^3,\ t^n = f(x,y) \neq 0\}$. Show $V_f \to U_f$, $(x,y,t) \mapsto (x,y)$ is a μ_n-torsor. Show that there is a natural free action of the group $\mathbf{G}_m$ such that equivariantly $V_f \cong \mathbf{G}_m \times Y_f$.*
ii) *Use* i) *to prove the equivalence of stacks $[V_f/\mathbf{G}_m] \cong Y_f$ and $[V_f/\mathbf{G}_m \times \mu_n] \cong [Y_f/\mu_n]$, and finally $[Y_f/\mu_n] \cong [U_f/\mathbf{G}_m]$.*

3.1.2. n points on the line. The example that follows is the most elementary moduli problem involving curves. It plays a basic role in Hurwitz theory, and was the source of many geometric considerations, see [36] for a nice illustration. Let us start with the classifying stack $\mathbf{BG}$, where $G = \mathrm{PGL}(2) = \mathrm{GL}(2)/\mathbf{G}_m$ is the projective linear group, i.e., the group of automorphisms of the projective line $\mathbb{P}^1$. The ground ring is here $A = \mathbb{Z}$.

We claim that this stack is isomorphic to $\mathcal{M}_0$, the moduli stack which classifies the smooth complete curve of genus 0. The claim seems strange because there is only one smooth complete rational curve, i.e., $\mathbb{P}^1$, a rigid object. We begin with the definition of the stack $\mathcal{M}_0$:

Definition 3.22. The sections over S of $\mathcal{M}_0$ are the proper flat families $\pi : E \to S$ with geometric fibers smooth curves of genus 0, i.e., isomorphic to $\mathbb{P}^1$. The morphisms in $\mathcal{M}_0$ are the cartesian diagrams

$$\begin{tikzcd} E' \arrow[r, "f"] \arrow[d, "\pi'"'] & E \arrow[d, "\pi"] \\ S' \arrow[r, "u"] & S. \end{tikzcd}$$

The projective line $\mathbb{P}^1$ defined over the ground ring yields a morphism

$$\operatorname{Spec} k \longrightarrow \mathcal{M}_0$$

which is easily seen to be an atlas. Indeed, for any section $\pi : E \to S$ of $\mathcal{M}_0$ the sheaf $\mathbf{Isom}_S(\mathbb{P}^1_S, E)$ is not only representable, but is a PGL(2)-torsor. This provides a 1-morphism of stacks

$$\mathcal{M}_0 \longrightarrow \mathbf{B}(\mathrm{PGL}(2)). \tag{3.9}$$

Proposition 3.23. *The morphism* (3.9) *is an isomorphism.*

Proof. The "quasi-inverse" morphism is obtained as follows. Let $P \to S$ be a $G = \mathrm{PGL}(2)$ torsor. We can twist the trivial bundle $\mathbb{P}^1 \times S \to S$ by mean of the torsor $P \to S$, according to the known operation (Section 2.2)

$$P/S \mapsto F(P/S) = P \times^{\mathrm{PGL}(2)} \mathbb{P}^1 \to S.$$

The fact that $P \times^{\mathrm{PGL}(2)} \mathbb{P}^1$ is a scheme follows from a descent argument explained in [64], namely descent with ample invertible sheaves. The result is clearly a section of $\mathcal{M}_0(S)$. That this construction provides a quasi-inverse functor comes from the functorial isomorphism $\mathbf{Isom}_S(S \times \mathbb{P}^1, P \times^G \mathbb{P}^1) \cong P$. The projective line $\mathbb{P}^1$ is a homogeneous space of G, therefore the result is a particular case of the more general and straightforward identification

$$P \cong \mathbf{Isom}_S(S \times (G/H), P/H)$$

for any G-principal bundle $P \to S$ (see Exercise 2.20). □

A similar construction works with the stack $\mathcal{M}_{0,1}$ classifying projective lines together with one marked point. A section of $\mathcal{M}_{0,1}$ over S is a proper flat family of projective lines $\pi : C \to S$ together with a section $P : S \to C$, that is $\pi \circ P = 1_S$:

$$\begin{tikzcd} E \arrow[r, "\pi"] & S. \arrow[l, bend right] \end{tikzcd}$$

Let $H = \mathbf{Aff}(\mathbf{1})$ be the affine group, i.e., the stabilizer of $\infty \in \mathbb{P}^1$ in $G = \mathrm{PGL}(2)$. The scheme $\mathbf{Isom}_S((C, P), (S \times \mathbb{P}^1, \infty))$ is clearly a torsor under H. Conversely, given $P \to S$ an H-torsor, the associated fiber space construction yields as in the previous case, a curve with a marked point

$$S = P \times^H \infty \hookrightarrow P \times^H \mathbb{P}^1 \to S.$$

It is easy to check that we get in this way an equivalence $\mathcal{M}_{0,1} \cong \mathbf{B}(\mathbf{Aff}(\mathbf{1}))$ (see Exercise 3.27). The same proof shows that $\mathcal{M}_{0,2} \cong \mathbf{B}\,\mathbf{G}_m$.

The comment that follows will be useful. Let $\pi : C \to S$ be a section of $\mathcal{M}_{0,1}$, with marked point $P : S \to C$. Let $D \subset C$ be the relative Cartier divisor[27] which is the image of P. It is a straightforward exercise invoking the base change theorem in cohomology ([33], Chap. III, Theorem 12.11) to check that:

$$\pi_*(\mathcal{O}_C) = \mathcal{O}_S \quad \text{and} \quad \mathrm{R}^1\,\pi_*(\mathcal{O}_E) = 0.$$

In addition the $\mathcal{O}_S$-module $\pi_*(\mathcal{O}(D))$ is locally free of rank 2 whose formation commutes with an arbitrary base change. Applying π_* to the exact sequence $0 \to \mathcal{O}_E \to \mathcal{O}(D) \to N_D \to 0$ we get

$$0 \to \mathcal{O}_S \longrightarrow \pi_*(\mathcal{O}(D)) \longrightarrow N_D \to 0. \tag{3.10}$$

From the known description of the functor of points of a projective bundle (Subsection 1.2.5), the map (3.10) yields a section (say the ∞-section) of the projective bundle $\mathbb{P}(\pi_*(\mathcal{O}(D))) \to S$. Thus we get a canonical isomorphism

$$\begin{tikzcd} C \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & \mathbb{P}(\pi_*(\mathcal{O}(D))) \arrow[dl] \\ & S & \end{tikzcd}$$

such that the section P goes to the ∞ section.

Let us consider now the fibered category in groupoids $\mathcal{M}_{0,n}$, $n \geq 3$ whose sections over S are the families of $\mathbb{P}^1$ over S, as above, together with n ordered disjoint sections $(P_i)_{1 \leq i \leq n}$. We can visualize such an object as

$$\begin{tikzcd} X \arrow[r, shift left, "\pi"] & S. \arrow[l, bend left, "P_i"] \end{tikzcd} \tag{3.11}$$

The morphisms are the morphisms of families of curves as before, but now preserving the labelled sections. On the other hand, let us denote U_n the open subset of n-tuples of pairwise distinct points of $\mathbb{P}^1$, i.e.,

$$U_n = \{(x_1, \ldots, x_n) \in \mathbb{P}^1,\ x_i \neq x_j \text{ if } i \neq j\}. \tag{3.12}$$

The group PGL(2) operates on U_n through its diagonal action on $(\mathbb{P}^1)^n$. Then we have the following result:

Proposition 3.24. *There is an isomorphism of fibered categories*

$$\mathcal{M}_{0,n} \cong [U_n / \operatorname{PGL}(2)]$$

as a corollary we get that for all $n \geq 0$, $\mathcal{M}_{0,n}$ is a smooth connected algebraic stack, indeed a scheme if $n \geq 3$.

[27] A **relative Cartier divisor** is an effective Cartier divisor $D \subset C \to S$ which is flat over S, i.e., inducing a Cartier divisor along the fibers [37].

Proof. We give an outline of two proofs. We set $G = \mathrm{PGL}(2)$. Suppose that the G-principal bundle $P \to S$ is given on an étale cover $(S_i \to S)$ by the cocycle of transition functions $g_{ij} : S_{ij} \to G$. Recall this means that $(s, g) \in S_{ij} \times G$ and $(s, g_{ji}(s)g)$ yield the same point of P over $s \in S$. Likewise the associated fiber bundle $P \times^G \mathbb{P}^1$ is obtained by the identification rule $(s, x) \sim (s, g_{ji}(s)x)$. therefore a section of $P \times^G \mathbb{P}^1 \to S$ amounts to a collection of functions $f_i : S_i \to \mathbb{P}^1$ such that on the overlap S_{ij}

$$f_j(s) = g_{ji}(s) f_i(s)$$

which in turn yields a morphism $\phi : P \to \mathbb{P}^1$ whose restriction to $S_i \times G$ is $\tilde{f}_i(s, g) = g^{-1} f_i(s)$. Clearly the result is a G-equivariant function $P \to \mathbb{P}^1$ is G. This construction provides us n equivariant functions $\phi_i : P \to \mathbb{P}^1$ pairwise disjoint, which in turn yields an equivariant morphism $P \to U_n$. This is our first sketchy proof.

A second proof goes as follows. We want to use the first section P_1 to realize the relative curve $C \to S$ as a projective bundle $C = \mathbb{P}(E) \to S$ associated to a rank 2 vector bundle E on S, together with a surjective map $E \to \mathcal{L}$, $\mathcal{L}$ a line bundle, with kernel $\mathcal{O}_S$. See the previous comment (3.10). Likewise the sections $P_2, \ldots, P_n$ correspond to surjective maps $\varphi_i : E \to \mathcal{L}_i$ up to $\varphi \mapsto \alpha_i \varphi_i$, $\alpha_i \in \Gamma(S, \mathcal{O}_S^*)$. The condition $P_i \cap P_j = \emptyset$ if $i \neq j$ has for consequence that the dotted arrow in the diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_S \arrow[r] \arrow[dr, dashed] & E \arrow[r] \arrow[d] & \mathcal{L}_1 \arrow[r] & 0 \\
& & \mathcal{L}_i & &
\end{tikzcd}$$

is bijective. This in turn yields $\mathcal{L}_i \cong \mathcal{O}_S$ for all $i = 1, \ldots, n$, and $E \cong \mathcal{O}_S^2$. We can assume that $P_i = (a_i : b_i)$, $a_i, b_i \in \Gamma(S, \mathcal{O}_S)$ and if $i \neq j$, $a_i b_j - a_j b_i \in \Gamma(S, \mathcal{O}_S^*)$. We infer from this that we can identify the category $\mathcal{M}_{0,n}$ with the category of n-tuples

$$\begin{tikzcd}
\mathcal{O}_S^2 \arrow[r, bend left, "\varphi_1"] \arrow[r, bend right, "\varphi_n"'] \arrow[r, phantom, "\vdots"] & \mathcal{O}_S
\end{tikzcd}$$

where the isomorphisms are given by diagrams:

$$\begin{tikzcd}
\mathcal{O}_S^2 \arrow[rr, "\begin{pmatrix} a & b \\ c & d \end{pmatrix}"] \arrow[dr, bend right, "\varphi_1"'] \arrow[dr, bend right=60, "\varphi_n"'] & & \mathcal{O}_S^2 \arrow[dl, bend left, "\varphi_n'"] \arrow[dl, bend left=60, "\varphi_1'"] \\
& \mathcal{O}_S &
\end{tikzcd} .$$

It is known that there exists a (unique) homographic transformation such that the points $\varphi_i = (a_i : b_i)$ for $i = 1, 2, 3$ go to $(0 : 1)$, $(1 : 0)$ and $(1 : 1)$. Assuming

this normalization we can normalize further assuming $\varphi_i = (a_i : 1)$ when $i \geq 4$. Therefore $a_i - a_j \in \Gamma(S, \mathcal{O}_S)^*$ if $i \neq j$. In turn this shows that $\mathcal{M}_{0,n}$ is a fibered category equivalent to the one coming from the functor of points of the scheme

$$V_n = \{(a_i)_{4 \leq i \leq n} \in (\mathbb{A}^1 - \{0,1\})^{n-3},\ a_i \neq a_j \text{ if} i \neq j\}.$$

This is the expected result. □

In the previous construction the marked points on $\mathbb{P}^1$ were labelled. We can try a similar construction but now with the P_i's unlabelled. The expected stack will be denoted $\mathcal{M}_{0,(n)}$. The sections over $S \in \mathbf{Sch}$ of this modified fibered category in groupoids are given by the diagrams

$$D \subset \mathbb{P}^1 \times S \xrightarrow{p_2} S \tag{3.13}$$

where D is a closed subscheme of $P^1 \times S$, finite étale of degree n over S. The morphisms over $f : S' \to S$ are the cartesian diagrams

$$\begin{array}{ccc} \mathbb{P}^1 \times S' & \xrightarrow{h} & \mathbb{P}^1 \times S \\ \downarrow & & \downarrow \\ S' & \xrightarrow{f} & S \end{array}$$

such that $h(D') = D$. We can write $h : D' \xrightarrow{\sim} f^*(D)$. We can find an étale covering $f : \tilde{S} \to S$, such that $f^*(D)$ splits as a sum of n disjoint sections. This means that we get the same result if we restrict the objects to those which are the sum $\sum_{i=1}^n P_i$ of n disjoint sections. In this way the stack $\mathcal{M}_{0,(n)}$ appears as the quotient of $\mathcal{M}_{0,n}$ by the natural action of the symmetric group S_n on $\mathcal{M}_{(0,n)}$, i.e., permutation of the P_i's. The most natural construction of $\mathcal{M}_{0,(n)}$ is as a quotient stack, even if a precise definition is missing in these notes (see [55]):

Definition 3.25. We set $\mathcal{M}_{0,(n)} = [\mathcal{M}_{0,n}/S_n]$.

Exercise 3.26. *Show that* $\mathcal{M}_{0,(3)} \cong \mathbf{B}\, S_3$, *and that* $\mathcal{M}_{0,(4)} \cong [\mathbb{A}^1 - (0,1)/\mathbf{S}_3] \times \mathbf{B}(\Delta)$, *where* Δ *is Klein's Viergruppe. Observe that* $\mathbf{S}_4/\Delta = \mathbf{S}_3$ (*see* [2], *Example* 3.8).

Exercise 3.27. *Let* H *be the group* $\mathbf{Aff}(\mathbf{1})$. *If* $P \to S$ *is an* H*-principal bundle, define a canonical isomorphism of* H*-bundles*

$$\mathbf{Isom}_S((S \times \mathbb{P}^1, \infty), P \times^H \mathbb{P}^1, P)) \cong P.$$

3.1.3. A warmup of formal deformation theory. In this subsection our aim is to explain quickly why formal deformation theory can be used to produce local formal charts for a scheme or an algebraic space, and in the next section for an algebraic stack. Throughout the ground ring is an algebraically closed field k.

The question we would like to address is about our understanding of the infinitesimal (or formal) structure of an algebraic stack $\mathcal{M}$ around a (geometric) point $\xi_0 \in \mathcal{M}(k)$. The formalism of formal deformations is well suited to this

goal. To begin with, let $\mathbf{Art}_k$ denote the category of local artinian k-algebras with residue field k, a morphism in this category being a morphism of k-algebras, necessary local.

Let $\mathcal{M}$ be a Deligne-Mumford stack[28] defined over the ground field k. We want to focus on the **restriction** of $\mathcal{M}$ to the subcategory $\mathbf{Art}_k^{\mathrm{op}} \subset \mathbf{Sch}_k$. The notation $\mathcal{M}(A)$ is a shortcut of $\mathcal{M}(\operatorname{Spec} A)$, and the pullback $\mathcal{M}(A) \to \mathcal{M}(k)$ will be denoted $\xi \mapsto \xi_{|k}$. Some definitions are necessary to ask the right questions, and eventually to get an answer.

Definition 3.28. Let A be in $\mathbf{Art}_k$.

1. By a lift of ξ_0 to A we mean a pair[29] (ξ, ϵ) with $\xi \in \mathcal{M}(A)$, and $\epsilon : \xi_{|k} \xrightarrow{\sim} \xi_0$.
2. A morphism $\phi : (\xi_1, \epsilon_1) \to (\xi_2, \epsilon_2)$ of liftings of ξ_0) to A, is a (iso)morphism $u \in \operatorname{Hom}_{\mathcal{M}}(\xi_1, \xi_2)$, making the triangle

$$\begin{tikzcd} \xi_{1|k} \arrow[rr, "u_{|k}", "\sim"'] \arrow[dr, "\epsilon_1"'] & & \xi_{2|k} \arrow[dl, "\epsilon_2"] \\ & \xi_0 & \end{tikzcd}$$

commutative.

The lifts of ξ_0 to A together with their (iso)morphisms define a category, precisely a groupoid (a set if $\mathcal{M}$ is an algebraic space) denoted $\mathcal{D}_{\xi_0}(A)$. Notice $\mathcal{D}_{\xi_0}(k)$ is a trivial groupoid (discrete). Let $A' \to A$ be a morphism of $\mathbf{Art}_k$. The pullback from A' to A, better cartesian arrows over $\operatorname{Spec} A \to \operatorname{Spec} A'$, defines a functor

$$\mathcal{D}_{\xi_0}(A') \longrightarrow \mathcal{D}_{\xi_0}(A). \tag{3.14}$$

The "presheaf in groupoids" $A \mapsto \mathcal{D}_{\xi_0}(A)$ which encodes the infinitesimal behaviour of ξ_0 is not quite a stack due to the lack of a natural topology on $\mathbf{Art}_k$, even if it shares some properties with stacks [53]. Classical deformation theory begins with a more down to earth definition [34]:

Definition 3.29. A **deformation** of ξ_0 to A is an isomorphism class of liftings, i.e., the set of deformations classes of ξ_0 to A is $\operatorname{Def}_{\xi_0}(A) = \mathcal{D}_{\xi_0}(A)/\sim$.

Clearly $A \mapsto \operatorname{Def}_{\xi_0}(A)$ is a covariant functor from $\mathbf{Art}_k$ to $\mathbf{Set}$. To describe more accurately this functor, let

$$\begin{tikzcd} R \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & U \arrow[r, "p"] & \mathcal{M} \end{tikzcd}$$

[28] This restriction is unnecessary: we make it just to limit the technical details. The natural framework to develop Schlessinger's formalism is that of fibered categories [53], [63].

[29] A better definition would be to consider an arrow $\xi_0 \to \xi$ over $\operatorname{Spec} k \to \operatorname{Spec} A$ as in [2] Section 4 or [53].

be a presentation of $\mathcal{M}$ as a quotient of an étale groupoid scheme. Since $k = \overline{k}$ there is a point $x_0 \in U(k)$ with $p(x_0) \cong \xi_0$. Choose such a point together with an isomorphism $\theta_0 : p(x_0) \cong \xi_0$. Assume given $\eta : \operatorname{Spec} A \to U$ with image x_0, then $p(\eta)$ is in a natural fashion a lift of ξ_0, indeed

$$p(\eta)_{|k} \xrightarrow{\sim} p(x_0) \xrightarrow{\theta_0} \xi_0. \tag{3.15}$$

Let $(\xi, \epsilon) \in \mathcal{D}_{\xi_0}(A)$ be a lift to A. Since an étale cover of $\operatorname{Spec} A$ is a sum of copies of $\operatorname{Spec} A$, there is a section, therefore we can find a morphism $\eta : \operatorname{Spec} A \to U$ such that $p(\eta) \cong \xi$. Denote $\eta(0)$ the k-point $\operatorname{Spec} k \hookrightarrow \operatorname{Spec} A \to U$. We have a well-defined isomorphism $\alpha_0 : p(\eta(0) \cong p(x_0)$, in turn a point $\gamma_0 \in R(k)$. As before we can lift γ_0 to $\gamma \in R(A)$. This shows that we can adjust the lift to get $\eta(0) = x_0$, i.e., η has center x_0. This show that each deformation class can be realized by a morphism $\eta : \operatorname{Spec} A \to U$ with center x_0, equivalently a morphism $\mathcal{O}_{U,x_0}/\mathcal{P}^{n+1} \to A$, if $\mathcal{P}$ stands for the maximal ideal of $\mathcal{O}_{U,x_0}$. This shows that the deformations of ξ_0 come from the one induced by (U, x_0). In order to propose a more precise conclusion, we need a quick review of Schlessinger's deformation theory.

In what follows all deformations are assumed centered at x_0 (Definition 3.29). Formal deformation theory in the Grothendieck-Schlessinger sense aims to describe completely, i.e., by generators and relations the complete local ring $\hat{\mathcal{O}}_{U,x_0}$ in terms of the deformation functor $\operatorname{Def}_{\xi_0}$. The process that makes this possible is a sort of gluing but not in a common sense, here gluing along "closed" subschemes, not along an open covering. The relevant definitions are as follows [6]:

Suppose we are given a diagram in $\mathbf{Art}_k$:

$$\begin{tikzcd} A' \arrow[dr, "u'"] & \\ & A \\ A'' \arrow[ur, "u''"'] & \end{tikzcd} \tag{3.16}$$

then we can perform the **fiber product** $B = A' \times_A A'' = \{(x', x''),\ u'(x') = u''(x'')\}$, leading to a cartesian diagram

$$\begin{tikzcd} & A' \arrow[dr, "u'"] & \\ B \arrow[ur, "q'"] \arrow[dr, "q''"'] & & A \\ & A''. \arrow[ur, "u''"'] & \end{tikzcd} \tag{3.17}$$

Dually, starting from the symmetric diagram leads to the coproduct of rings

$$\begin{tikzcd} & A' \arrow[dr, "u'"] & \\ B \arrow[ur, "q'"] \arrow[dr, "q''"'] & & A' \otimes_B A'' \\ & A'' \arrow[ur, "u''"'] & \end{tikzcd} \tag{3.18}$$

The following elementary lemma plays a significant role in Schlessinger's theory:

Lemma 3.30. *Assume that u' is surjective, then the cartesian diagram* (3.17) *is also cocartesian, i.e., $A \cong A' \otimes_B A''$, and conversely.*

Proof. Let $A = A'/I$. Then q'' is also onto with kernel $J = I \times \{0\}$, which in turn yields $q'(J) = I = JA'$. Then $A' \otimes_B A'' = B_J \otimes_B A'' \cong A''/JA'' = A''/I = A$. Likewise in the opposite direction. □

To extract a versal formal deformation space from the functor Def_{ξ_0}, Schlessinger's theory says it suffices to check the following properties (Schlessinger's H_i conditions):

Given a diagram (3.16) we have the natural map

$$\begin{tikzcd} & \mathrm{Def}_{\xi_0}(A') \arrow[dr, "u'"] & \\ \mathrm{Def}_{\xi_0}(B) \arrow[ur, "q'"] \arrow[dr, "q''"'] & & \mathrm{Def}_{\xi_0}(A) \\ & \mathrm{Def}_{\xi_0}(A''). \arrow[ur, "u''"'] & \end{tikzcd} \tag{3.19}$$

(H1) The map (3.19) is surjective whenever u' is onto.
(H2) The map (3.19) is a bijection when $A = k$, and $A' = k[\epsilon]$.
(H3) The tangent space $\mathrm{Def}_{\xi_0}(k[\epsilon])$ is finite dimensional over k.
(H4) The map (3.19) is a bijection whenever $A' = A''$ and $A' \to A$ is a tiny extension.

The notation $k[\epsilon]$ stands for the algebra of dual numbers, i.e., $k[X]/(X^2)$, $\epsilon = \overline{X}$. Recall that a surjective morphism $u' : A' \to A$ is a **tiny extension** if $\ker(u') = (t)$ and $\mathcal{M}_{A'}t = 0$. Any surjection factors as a product of tiny extensions. Thus it suffices to check H1 for $A' \to A$ a tiny extension. In H3, it is suggested that $\mathrm{Def}_{\xi_0}(k[\epsilon])$ has a natural structure of k-vector space. This follows formally from H2, see Exercise 3.36.

The theorem is:

Theorem 3.31. *If conditions* H1, H2, H3 *hold, then there is a complete local k-algebra $\mathcal{O}_{\mathrm{ver}}$ with maximal ideal $\mathcal{M}$, residue field k, together with a versal deformation $\xi = (\xi_n)_{n>0}$*

$$\xi_n \in \mathrm{Def}_{\xi_0}(\mathcal{O}_{\mathrm{ver}}/\mathcal{M}^{n+1}) \tag{3.20}$$

with $\xi_{n+1} \mapsto \xi_n$ under $\mathcal{O}_{\mathrm{ver}}/\mathcal{M}^{n+2} \to \mathcal{O}_{\mathrm{ver}}/\mathcal{M}^{n+1}$, such that the versality property holds:

1. *For any $A \in \mathbf{Art}_k$, and $\eta \in \mathrm{Def}_{\xi_0}(A)$, there is an algebra morphism (not unique) $f : \mathcal{O}_{\mathrm{ver}} \to A$ such that $\xi = (\xi_n) \mapsto \eta$, precisely if f factors through $\mathcal{O}_{\mathrm{ver}}/\mathcal{M}^{n+1}$ then $\xi_n \mapsto \eta$. It is assumed that the first-order part $\overline{f} : \mathcal{O}_{\mathrm{ver}}/\mathcal{M}^2 \to A/\mathcal{M}_A^2$ of f is unique, i.e., a versal deformation is 1-universal.*
2. *The pair $(\mathcal{O}_{\mathrm{ver}}, \{\xi_n\})$ is unique up to a non unique isomorphism. However:*
3. *If* H4 *holds, i.e.,* (3.19)*) is a bijection whenever $A' = A''$ is a tiny extension of A, then the morphism f is unique, that is Def_{ξ_0} is pro-representable, the versal deformation is universal.*

This can be rephrased as follows:

1. The linear map $(\mathcal{M}/\mathcal{M}^2)^* = \mathrm{Hom}_k(\mathcal{O}_{\mathrm{ver}}, k[\epsilon]) \longrightarrow \mathrm{Def}_{\xi_0}(k[\epsilon])$ is bijective, and
2. The functorial morphism $\mathrm{Hom}_k(\mathcal{O}_{\mathrm{ver}}, -) \longrightarrow \mathrm{Def}_{\xi_0}(-)$ is smooth, indeed étale, that is, given a tiny extension $A' \to A$, the map

$$\mathrm{Hom}_k(\mathcal{O}_{\mathrm{ver}}(A') \longrightarrow \mathrm{Hom}_k(\mathcal{O}_{\mathrm{ver}}(A) \times_{\mathrm{Def}_{\xi_0}(A)} \mathrm{Def}_{\xi_0}(A')$$

is onto.

Coming back to our initial setting, the pair $(\hat{\mathcal{O}}_{U,x_0}, (p_n))$ where p_n is the image of p in $\mathcal{M}(\mathcal{O}_{U,x_0}/\mathcal{M}^{n+1})$ is a candidate to be a versal deformation of ξ_0. This amounts to checking that the deformation functor Def_{ξ_0} as defined at the beginning satisfies the conditions H1, H2, H3.

Proposition 3.32. *The deformation functor of $\xi_0 \in \mathcal{M}(k)$ given by restricting the stack to $\mathbf{Art}_k$, has a good deformation theory, meaning that the Schlessinger conditions* H1, H2, H3 *hold. The complete local ring $\hat{\mathcal{O}}_{U,x_0}$, for one* (*and then for all*) *étale atlas $U \to \mathcal{M}$, is the formal deformation space of ξ_0, in fact universal.*

Proof. (abridged) The idea is to lift all the data to U, and observe that the result is in this case clear. The second step is to see why $\mathrm{Def}_{\xi_0}(k[\epsilon]) \xrightarrow{\sim} T^*_{u,x_0}$ as a vector space. This amount to check that the map sending $(\eta : \mathrm{Spec}\, k[\epsilon] \to U,\ \eta(0) = x_0)$ to the deformation class $[p(\eta)]$ is one-to-one. We saw that it is onto. Assume given η_i $(i = 1, 2)$ two such points of U, i.e., tangent vectors at x_0 such that the induced deformations $[p(\eta_1)]$ and $[p(\eta_2)]$ coincide. Thus there is an isomorphism $\theta : p(\eta_1) \xrightarrow{\sim} p(\eta_2)$ inducing the identity on ξ_0. This provides us with a morphism

$$(\eta_1, \eta_2, \theta) : \mathrm{Spec}\, k[\epsilon] \longrightarrow R = U \times_{\mathcal{M}} U$$

sending the closed point to $(x_0, x_0, Id) = \Delta(x_0)$. But the diagonal $\Delta : U \longrightarrow R = U \times_{\mathcal{M}} U$, $(x \mapsto (x, x, id)$ is étale at x_0, this forces to have $\eta_1 = \eta_2$, see Exercise 3.36. The last remark follows from the fact $\mathcal{M}$ being a Deligne-Mumford stack, we are able to lift infinitesimally the automorphisms. □

This discussion applies obviously to the functor of points $F = h_X$ of a scheme X, a k-scheme to simplify. Let $x \in F(k)$ be a k-point of X. Restricting F to $\mathbf{Art}_k$, you get

$$F_x : \mathbf{Art}_k \longrightarrow \mathbf{Set}, \quad A \mapsto F(A) \tag{3.21}$$

the functor of infinitesimal deformations of the point x. Let $\hat{\mathcal{O}}_{X,x}$ be the complete local ring of X at x. There is a canonical section $\xi \in F(\hat{\mathcal{O}}_{X,x})$. Then clearly the previous discussion in turn yields that the pair $(\hat{\mathcal{O}}_{X,x}, \xi)$ pro-represents the functor F_x. If F is an étale sheaf represented by an algebraic space, what was said previously when $F = \mathcal{M}$ is an algebraic stack yields obviously an identical result. If $U \to F$ is an étale presentation, you can choose a lift of ξ_0 to U, then $\hat{\mathcal{O}}_{U,\xi_0}$ pro-represents the functor F_x.

Remark 3.33. In many cases, the deformation functor is defined on the larger category $\widehat{\mathbf{Art}_k}$ of complete local noetherian k-algebras with residue field k. Let $(\mathcal{O}_{\text{ver}}, (\xi_n))$ be a versal deformation (Theorem 3.31). Then the collection of infinitesimal deformations (ξ_n) does not define in general an element of $\mathrm{Def}_{\xi_0}(\hat{\mathcal{O}}_{\text{ver}})$. When this is so, i.e., if there exists $\xi \in \mathrm{Def}_{\xi_0}(\hat{\mathcal{O}}_{\text{ver}})$ inducing the ξ_n's from the morphisms $\hat{\mathcal{O}}_{\text{ver}} \to \hat{\mathcal{O}}_{\text{ver}}/\mathcal{M}^n$, then we say that the versal deformation (ξ_n) is **effective**. This is clearly the case if one starts with the functor of infinitesimal deformations of a point x of a k-scheme X. Indeed the point $\xi : \operatorname{Spec} \hat{\mathcal{O}}_x \to \operatorname{Spec} \mathcal{O}_x \to X$ is the solution.

A further remark about a versal formal deformation is that such an object is purely formal, living essentially on a complete local notherian ring. The formal deformation is called **algebraizable** if there exists a k-algebra of finite type R, a maximal ideal $\mathcal{M}$ of R, such that $\hat{\mathcal{O}}_{\text{ver}} \cong \hat{R}_{\mathcal{M}}$, and ξ comes from $\mathrm{Def}_{\xi_0}(R_{\mathcal{M}})$. Algebraizability is a difficult problem (see [4]). Let us say a covariant functor $D : \mathbf{Ann} \to \mathbf{Set}$ is **locally of finite presentation** if for any inductive system of rings (A_i), one has

$$\varinjlim D(A_i) = D(\varinjlim A_i). \tag{3.22}$$

Then an important result of M. Artin ([4], Théorème 3.15) is the following.

Theorem 3.34. *Assume that the deformation functor* Def_{ξ_0} *is locally of finite presentation. If there exists an effective versal deformation, it is algebraizable.*

This result leads to a new criterion of representability of an étale sheaf by an algebraic space, or an existence theorem for algebraic stacks, the so-called. Artin's criterion (loc. cit.). Roughly, the key point is the fact that in Definition 1.57, deformation theory produces smooth (étale) neighborhoods of points, which in turn yields a smooth (étale) atlas.

Finally we can test the smoothness of a DM stack in terms of formal deformation theory of its geometric points, here for ease, we are working over a field $k = \overline{k}$. Recall an algebraic stack $\mathcal{M}$ is smooth (over k), if there is an atlas $U \to \mathcal{M}$ with U smooth over k. It is straightforward to see this definition is independent of the choice of U. The relationship between local properties of U, at some closed point and the deformation functor of the corresponding point of $\mathcal{M}$ yields:

Proposition 3.35. *The stack $\mathcal{M}$ is smooth if and only if the formal deformation space of its geometric points are smooth, i.e., the rings $\mathcal{O}_{\mathrm{ver}}$ are formal power series rings. This occurs precisely when the deformations functors of points are unobstructed.*

Since the (formal) spectrum of a versal deformation space yields a (formal) local chart around the chosen point, the smoothness of these deformations spaces yields the smoothness of any atlas. To understand what are the obstructions of smoothness of a versal deformation space, one need to study the obstructions to lift infinitesimally a deformations. See for all the relevant definitions and details ([63], Section 6), or [53].

An elementary example was previously discussed in Subsection 3.1.1. The weighted projective line $\mathbb{P}^1(p,q)$ has two stacky points, then a local chart around each of these points was obtained by taking a local-étale slice, showing in particular it is a smooth stack.

Exercise 3.36. *Prove that with the notations of Theorem* 3.31 *the set* $\mathrm{Def}_{\xi_0}(k[\epsilon])$ *has a natural structure of k-vector space. Use H_2 to identify* $\mathrm{Def}_{\xi_0}(k[\epsilon]\otimes_k k[\epsilon])$ *with* $\mathrm{Def}_{\xi_0}(k[\epsilon]\times \mathrm{Def}_{\xi_0}(k[\epsilon]$*, then compose with the "diagonal".*

3.1.4. Coarse moduli space. Since the first days of moduli problems (Riemann), the object of interest was the *coarse moduli space*. This is a hypothetical algebraic space (better a scheme) M such that the (geometric) points of M are in one-to-one correspondence with the isomorphism classes of objects under consideration. The foundational example is certainly the coarse moduli space of compact Riemann surfaces of genus $g \geq 1$ ($k = \mathbb{C}$), when $g = 1$, we must work with elliptic curves, i.e., genus 1 smooth curves together with a marked point. It appears often more convenient to work in an analytical framework and try to classify the surfaces enriched by a **Teichmüller structure** to kill the automorphisms ([48], Chap. 5). As a consequence the moduli space that we get, i.e., the Teichmüller space is not algebraic, indeed a complex ball of dimension $3g-3$ if $g \geq 2$, 1 if $g = 1$. To work completely in an algebraic framework we need to choose a finite level n, as in [15]. If $n \geq 3$, a level structure suffices to kill all the automorphisms, and then a fine moduli space (non compact) can be construct. The next fundamental problem is to compactify this moduli space in a way that preserves the modular origin.

Returning to a more general setting, let $\mathcal{X}$ be an algebraic stack, say defined over $(\mathbf{Sch}_k)_{et}$, and of DM type to simplify. It is tempting to consider not exactly the groupoid $\mathcal{M}(S)$, but the set of its connected components, i.e., $\mathcal{M}(S)/\sim$. The

difficulty is that

$$\overline{\mathcal{X}}(S) = \pi_0(\mathcal{X}(S) = \mathcal{X}(S)/\sim \tag{3.23}$$

defines only a presheaf of sets. If the étale associated sheaf is representable, i.e., of the form $h_M(-)$, then the scheme obtained is the expected **fine moduli space** M. In most interesting cases this sheaf may not be representable. A fundamental example is that of a quotient stack $[X/G]$, where G is a (reductive) smooth affine group scheme acting on a scheme X [48]. The goal of the GIT theory of Hilbert-Mumford is to define a quotient in an appropriate sense within the category of schemes, the so-called **GIT quotient**. If a GIT-quotient exists, it is classically denoted $X//G$, and there is a canonical (quotient) map $p : X \to X//G$ ([48], Theorem 1.10). Strong properties of the pair $(X//G, p)$ are expected, in part to ensure its uniqueness.

The most reasonable property is that it must be a **categorical quotient**. This means that $p : X \to X//G$ is equivariant, the group G acting trivially on $X//G$, and furthermore that any morphism $f : X \to Y$ which is equivariant with respect to the trivial action on a scheme Y factors uniquely through $X//G$, as pictured below:

$$\begin{tikzcd} X \arrow[r, "p"] \arrow[dr, "f"'] & X//G \arrow[d, dashed, "\overline{f}"] \\ & Y. \end{tikzcd}$$

If such a categorical quotient exists, it is unique up to a unique isomorphism.

Another expected strong property of a GIT quotient is that $X//G$ must parameterize the G-orbits of X, through the map $y \in X//G \mapsto p^{-1}(y)$. In turn the G-orbits of points of X must be closed, and p is affine. To get such a result we must work not on the whole X, but on a suitable G-invariant open subset, the so-called set of **stable** points X^s (see [48], Definition 1.7). To be able to find the open set X^s of stable points, we must choose a G-linearized invertible sheaf on X (see (1.47)). The stability condition is sensitive to this choice. Under the same assumptions, we can work on a larger open set X^{ss}, the set of **semi-stable** points ([48], Definition 1.7.b)). A key result of this theory asserts that a (categorical) quotient of this G-stable open subset makes sense $X^{ss} \to X^{ss}//G$, under a weakened form.

Notice that whenever a GIT quotient makes sense, there is a obvious morphism

$$\phi : [X/G] \longrightarrow X//G. \tag{3.24}$$

Indeed if $(P \to S, f)$ is a section of $[X/G]$ over S, then $pf : P \to X//G$ is G-equivariant therefore factors uniquely through $S = P/G$.

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d] & X \arrow[d, "p"] \\ S \arrow[r] & X//G. \end{tikzcd}$$

Another key feature of the GIT quotient $X//G$, is the fact that the sheaf of regular functions $\mathcal{O}_{X//G}$ is precisely the sheaf of functions on the quotient stack $[X.G]$, i.e., the sheaf of G-invariant functions on X

$$0 \to \mathcal{O}_{X//G} \longrightarrow p_*(\mathcal{O}_X) \overset{d_0^*-d_1^*}{\longrightarrow} (pd_i)_*(\mathcal{O}_{G\times X}).$$

The right definition that follows is due to Mumford ([48], Definition 5.6).

Definition 3.37. A **coarse moduli space** for the algebraic stack $\mathcal{X}$ is a pair (X, π) where X is a scheme, and $\pi : \mathcal{X} \to X$ a 1-morphism. where X is an algebraic space defined over k, such that

i) π is universal for morphisms to algebraic spaces,
ii) For $K = \overline{K}$, the obvious map $\mathcal{X}(K)/(\text{isom}) \to X(K)$ is bijective.

Property i) ensures that if (X, π) exists, it is unique up to a unique isomorphism. The existence, in a sufficiently general setting is a theorem ([38], Theorem 1.1, Cor. 1.3). In loc. cit. geometric quotients are studied in a framework of groupoid schemes and algebraic spaces.

Theorem 3.38. *Assume that the diagonal $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is finite*[30]*, equivalently for all pairs $(x_1, x_2) \in \mathcal{X}(S)^2$, the space $\mathrm{Isom}_S(x_1, x_2) \longrightarrow S$ is finite (in particular it is a scheme). Then there exists a coarse moduli space $\pi : \mathcal{X} \to X$. Additionally:*

i) *X is separated and locally of finite type,*
ii) *π is proper (to be defined later)*
iii) *the formation of $\pi : \mathcal{X} \to X$ commutes with any flat base change*[31] *$X' \to X$.*

For a proof of the theorem we refer to [38]. Roughly, the proof goes as follows. After many reduction steps the proof can be reduced to this ultimate case. Let $\mathcal{X}$ be an algebraic stack and assume there is a finite locally free surjective map $U \to \mathcal{X}$ with U a scheme. Let $R = U \times_{\mathcal{X}} U \overset{d_1}{\underset{d_2}{\rightrightarrows}} U$ be the projections. If the "orbits" $p_1(p_2^{-1}(u))$ are contained in an affine open in U, then Grothendieck's quotient construction (Theorem 1.58) works and in turn yields a scheme X with a morphism $\mathcal{X} \to X$. This morphism satisfies the universal property i) above, but restricted to schemes. □

A foundational example that will be studied in Section 4.1.2, see also below, is that of the moduli space of pointed curves of genus 1. As we shall see the stack $\mathcal{M}_{1,1}$ is definitively different from its coarse moduli space $M_{1,1}$. The finiteness assumption in Theorem 3.38 is essential and cannot be weakened. In [56] one can find an example of an Artin stack with quasi-finite diagonal that does not admit a coarse moduli space.

Our aim in the rest of this section is to study various examples, especially the case of a quotient stack $[X/G]$.

[30] The stack is not necessarily Deligne-Mumford. Note that this is the case if $\mathcal{X}$ is a separated DM stack (see the definition below).
[31] But not with an arbitrary base change in general.

Example 3.39. *The case of a finite group.* Let G be a finite constant group which acts on a quasi-projective scheme X of finite type over a field k, with action $\mu : G \times X \to X$. The quasi-projectivity assumption asserts that we may cover X by finitely many invariant affine open subsets[32]. Classical invariant theory tells us how to construct in this special example a GIT quotient $X//G$. When $X = \operatorname{Spec} R$, where R is a finitely generated algebra over k, it is well known that the subalgebra of G-invariants R^G is finitely generated, then we take $X//G = \operatorname{Spec} R^G$, and $\pi : \operatorname{Spec} R \to \operatorname{Spec} R^G$. It is a classical result that the fibers of π are the G-orbits [20]. An easy reduction reduces the general case to the affine case. For a finite group we can check without difficulty:

Proposition 3.40. *The geometric quotient $X//G$ is the coarse moduli space of the quotient stack $[X/G]$.*

Proof. Proposition 3.9 says that the canonical 1-morphism $[X/G] \to X//G$ (3.8) is a "weak" categorical quotient. At the level of geometric points, the construction of $X//G$ shows that the fibers of π are exactly the G-orbits in X. The claim follows. If the action is free, then $\pi : X \to X//G$ is a G-torsor which in turn shows that we have $[X/G] \cong X//G$. □

This elementary result can be greatly extended, see [17], Chap. III, § 2, no. 3, [58], Exposé V, Thm. 4.1, or [37], Appendix to Chap. 7. In this generalization the group G becomes a finite flat groupoid acting on a scheme $X \in \mathbf{Sch}/S$, i.e., d_i is finite locally free. It is assumed as above that each orbit $d_1(d_0^{-1}(x))$ is contained in an affine open subset. Then a quotient scheme $X//R$ can be defined, essentially with the same properties as above. The proof consists in treating first the affine case, then use as before the unicity property to glue together these local quotients.

An important corollary of this construction in when R (or G) acts **freely**. We say that R acts freely if $(d_0, d_1) : R \to X \times_S X$ is a monomorphism, i.e., if $R(T)$ acts freely on $X(T)$ for any $T \to S$, see the discussion in [17], Chap. III, § 2, no. 2.

Proposition 3.41. *Let R act freely and admissibly on X. Then $\pi : X \to X//R$ is finite locally free, indeed an R-torsor, i.e., $(d_0, d_1) : R \longrightarrow X \times_S X$ is an isomorphism. Furthermore the formation of $X//R$ commutes with an arbitrary base change.* □

See [37], Appendix to Chap. 7 for a clear treatment of the formation of quotients and their commutation with base change for finite group actions. Note that it is obviously true that if $X \to S$ is a G-bundle, then $S = X//G$ (Proposition 2.12).

Assuming as always that G is an affine smooth algebraic group acting on the scheme X, we know from a general principle that the 1-morphism $X \to [X/G]$ is in some sense a categorical quotient (Proposition 3.9). I think it is instructive to give a slightly different proof, focusing on the group action. The main point is:

[32] The construction works more generally if G acts admissibly on X, i.e., if one can cover X by affine G-stable open subsets.

Lemma 3.42. *Let $\phi : [X/G] \to M$ be a 1-morphism with M a scheme. Then the composite morphism $X \to [X/G] \to M$ is constant along the G-orbits.*

Proof. To check this consider the trivial G-principal bundle $p : X \times G \to X$. The map $f : X \times G \longrightarrow X$ given at the level of points by $(x, s) \mapsto s^{-1}.x$ is clearly G-equivariant, this in turn yields a section (i.e., atlas) $X \to [X/G]$. For any $\lambda \in G$ we have the commutative diagram, where we set $h(x, s) = (\lambda x \lambda s)$

$$\begin{tikzcd}
 & X & \\
X \times G \arrow[ur, "f"] \arrow[rr, "h"] \arrow[d] & & X \times G \arrow[ul, "f"'] \arrow[d] \\
X \arrow[rr, "\lambda"] & & X.
\end{tikzcd}$$

This diagram provides us with an endomorphism of $[X/G]$ above λ. Therefore if $\phi : [X/G] \to M$ is a 1-morphism, the composite $\varphi : X \to M$ verifies $\varphi(\lambda x) = \varphi(x)$, therefore is constant along the G-orbits. □

The same result as in Proposition 3.40 holds true for a GIT-quotient $X^s \to X^s//G$ [48]. A stack of the form $[X^s/G]$ will be sometimes called a **GIT stack**. The morphism

$$[X^s/G] \longrightarrow X^s//G$$

is a coarse moduli space.

Exercise 3.43. *Check the last assertion of Proposition* 3.41.

Example 3.44. Let $\mathbf{G}_m$ act as usual on the affine line $\mathbb{A}^1$ over a field k. Then the morphism $[\mathbb{A}^1/\mathbf{G}_m] \longrightarrow \operatorname{Spec} k$ satisfies assertion i) of Definition 3.37, but not ii). Indeed, assertion i) is known. To check ii) notice $\mathbf{G}_m \subset \mathbb{A}^1$ is a dense orbit, from which we infer that with the previous notation, φ maps $\mathbb{A}^1$ onto a point. Let $y : S \to [\mathbb{A}^1/\mathbf{G}_m]$ be an arbitrary section, then we know that after a smooth cover $S' \to S$ the composite morphism $S' \to S \to [\mathbb{A}^1/\mathbf{G}_m] \to X$ factors through $\mathbb{A}^1$, therefore it maps to a point. But $S' \to S$ is an epimorphism (Theorem 1.42), this implies that $\phi \circ y$ maps to a point. Our claim follows.

More interesting is the following positive result:

Proposition 3.45. *The coarse moduli space of the stacky projective line $\mathbb{P}^1(p, q)$ is the ordinary projective line.*

Proof. First we record that there is a natural morphism $\phi : \mathbb{P}^1(p, q) \longrightarrow \mathbb{P}^1$ (see (3.8)). The second step is to check that this morphism is universal, i.e., that any

morphism $f : \mathbb{P}^1(p,q) \longrightarrow X$ where $X \in \mathbf{Sch}$ factors uniquely through $\mathbb{P}^1$.

$$\begin{tikzcd} \mathbb{P}^1(p,q) \arrow[r, "\phi"] \arrow[dr, dashed, "\tilde{f}"'] & \mathbb{P}^1 \arrow[d, "f"] \\ & X. \end{tikzcd}$$

This is a very special case of the GIT theory, which asserts that $\mathbb{P}^1 = \mathbb{A}^2 - \{(0,0)\}//\,\mathbf{G}_m$ is defined as categorical and even geometric quotient [48]. Here we just suggest a direct proof exploiting the concrete description of $\mathbb{P}^1(p,q)$. We can cover $\mathbb{A}^2 - \{(0,0)\}$ by a collection of invariant affine open subsets U, such that $f(U)$ is include in an affine open subset $V = \operatorname{Spec} A$ of X. We can take $U = \operatorname{Spec} k[x,y,1/P]$ where $P = P(x,y)$ is some (quasi)-homogeneous polynomial function of degree say N. There is no loss of generality in assuming that either x or y divides P. If x/P, then we leave as an exercise to check that the $\mathbf{G}_m$-invariant subring is

$$k[x,y,1/P]^{\mathbf{G}_m} = k[t,1/\tilde{P}]$$

for a suitable polynomial $\tilde{P}(t)$ deriving from P. It is then easy to check that the restriction of f to U factors uniquely through the open subset $\operatorname{Spec} k[t,\frac{1}{\tilde{P}}]$ of $\mathbb{P}^1$. The conclusion follows. To end the proof that $\mathbb{P}^1$ is a coarse moduli space, we have to check that over $K = \overline{K}$, the K-points of $\mathbb{P}^1$ are in one-to-one correspondence with the isomorphism classes of sections over K of the stack $\mathbb{P}^1(p,q)$ equivalently with the $\mathbf{G}_m$-orbits in $\mathbb{A}^2 - \{(0,0)\}$. This is clear. $\square$

If $\mathcal{X}$ is not representable, assuming a coarse moduli space X exists, i.e., the hypothesis of Theorem 3.38 are fulfilled, one may try to understand the relationship between $\mathcal{X}$ and X, through the canonical morphism $\phi : \mathcal{X} \to X$. What is clear is that an S-point of X in general does not correspond to a section of $\mathcal{X}$ over X, even if $S = \operatorname{Spec} K$, unless $K = \overline{K}$. This leads us to a natural and subtle problem about **fields of definition** for points (sections) of $\mathcal{X}$.

To begin with assume that X is a scheme of finite type over a perfect field k. Let $\overline{k}$ an algebraic closure, and $\Gamma = \Gamma_{\overline{k}/k}$ the Galois group. There is a natural action of Γ on $X \otimes_k \overline{k}$ such that in some sense $X \otimes_k \overline{k}//\Gamma = X$. This is a well-known fact ([46], Chap. II, § 4). More precisely, given a geometric point $\overline{x} \in X(\overline{k})$, its Γ-orbit is finite. Let $\Gamma_{\overline{x}}$ be the stabilizer, and $x \in X$ the image (center) of $\overline{x}$, then

$$\overline{k}^{\Gamma_{\overline{x}}} = k(x).$$

Thus $k(x)$ is both a field of definition and a field of moduli for x. Suppose now that $\mathcal{X}$ is a DM stack, say of finite type over k. Let $x \in X$ be a K-point of X where $k \subset K \subset \overline{k}$ is an arbitrary subfield. The definition of a coarse moduli space

(Definition 3.37) does not guarantee that we can lift x to a section $\xi \in \mathcal{X}(K)$. Such a lift is possible only over $\overline{k}$, say $\overline{\xi}$. In view of the preceding remark when $\mathcal{X} = X$, we can try the following definition. First notice that the Galois group Γ "acts" on the groupoid $\mathcal{X}(\overline{k})$ according to the rule

$$\sigma.\eta = \sigma^*(\eta).$$

We write abusively σ the morphism $\operatorname{Spec}\overline{k} \xrightarrow{\sim} \operatorname{Spec}\overline{k}$ associated to the field isomorphism σ^{-1}. Then we set

$$\Gamma_{\overline{\xi}} = \{\sigma \in \Gamma\,,\ \sigma.\overline{\xi} \cong \overline{\xi}\}. \tag{3.25}$$

Definition 3.46. The **field of moduli** of $\overline{\xi}$ is the fixed field $\overline{k}^{\Gamma_{\overline{\xi}}}$.

Obviously $\Gamma_{\overline{\xi}} = \Gamma_{\overline{x}=\phi(\overline{\xi})}$, therefore the field of moduli of $\overline{\xi}$ is the residual field $k(x)$ of the image $x \in X$ of the geometric point $\overline{x}$.

The term "field of moduli" comes up in different scenarios [14]. What was observed is that in some cases the field of moduli is not a field of definition. This sugests that the right question is to understand how the possible lifts of $x : \operatorname{Spec} K \to X$ over arbitrary fields extensions L/K (finite separable) organize. Otherwise said, we must focus on the stack given by a 2-fiber product

$$\begin{tikzcd} \mathcal{X} \arrow[r, "\phi"] & X \\ \mathcal{G}_x \arrow[u] \arrow[r] & \operatorname{Spec} K. \arrow[u, "x"'] \end{tikzcd}$$

We are here over the small étale site of $\operatorname{Spec} K$. If L/K is a finite separable algebra, then $\mathcal{G}_\xi(L)$ stands for the full subgroupoid of $\mathcal{X}(L)$ whose sections are the lifts ξ of $x_{|L}$, i.e., $\phi(\xi) = \operatorname{Spec} L \to \operatorname{Spec} K \to X$. Here is a naive answer, compare with [14]:

Proposition 3.47. *Assume that $\mathcal{X}$ is a separated DM stack of finite type over* $\mathbf{Sch}_k$. *Assuming K is perfect, then $\mathcal{G}_x$ is a gerbe over the small étale site of* $\operatorname{Spec} K$ *($K = k(x)$), called the residual gerbe at x. In most cases of interest, $K = k$, for example $k = \mathbb{Q}$.*

Proof. We must check first that x has a lift over a suitable finite separable extension L/K. Let $U \to \mathcal{X}$ be an (étale) atlas. Clearly we can lift x to a point $\operatorname{Spec}\overline{K} \to U$, and then using the fact that U is a k-scheme of finite type, descent $\overline{x}$ over a finite extension, then separable, L/K, thereby proving our first claim. Let us now assume we are given two lifts ξ_i $(i = 1, 2)$ in $\mathcal{X}(L)$, the extension L/K being finite separable. The ii) part of Definition 3.37 says that after pullback to $\overline{K}$, we have $\xi_1 \cong \xi_2$, i.e., the finite scheme $\mathbf{Isom}_L(\xi_1, \xi_2)$ is non empty. But this scheme is finite unramified over L. Thus if we pick any point α, the residual field of this point is finite and separable over L. Therefore we can find an isomorphism defined over a larger finite separable extension of L/K. This says exactly that $\mathcal{G}_x$ is a gerbe over K (Subsection 1.3.4). $\square$

The problem to decide the triviality or not of the gerbe $\mathcal{G}_x$ is of interest. Triviality means we can find a section of $\mathcal{C}$ over $K = k(x)$ lying over x (see Subsection 1.3.4). A **field of definition** L, i.e., such that x lifts to L contains K, but in general we cannot take $L = K$. This problem, cohomological in nature, is discussed with examples mainly for the stacks of curves or covers in [14]. An interesting example comes from the stack of hyperelliptic curves of a given genus.

More generally, let us start with an algebraic (étale) gerbe $\pi : \mathcal{G} \to X$, i.e., an algebraic stack $\mathcal{G}$ that is a gerbe over X. Under mild finiteness restriction, one has:

Proposition 3.48. *X is a coarse moduli space for $\mathcal{G}$.*

Proof. Let $\psi : \mathcal{G} \to Y$ be a morphism for some scheme Y. We want to prove that ψ factors through X. Notice if $x \cong y$ in $\mathcal{G}(S)$ then $\psi(x) = \psi(y) \in \mathrm{Hom}(S, Y)$. Furthermore this S-point of Y depends only of x, not on the isomorphism. From this remark we see that if now $x, y \in \mathcal{G}(S)$ are only locally isomorphic, then $\psi(x) = \psi(y)$; use for this the fact that $\mathrm{Hom}(-, Y)$ is a sheaf. In turn this show that if $x_i \in \mathcal{G}(U_i)$ are local sections of p, for some cover $(U_i \to X)$, then the $\psi(x_i)$ glue together into a morphism $\overline{\psi} : X \to Y$. Clearly $\psi = \overline{\psi}p$.

Let $K = \overline{K}$ be an algebraically closed field. Since we are working with the étale topology, an étale cover of $\mathrm{Spec}\, K$ has a section, so any $\xi \in X(K)$ lifts to $\mathcal{G}$. Now if $x, y \in \mathcal{G}(K)$ are such that $p(x) = p(y)$, then x and y are locally isomorphic, but as before, they are indeed isomorphic. This shows that X is the coarse moduli space of $\mathcal{G}$, as expected. □

We cannot expect in general that the canonical morphism $\mathcal{X} \to X$ from a separated DM stack to its coarse moduli space be an étale gerbe. This is due to the fact that the stabilizers of points can jump. Generically this is true. A more precise result is (Olsson [51], Proposition 2.1):

Proposition 3.49. *Let $\mathcal{X}$ be a normal*[33] *Deligne-Mumford stack separated and of finite type over noetherian base scheme S. Let X denote the coarse moduli space. Then the morphism $\phi : \mathcal{X} \to X$ factors canonically as*

$$\mathcal{X} \xrightarrow{\psi} \overline{\mathcal{X}} \xrightarrow{\varphi} X$$

where $\overline{\mathcal{X}}$ is a separated DM stack of finite type over S, X is the common coarse moduli space of $\mathcal{X}$ and $\overline{\mathcal{X}}$, φ is an isomorphism over a dense open set of X, and $\psi : \mathcal{X} \to \overline{\mathcal{X}}$ is an étale gerbe.

Exercise 3.50. *Let $\mathbb{P}^1(p, q)$ be the weighted projective line (Subsection 3.1.1) over a field k with $p, q \geq 1$, and with coarse moduli space $\mathbb{P}^1$. Check that the canonical morphism $\mathbb{P}^1(p, q) \longrightarrow \mathbb{P}^1$ derives from $(x : y) \mapsto t = \frac{x^q}{y^p}$. Identify the gerbe of liftings of ∞ with $\mathbf{B}\mu_p$.*

[33] See Section 3.2.

3.2. Geometry on stacks

In this section, we would like to give an outline of the geometric dictionary:

Schemes $\leftrightarrow$ **Algebraic stacks**.

Geometry means that we are able to define open or closed substacks, vector bundles, differentials forms etc. Throughout this section by an algebraic stack we shall mean a DM stack, this restriction suffices for our purposes; see [62], [44], for a wider approach. Recall that we are working essentially with the étale topology, therefore the adjective local will mean local in the étale sense.

3.2.1. Substacks. In this section we review the basic geometric concepts within the 2-category of algebraic stacks. In most cases we limit ourselves to DM stacks. In the next section about the moduli stack of covers these definitions will become more natural.

To begin with, we first notice that an algebraic stack has an associated topological space of points. Let $\mathcal{X}$ be an (algebraic) stack, and $x \in \mathcal{X}(K)$, $x' \in \mathcal{X}(K')$, for two fields K, K'. Write $x \sim x'$ if there is a field L and two embeddings $K \hookrightarrow L$, $K' \hookrightarrow L$ such that $x_{|L} \cong x'_{|L}$. The relation $\sim$ is an equivalence relation. Indeed if $x \sim x'$, $x' \sim x''$, with obvious notations, we have two fields L, L' as above, with in particular $K' \hookrightarrow L$, $K' \hookrightarrow L'$. We can form the compositum $L'' = L.L'$, i.e., a quotient field of $L \otimes_{K'} L'$, and notice that over L'', we have $x \sim x' \sim x''$. This justifies the following:

Definition 3.51. A **point** x of the stack $\mathcal{X}$ is an equivalence class of sections $x \in \mathcal{X}(K)$, where K is a field. We denote by $|\mathcal{X}|$ the set of points.

Notice that a 1-morphism of stacks $F : \mathcal{X} \to \mathcal{Y}$ defines a map $|F| : |\mathcal{X}| \to |\mathcal{Y}|$.

Definition 3.52. Assume given a stack $\mathcal{X}$. A substack $\mathcal{Y} \subset \mathcal{X}$ is an **open substack** (resp. **closed substack**) if the inclusion is representable by an open immersion (resp. by a closed immersion).

Notice that an open or closed substack is also an algebraic stack. This follows from the easy remark, which says that if $F : \mathcal{X} \to \mathcal{Y}$ is representable, then $\mathcal{Y}$ representable implies that $\mathcal{X}$ is representable. It is important to translate these definitions into the setting of groupoids. Let $\mathcal{X} = [U/R]$. If $\mathcal{Y}$ is an open substack of $\mathcal{X}$, we can draw a diagram

$$\begin{tikzcd} R \arrow[r, shift left] \arrow[r, shift right] & U \arrow[r, "P"] & \mathcal{X} \\ ? \arrow[r, shift left] \arrow[r, shift right] & V \arrow[u, "\imath"] \arrow[r, "Q"] & \mathcal{Y} \arrow[u] \end{tikzcd}$$

with $V = U \times_{\mathcal{X}} \mathcal{Y}$. Then $\imath$ is an open immersion, and V is invariant under the R-action. Let $R_V =?$ be the induced groupoid, which acts on V. Then it is not difficult to check that:

Lemma 3.53. *We have* $\mathcal{Y} = [V/R_V]$.

The case of closed substacks is analogous. In turn this yields another way to think about substacks:

Proposition 3.54. *Let $\mathcal{X} = [U/R]$ be a quotient stack. Then the open (resp. closed) substacks of $\mathcal{X}$ are the quotient $[V/R_V]$, for $V \subset U$ an open (resp. closed) R-invariant open (resp. closed) subscheme.*

Finally we define the Zariski topology on the set $|\mathcal{X}|$ by defining its open subsets which are the subsets $|\mathcal{U}|$ where $\mathcal{U}$ runs over all open substacks. There is a link between $|X|$ and the atlas U. If $x \in U$, then define $p(x) \in |\mathcal{X}|$ as the point represented by $\operatorname{Spec} k(x) \to U \to \mathcal{X}$.

Proposition 3.55. *The map $p : U \to |\mathcal{X}|$ is open surjective, and identifies the topological space $|\mathcal{X}|$ with the quotient space of U by the equivalence relation induced by R.*

Proof. Since $p : U \to \mathcal{X}$ is an epimorphism, any $x \in \mathcal{X}(K)$ lifts to $\operatorname{Spec} K' \to U$ for a suitable extension K'/K (finite separable). Thus we get the onto property. Now if $\mathcal{U} \subset \mathcal{X}$ is an open substack, then as before $V = U \times_{\mathcal{X}} \mathcal{U}$ is open in U and R-stable. Further we have set-theoretically $V = p^{-1}(|\mathcal{U}|)$. The conclusion follows. □

Exercise 3.56. *Prove that the disjoint sum $\mathcal{X} = \mathcal{X}_1 \coprod \mathcal{X}_2$ of two algebraic stacks $\mathcal{X}_i = [U_i/R_i], i = 1, 2$ is given by $\mathcal{X} = [U_1 \coprod U_2/R_1 \coprod R_2]$, compare with exercise* 1.68.

Definition 3.57.

i) An algebraic stack $\mathcal{X}$ is **connected** if it is not the disjoint union $\mathcal{X}_1 \coprod \mathcal{X}_2$ of two non-empty open substacks. It is **irreducible** if for all pairs of non-trivial open substacks $(\mathcal{U}_1, \mathcal{U}_2)$, the intersection $\mathcal{U}_1 \cap \mathcal{U}_2 := \mathcal{U}_1 \times_{\mathcal{X}} \mathcal{U}_2$ is non-empty.
ii) $\mathcal{X}$ is **reduced** (resp. **quasi-compact**, **smooth** over a ground field k, **normal**) if one can find[34] an atlas $U \to \mathcal{X}$, such that U is reduced (resp. quasi-compact, smooth, normal).
iii) Let $\mathcal{X}$ be of DM type. Then we say that $\mathcal{X}$ is **purely of dimension** n if and only if there exists an atlas (then for all atlases) which is purely of dimension n.

We are assuming that schemes are locally noetherian, therefore a quasi-compact algebraic stack is noetherian. In the same vein $\mathcal{X}$ is integral if reduced and irreducible.

Proposition 3.58. *Let $\mathcal{U}$ be an open substack of $\mathcal{X}$, then there is a unique reduced closed substack $\mathcal{Y}$ such that $|\mathcal{Y}| = |\mathcal{X}| - |\mathcal{U}|$. If $\mathcal{X} = \emptyset$, then $\mathcal{Y} = (\mathcal{X})_{\mathrm{red}}$.*

[34] If $\mathcal{P}$ is a property of schemes which is local for the smooth topology (étale topology in the DM-case), i.e., if $V \to U$ is smooth surjective, V has $\mathcal{P}$ if and only if U has $\mathcal{P}$. If this is the case, if $\mathcal{P}$ is true for one atlas then it is true for all atlases. This amounts to checking for example that if $U \to V$ is étale surjective, then V is reduced, normal, smooth over k, regular, if and only if U has the same property. This rests on permanence properties under an étale morphism, as listed in the section "Permanence properties" in the chapter "Étale Morphisms of Schemes" of [62]. Notice that quasi-compacity, or irreducibility, is not a local property in this sense.

Proof. Let us take an atlas $p : U \to \mathcal{X}$. Then $V = p^{-1}(\mathcal{U})$ is as seen above open and R-invariant. Let Z be the unique reduced subscheme with support $U - V$. It is R-stable, in turn we have a groupoid $R_Z = d_0^{-1}(Z) = d_1^{-1}(Z) \rightrightarrows Z$. The quotient stack $\mathcal{Y} = [Z/R_Z]$ is the answer. □

This generalizes the case where the groupoid comes from a group action. With similar techniques we can readily show that a noetherian, i.e., quasi-compact stack is covered by its irreducible components (see [44]).

Proposition 3.59. *Let $\mathcal{X}$ be a noetherian stack. Then there exists a unique finite collection of irreducible reduced closed substacks $\mathcal{Z}_i$, such that*

i) *$\mathcal{Z}_i$ is maximal among reduced closed substacks,*
ii) $|\mathcal{X}| = \cup_i |\mathcal{Z}_i|$.

Proof. The proof mimics the usual proof for noetherian schemes, as in [33]. □

The following comment comes from [29]. Assume that the stack $\mathcal{X}$ is DM, so that $\mathcal{X} = [U/R]$ with $R \rightrightarrows U$ an étale groupoid. Let $Y \subset U$ be a reduced closed subscheme. We can saturate it, i.e., take

$$R.Y = \overline{d_1(d_0^{-1}(Y))}. \tag{3.26}$$

Equivalently if y is the generic point of Y, and if $d_0^{-1}(y) = \{z_1, \ldots, z_m\}$, then $R.Y = \cup_j \overline{d_1(z_j)}$. The points $d_1(z_j)$ are the end points of the arrows starting at z_j. The closed subset $R.Y$ is R-stable, and $\dim Y = \dim R.Y$. This subscheme yields a closed substack $p(Y) \subset \mathcal{X}$. If $Z \not\subseteq Y$ is closed and irreducible, then $R.Z \neq R.Y$. In turn this shows that $p(Z) \neq p(Y)$. It is easy to check that the combinatorial dimension of topological space $|\mathcal{X}|$ is the same as the dimension of U.

Definition 3.60. A reduced DM stack $\mathcal{X}$ over $\mathbf{Sch}_k$ is **punctual** if $|\mathcal{X}|$ is a point.

This implies that at least one atlas (and then all atlases is a disjoint union of finitely many reduced closed points. For example if G is a finite constant group, then **BG** is punctual.

From a combinatorial viewpoint the number of points of a punctual stack is $|\operatorname{Aut}(\xi)|^{-1}$ where ξ is a geometric point of $\mathcal{X}$ (i.e., $\xi \in \mathcal{X}(K)$ with $K = \overline{K}$). Since $\mathcal{X}$ is a DM stack, the group scheme $G_\xi = \operatorname{Aut}(\xi)$ is a finite unramified over K, so a finite constant group. The number of elements of this group does not depend of the choice of ξ. We can count the number of points of a "finite" stack, i.e., a DM stack whose irreducible (connected) components are punctual.

Definition 3.61. The **number of points** of a reduced stack $\mathcal{X}$ with $|\mathcal{X}|$ finite, denoted $\#\mathcal{X}$, is defined by the *mass formula*

$$\#\mathcal{X} = \sum_{z \in |\mathcal{X}|} \frac{1}{|\operatorname{Aut}(z)|}. \tag{3.27}$$

We warn the reader that the number of points of a (finite) DM stack is a rational number. For example if G is a finite constant group, and if we work over a ground field k, then $\#\,\mathbf{BG} = |G|^{-1}$. More generally if X is a finite set acted on by the finite group G, then viewing X as the set of points of a finite reduced scheme, the Burnside formula yields

$$\#[X/G] = \sum_{\xi \in X/G} \frac{1}{|G_\xi|} = \frac{|X|}{|G|}.$$

With this definition, we will be able to count the number of (ramified) covers $\pi : C \to D$, with fixed degree d, of a fixed smooth projective algebraic curve D, by a smooth curve of fixed genus, and with preassigned branch points, a so-called **Hurwitz number**.

Exercise 3.62. *Let $\mathcal{X}$ be a smooth quasi-compact algebraic stack. Prove that the irreducible components are identical to the connected components.*

Exercise 3.63. *Let $\mathcal{Y}_1, \dots, \mathcal{Y}_r$ be a finite collection of closed substacks of the algebraic stack $\mathcal{X}$. Prove that there exists a smallest closed substack $\mathcal{Y}$ containing the $\mathcal{Y}_i$'s. Check that $|\mathcal{Y}| = |\mathcal{Y}_1| \cup \cdots \cup |\mathcal{Y}_r|$.*

Exercise 3.64. *Suppose that $\gcd(p,q) = 1$. Show that the complementary open substack $\mathbb{P}^1(p,q)$ minus the two stacky points is a scheme.*

3.2.2. Morphisms. Throughout, stacks are algebraic stacks. A notable difference between schemes and stacks is the fact that stacks are objects of a 2-category. A 1-morphism is either representable (schematic if your interest is about DM stacks), or not.

We are interested in 1-morphisms and their expected properties. Let $\mathcal{P}$ be a property of morphisms in the category **Sch** which is local for the smooth (resp. étale) topology on the target, and stable by base change (finite, finite type, proper, smooth, flat, ...).

Recall that a representable 1-morphism $F : \mathcal{X} \to \mathcal{Y}$ has property $\mathcal{P}$ if for any scheme S the induced morphism $F_S : \mathcal{X} \times_{\mathcal{Y}} S \longrightarrow S$ has $\mathcal{P}$. This makes sense since the fiber product is an algebraic space (a scheme in DM case). Sometimes it suffices to check this for simply $S = V$ an atlas of $\mathcal{Y}$. If now $\mathcal{P}$ is local on the domain and the target, then an arbitrary F has $\mathcal{P}$ if for any diagram

$$\begin{tikzcd} \mathcal{X} \arrow[r, "F"] & \mathcal{Y} \\ U \arrow[u, "P"] \arrow[r, "f"] & V \arrow[u, "Q"'] \end{tikzcd} \tag{3.28}$$

where P, Q are atlases for $\mathcal{X}, \mathcal{Y}$, then f has the property $\mathcal{P}$.

One can define a separated morphism (then a separated stack) by a simple transcription of the usual definition for schemes.

Definition 3.65. A 1-morphism of algebraic stacks $F : \mathcal{X} \to \mathcal{Y}$ is called **separated** if the diagonal $\Delta_{\mathcal{X}/\mathcal{Y}} : \mathcal{X} \longrightarrow \mathcal{X} \times_{\mathcal{Y}} \mathcal{X}$ is universally closed[35]. Equivalently, for all morphisms $S \to \mathcal{Y}$, with S a scheme, the algebraic space (or scheme in DM case) $\mathcal{X} \times_{\mathcal{Y}} S$ is separated.

We know that the relative diagonal is representable (Exercise 3.7). Under the same assumptions, a DM stack of finite type is separated if an only if its diagonal if finite (then finite unramified). We know this implies the existence of a coarse moduli space (Theorem 3.38). More subtle is the definition of a proper morphism. These definitions both rely on a valuative criterion, but with a difference. The semi-stable reduction theorem of curves [54] is already a manifestation of this difference.

Definition 3.66. Let $F : \mathcal{X} \to \mathcal{Y}$ be a morphism of stacks.

i) It is **separated** if it is of finite type, and if for any discrete valuation ring R with fraction field K, for any 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} R \arrow[r, "x_1"] \arrow[d, "x_2"'] & \mathcal{X} \arrow[d, "F"] \\ \mathcal{X} \arrow[r, "F"] & \mathcal{Y} \end{tikzcd} \tag{3.29}$$

including an isomorphism $\theta : F(x_1) \cong F(x_2)$, any isomorphism $\alpha : x_{1|K} \cong x_{2|K}$ inducing $\theta_{|K}$ extends to an isomorphism $\beta : x_1 \xrightarrow{\sim} x_2$ with $F(\beta) = \theta$.

ii) The morphism F is **proper** if it is separated, of finite type, and if for any 2-commutative diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r, "v"] \arrow[d, "\imath"'] & \mathcal{X} \arrow[d, "F"] \\ \operatorname{Spec} R \arrow[r, "u"] & \mathcal{Y} \end{tikzcd} \tag{3.30}$$

with R a discrete valuation ring, K its field of fractions, there is a finite extension K'/K such that if R' is the normalization of R in R', then we can find a morphism $w : \operatorname{Spec} R' \to \mathcal{X}$ that makes the previous diagram 2-commutative.

For our most important example, the stack $\overline{\mathcal{M}}_g$, $(g \geq 2)$, both properties of separatedness and properness are a translation of the main results of lectures by M. Romagny on models of curves [54].

Useful, for example to build an intersection theory, is the result below given without proof ([44], Theorem 16.6):

Proposition 3.67. *Let $\mathcal{X}$ be a noetherian (i.e., quasi-compact) DM stack. Then there exists a scheme X together a finite surjective morphism $X \to \mathcal{X}$. Then a morphism $F : \mathcal{X} \to \mathcal{Y}$ is proper if and only if the (representable) composition $X \to \mathcal{X} \to \mathcal{Y}$ is proper.* □

[35] The diagonal $\Delta_{\mathcal{X}/\mathcal{Y})}$ is separated of finite type, so universally closed means proper [33].

Under more specific assumptions on $\mathcal{X}$, more information is available on such a cover X (see [43], Theorem 1). For example, using level structures, Mumford proved that the moduli stack of stable curves (to be discussed later) has a finite flat cover by a scheme.

Exercise 3.68. *Assume that P is a $\mathbf{G}_m$-principal bundle over $S \in$ **Sch**. Prove that $\mathbf{Isom}_S(P,P) = \mathbf{G}_m \times S$. In particular $\mathbf{B}\,\mathbf{G}_m$ is not separated.*

3.2.3. Sheaves. To pursue the dictionary between schemes and stacks we must explain what we mean by a quasi-coherent sheaf (resp. coherent, locally free) on an algebraic stack[36]. If X is a scheme, a quasi coherent sheaf $\mathcal{F}$ on X is implicitly understood as a sheaf of modules on X_{zar}, the Zariski site, viz.

$$(f : S \to X) \mapsto \Gamma(S, f^*(\mathcal{F})).$$

It is sufficient to know $\mathcal{F}(S)$ on the affine opens.

Descent theory (Subsection 1.2.4, Theorem 1.44) tells us that a quasi-coherent sheaf $\mathcal{F}$ on a scheme X is a sheaf on the (small) étale site X_{et}, even fpqc, see Theorem 1.44, ([64], 4.2.2). Therefore on a scheme X the quasi-coherent sheaves are the same if we work either with the Zariski site, or the étale site. Let $U \to X$ an étale surjective morphism, essential is the fact that we have an equivalence of categories between the category $\mathbf{Qcoh}(X)$ on one hand, and on the other hand the category of quasi-coherent sheaves on U together with a descent datum relative to $U \to X$ (Theorem 1.44). Recall this means that if $R = U \times_X U$, with projections $d_0, d_1 : \; R \rightrightarrows U$, then we have an isomorphism

$$\phi : p_1^*(\mathcal{F}) \xrightarrow{\sim} p_2^*(\mathcal{F}) \tag{3.31}$$

with the usual cocycle condition (1.30)

$$p_{31}^*(\phi) = p_{32}^*(\phi) \circ p_{21}^*(\phi). \tag{3.32}$$

Let us try to extend this definition to an algebraic stack $\mathcal{C}$. The previous discussion suggests the following as a plausible definition:

Definition 3.69. A **quasi-coherent sheaf on** $\mathcal{C}$ is the assignment of a quasi-coherent sheaf $\mathcal{F}_{(S,x)}$ to any section $x \in \mathcal{C}(S)$ satisfying the following compatibility condition with respect to cartesian squares. For any morphisms $f : S' \to S$ in **Sch**, and $u : x' \to x$ with $p(u) = f$, there exists an isomorphism[37]

$$\alpha_u : f^*(\mathcal{F}_{(S,x)}) \xrightarrow{\sim} \mathcal{F}_{(S',x')} \tag{3.33}$$

[36] One can define more general sheaves, abelian sheaves etc. [44], [50].
[37] When such isomorphisms exist, the sheaf is called **cartesian**.

such that if $x'' \xrightarrow{v} x' \to x$ makes sense with $p(v) = g : S'' \to S'$, then we have $\alpha_{uv} = \alpha_v . g^*(\alpha_u)$, that is, the following diagram is commutative:

$$\begin{tikzcd} g^* f^*(\mathcal{F}_{(S,x)}) = (fg)^*(\mathcal{F}_{(S,x)}) \arrow[r, "g^*(\alpha_u)", "\sim"'] \arrow[rr, bend right, "\sim", "\alpha_{uv}"'] & g^*(\mathcal{F}_{(S',x')}) \arrow[r, "\alpha_v", "\sim"'] & \mathcal{F}(S'', x'') \end{tikzcd} . \tag{3.34}$$

Let $\xi : U \to \mathcal{C}$ be an atlas, and let $d_0, d_1 : R \rightrightarrows U$ be the associated groupoid. Given a sheaf $\mathcal{F}$ on $\mathcal{C}$, its restriction to U will be denoted $\mathcal{F}_{(U,\xi)}$, or for brevity $\mathcal{F}_\xi$ or $\mathcal{F}_U$. We are going to show that this sheaf comes together with an additional structure from which we can recover the whole datum encapsulated in $\mathcal{F}$. Indeed let

$$m : R \times_{d_1,U,d_0} R = U \times_{\mathcal{C}} U \times_{\mathcal{C}} U \longrightarrow R$$

be the composition law of the groupoid. In this identification m becomes the projection p_{31} and the projections on the two factors R, p_{21}, p_{32} respectively. We have a canonical isomorphism $d_0^*(\xi) \cong d_1^*(\xi)$ which in turn gives an isomorphism $\phi : d_0^*(\mathcal{F}_\xi) \xrightarrow{\sim} d_1^*(\mathcal{F}_\xi)$. Using the commutative diagrams (2.4) together with the associativity property (3.34) we get the commutative diagram of isomorphisms

$$\begin{tikzcd} m^* d_0^*(\mathcal{F}_\xi) \arrow[r, "m^*(\phi)"] \arrow[d] & m^* d_1^*(\mathcal{F}_\xi) \\ p_1^* d_0^*(\mathcal{F}_\xi) \arrow[d] & p_2^* d_1^*(\mathcal{F}_\xi) \arrow[u] \\ p_1^* d_0^*(\mathcal{F}_\xi) \arrow[r, "p_1^*(\phi)"] & p_1^* d_1^*(\mathcal{F}_\xi) \arrow[u, "p_2^*(\phi)"'] \end{tikzcd}$$

from which we extract the relation

$$m^*(\phi) = p_2^*(\phi) p_1^*(\phi) \tag{3.35}$$

where $p_1, p_2 : R \times_{d_1,U,d_0} R \to R$ are the names for the projections on the two factors. This relation is nothing but a descent datum for $\mathcal{F}$ relative to the representable morphism $\xi : U \to \mathcal{C}$. Let now $x : S \to \mathcal{C}$ be a section. Let $S' = U \times_{\xi,\mathcal{C},x} S$ be the fiber product, and $x' : S' \to U$ the first projection. It is easy to see that S' fits into a diagram

$$\begin{tikzcd} R \times_{d_1,U,d_0} R \arrow[r, shift left] \arrow[r, shift right] \arrow[r, shift left=2] & R \arrow[r, shift left] \arrow[r, shift right] & U \arrow[r] & \mathcal{C} \\ S' \times_S S' \times_S S' \arrow[u] \arrow[r, shift left] \arrow[r, shift right] \arrow[r, shift left=2] & S' \times_S S' \arrow[u, "y"] \arrow[r, shift left] \arrow[r, shift right] & S' \arrow[u, "x'"'] \arrow[r] & S. \arrow[u, "x"'] \end{tikzcd}$$

The sheaf $x'^*(\mathcal{F})$ comes together with a descent datum $x'^*(\phi)$, therefore applying descent theory for sheaves (Theorem 1.43) yields the expected sheaf $\mathcal{F}_{(S,x)}$ on S.

Another way to define a sheaf on $\mathcal{C}$ is to choose a site. The most natural topology associated to an algebraic stack $\mathcal{C}$ is the lisse-étale topology, see [44] or [50], Definition 3.1 for a complete discussion. To keep this introduction simple enough, and this will be sufficient for our purposes, we limit ourselves to the case of Deligne-Mumford stacks. In this case the étale topology is the good choice:

Definition 3.70. Let $\mathcal{C}$ be a Deligne-Mumford stack. The objects of the **small étale site** $\mathcal{C}_{et}$ are the étale morphisms $U \to \mathcal{C}$ (necessarily representable). A morphism $(f, \alpha) : (U \to \mathcal{C}) \longrightarrow (V, \mathcal{C})$ is a 2-commutative diagram

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr] & & V \arrow[dl] \\ & \mathcal{C}, & \end{tikzcd}$$

i.e., $u \overset{\alpha}{\Longrightarrow} vf$. A covering of $U \to \mathcal{C}$ is a collection of morphisms $(U_i \to \mathcal{C}) \longrightarrow (U \to \mathcal{C})$, such that $(U_i \to U)_i$ is an étale covering of U.

We refer to [44], Chap. 12 and [50]. The previous discussion (in principle) shows the equivalence of Definitions 3.69 and 3.70. Finally without proof let me mention another alternative definition of a quasi-coherent sheaf ([50], Definition 6.1): a quasi-coherent sheaf on $\mathcal{C}$ is an étale cartesian sheaf on the site $\mathcal{C}_{\mathrm{et}}$ such that for any étale $U \to \mathcal{C}$, its restriction to U_{et} is quasi-coherent. What we mean by restriction along $U \to \mathcal{C}$ is hopefully clear.

Once we have in our hands the definition of quasi-coherent sheaves, it is clear how to define coherent sheaves, etc. A sheaf $\mathcal{F}$ on the stack $\mathcal{C}$ is **coherent** (resp. **locally free** of rank n) if $\mathcal{F}(x)$ is coherent (resp. locally free of rank n) for any section x of $\mathcal{C}$. A locally free sheaf of rank 1 is also called (abusively) a line bundle. If $\mathcal{F}$ is described on an atlas $u : U \to \mathcal{C}$ (together with a descent datum), this is equivalent to saying that $\mathcal{F}_{(U,u)}$ is a coherent sheaf (resp. locally free of rank n) on U. For example the **structural sheaf** of $\mathcal{O}_{\mathcal{C}}$ is the assignment $(x : X \to \mathcal{C}) \mapsto \Gamma(X, \mathcal{O}_X)$, equivalently $(x : X \to \mathcal{C}) \mapsto \mathcal{O}_X$. A quasi-coherent sheaf is therefore a $\mathcal{O}_{\mathcal{C}}$-module. Given $\mathcal{F}$, $\mathcal{G}$ two quasi-coherent sheaves, the definition of a morphism $u : \mathcal{F} \to \mathcal{G}$ is obvious. We write $\mathbf{Qcoh}_{\mathcal{C}}$ the category of quasi-coherent sheaves on $\mathcal{C}$.

Exercise 3.71. *Prove that* $\mathbf{Qcoh}_{\mathcal{C}}$ *is an abelian category* (*see* [44], *Lemme* 13.1.3).

One can ask how to define the **global sections** of a quasi-coherent sheaf $\mathcal{F}$ on a stack $\mathcal{C}$. Our first definition will be indirect, making use of an atlas:

Definition 3.72. The group of global sections $\Gamma(\mathcal{C}, \mathcal{F})$ is the group defined via an atlas $\xi : U \to \mathcal{C}$ by

$$\Gamma(\mathcal{C}, \mathcal{F}) = \ker \left(\Gamma(U, \mathcal{F}_U) \overset{d_1^* - \phi d_0^*}{\longrightarrow} \Gamma(R, \mathcal{F}_R) \right). \tag{3.36}$$

where $(\mathcal{F}_U, \phi)$ denotes the realization of $\mathcal{F}$ on U endowed with its descent datum, and $\mathcal{F}_R = d_1^*(\mathcal{F}_U)$.

The trouble is that there is no canonical choice of a presentation. Thus the definition will be satisfactory only if we are able to check that two atlases yield canonically isomorphic results. To check this, it will be sufficient to give an equivalent definition, but now without the choice of an atlas. Let $x' \to x$ be a morphism of the category $\mathcal{C}$ over $f : X' \to X$. From the isomorphism $f^*(\mathcal{F}(x)) \xrightarrow{\sim} \mathcal{F}(x')$ we infer the existence of a natural map

$$\Gamma(X, \mathcal{F}(x)) \to \Gamma(X, f^*(\mathcal{F}(x)) \cong \Gamma(X', \mathcal{F}(x')). \tag{3.37}$$

It is not difficult to check this defines a contravariant functor, i.e., an inverse system $\mathcal{C} \longrightarrow \mathbf{Ab}$. The claim is that there exists an isomorphism

$$\Gamma(\mathcal{C}, \mathcal{F}) \xrightarrow{\sim} \varprojlim_{x:X\to\mathcal{C}} \Gamma(X, \mathcal{F}(x)). \tag{3.38}$$

To check this, observe that the definition yields a morphism

$$\varprojlim_{x:X\to\mathcal{C}} \Gamma(X, \mathcal{F}(x)) \longrightarrow \ker\left(\Gamma(U, \mathcal{F}_U) \xrightarrow{d_1^* - \phi d_0^*} \Gamma(R, \mathcal{F}_R)\right). \tag{3.39}$$

In the opposite direction, let us take a section $x : X \to \mathcal{C}$. We can find an étale covering $h : X' \to X$ such that $h^*(x)$ lifts to U through $f : X' \to U$, that is $\theta : h^*(x) \xrightarrow{\sim} f^*(\xi)$. From this we infer a 2-commutative diagram

$$\begin{tikzcd}
X \arrow[r, "x"] & U \\
X' \arrow[u, "h"] \arrow[r, "f"] & U \arrow[u, "\xi"'] \\
X' \times_X X' \arrow[u, shift left, "p_0"] \arrow[u, shift right, "p_1"'] \arrow[r, "g"] & R. \arrow[u, shift left, "d_0"] \arrow[u, shift right, "d_1"']
\end{tikzcd} \tag{3.40}$$

Notice that the bottom square is commutative with g coming from the pair of isomorphisms

$$p_0^*(\theta) : p_0^* h^*(x) \xrightarrow{\sim} p_0^* f^*(\xi) \quad , \quad p_1^*(\theta) : p_0^* h^*(x) = p_1^* h^*(x) \xrightarrow{\sim} p_1^* f^*(\xi),$$

more precisely coming from

$$p_1^*(\theta).(p_0^*(\theta))^{-1} : (fp_0)^*(\xi) \longrightarrow (fp_1)^*(\xi).$$

Writing $X'' = X' \times_X X'$ and using simplified notations, this yields a morphism

$$\ker\left(\Gamma(U, \mathcal{F}_U) \xrightarrow{d_1^* - \phi d_0^*} \Gamma(R, \mathcal{F}_R)\right) \longrightarrow \ker\left(\Gamma(X', \mathcal{F}_{X'}) \xrightarrow{p_1^* - p_0^*} \Gamma(X'', \mathcal{F}_{X''})\right). \tag{3.41}$$

But descent theory of quasi-coherent sheaves tells us that

$$\Gamma(X, \mathcal{F}(x)) \xrightarrow{\sim} \ker\left(\Gamma(X, h^*(\mathcal{F}(x))\right) \xrightarrow{p_1^* - p_0^*} \Gamma(X'', \mathcal{F}_{X''})$$

and these facts put together yield the expected map going in the direction opposite to (3.39). This suffices to prove the claim.

The definition applies to the structural sheaf $\mathcal{O}_{\mathcal{C}}$, giving the group of global sections $\Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}})$. One can understand these sections as the regular functions on a given atlas U, invariant under the action of the groupoid R.

Exercise 3.73. *Show that global sections of $\mathcal{O}_{\mathcal{C}}$ correspond to 1-morphisms $\mathcal{C} \to \mathbb{A}^1$.*

Exercise 3.74. *Let $\mathcal{X} \to \mathcal{C}$ be an étale morphism of (DM)-stacks. Given a quasi-coherent module $\mathcal{F}$ on $\mathcal{C}$ define the restriction of $\mathcal{F}$ to $\mathcal{X}$.*

Example 3.75. *Linear representations versus quasi-coherent modules on* **BG**. Let k be a field, and work in $\mathbf{Sch}_k$. Let G be an affine smooth algebraic group over k. We are going to see that the category of quasi-coherent sheaves on **BG** is equivalent to the category of linear representations of G. Let V be a quasi-coherent module over **BG**. Recall the groupoid scheme defining **BG** is

$$R = G \underset{d_1}{\overset{d_0}{\rightrightarrows}} U = \operatorname{Spec} k$$

which in turn yields

$$R_{d_1 \times U_{d_0}} R = G \times G.$$

Caution: the composition morphism is $m(g,h) = hg$ (2.5). The structure of a **BG**-quasi-coherent sheaf on V amounts to an isomorphism

$$\phi : M \otimes k[G] \xrightarrow{\sim} M \otimes k[G]$$

which fulfills the descent datum condition $p_2^*(\phi) \circ p_1^*(\phi) = m^*(\phi)$. There in an induced map $\rho : M \to M \otimes k[G] \xrightarrow{\phi} M \otimes k[G]$. It is easy to see that the cocycle condition says that this map is a "coaction" of G on V, making V a linear representation of G. This is clear if G is a finite constant group, then $k[G] = \oplus_{g \in G} k\delta_g$ with $\delta_g(h) = \delta_{g,h}$, and the coaction amounts to a collection of linear isomorphisms $\rho_g \in \mathrm{GL}_k(V)$, with $\rho_g \rho_h = \rho_{gh}$, making V a left G-module, not necessarily finite dimensional. The converse is straightforward. In this relationship coherent modules correspond to finite-dimensional representations.

More generally, when the stack is a quotient stack $[X/G]$, a quasi-coherent sheaf once interpreted as a quasi-coherent sheaf on X plus an extra condition is the same object as a G-linearized sheaf, or a G-sheaf ([48], definition 1.6).

Definition 3.76. Let G be a group scheme with multiplication $m : G \times G \to G$ and $\mathcal{F}$ a quasi-coherent sheaf on a scheme X endowed with an action $\sigma : G \times X \to X$. A G-**linearization** of $\mathcal{F}$ consists in an isomorphism

$$\phi : \sigma^*(\mathcal{F}) \xrightarrow{\sim} p_2^*(\mathcal{F}) \tag{3.42}$$

of sheaves on $G \times X$, satisfying the condition

$$(m \times 1_X)^*(\phi) = p_{23}^*(\phi) \circ (1_G \times \sigma)^*(\phi)$$

on $G \times G \times X$.

Suppose that $\mathcal{F}$ is locally free of finite rank. The vector bundle with $\mathcal{F}$ as sheaf of sections is $\mathbb{V}(\mathcal{F}) = \mathrm{Spec}(\mathrm{Sym}(\mathcal{F}^\vee)) \to X$ ([46], III, **2**). A G-linearization of $\mathcal{F}$ can be seen as a lift of the action of G on X to the bundle $\mathbb{V}(\mathcal{F})$, viz. a commutative diagram

$$\begin{tikzcd} G \times \mathbb{V}(\mathcal{F}) \arrow[r, "\hat{\sigma}"] \arrow[d] & \mathbb{V}(\mathcal{F}) \arrow[d] \\ G \times X \arrow[r, "\sigma"] & X. \end{tikzcd} \tag{3.43}$$

Exercise 3.77. *Let there be given an action of a finite constant group (or smooth affine) G on X. Write the details of the fact that the category of quasi-coherent sheaves on the stack $[X/G]$ is equivalent to the category of quasi-coherent G-linearized sheaves on X.*

Exercise 3.78. *Let V a quasi-coherent module over* $\mathbf{BG}$. *Check that* $\Gamma(\mathbf{BG}, V) = V^G$ *(see Example* 3.75*).*

As for a scheme, we define the **Picard group** $\mathrm{Pic}(\mathcal{C})$ of $\mathcal{C}$ as the group of locally free modules of rank 1, the group law given by the tensor product. Our previous description of the coherent sheaves on $\mathbf{BG}$ (G a constant finite group) yields

$$\mathrm{Pic}(\mathbf{BG}) = \mathrm{Hom}(G, k^*) = \hat{G},$$

the group of characters of G. Interesting is the case of a stacky projective line.

Proposition 3.79. *Assume $gcd(p, q) = 1$. We have* $\mathrm{Pic}(\mathbb{P}^1(p, q)) = \mathbb{Z}$.

Recall the definition $\mathbb{P}^1(p, q) = [\mathbb{A}^2_*/\,\mathbf{G}_m]$ (Subsection 3.1.1). As previously seen the line bundles on $\mathbb{P}^1(p, q)$ are described by $\mathbf{G}_m$-linearized line bundles on the punctured plane $\mathbb{A}^2_*$. A line bundle on the punctured affine plane is associated to a divisor. It is clear that a divisor of $\mathbb{A}^2_*$ is the restriction of a divisor on of the whole plane, therefore is the divisor of a function. As a consequence a line bundle on $\mathbb{A}^2_*$ is trivial. We must now describe the $\mathbf{G}_m$-linearizations of the trivial bundle $\mathbb{A}^2_* \times \mathbb{A}^1 \to \mathbb{A}^2_*$. Such a linearization comes from a $\mathbf{G}_m$-linearization of the trivial bundle on the whole plane $\mathbb{A}^2$. Indeed the composite morphism

$$\mathbf{G}_m \times \mathbb{A}^2_* \times \mathbb{A}^1 \xrightarrow{\tilde{\sigma}} \mathbb{A}^2_* \times \mathbb{A}^1 \hookrightarrow \mathbb{A}^2 \times \mathbb{A}^1$$

must factor through $\mathbf{G}_m \times \mathbb{A}^2 \times \mathbb{A}^1$, thus inducing an action on the trivial bundle over $\mathbb{A}^2$. A linearization of the trivial bundle over $\mathbb{A}^2$ is easily seen given by a character $\lambda \mapsto \lambda^r$ of $\mathbf{G}_m$, viz. at the level of points

$$\hat{\sigma}(\lambda, x, y, e) = (\lambda^p x, \lambda^q y, \lambda^r e).$$

Therefore $\mathrm{Pic}(\mathbb{P}^1(p, q)) = \mathrm{Hom}(\mathbf{G}_m, \mathbf{G}_m) = \mathbb{Z}$. □

Line bundles on a scheme are associated to Cartier divisors, see for example [33], Chap. II, Section 6. Let X be a reduced noetherian scheme, $\mathcal{L}$ a line bundle on X. By a **meromorphic section** of $\mathcal{L}$ we mean a section $\sigma \in \Gamma(U, \mathcal{L})$ regular over an open set U containing the generic points of the irreducible components, and non-zero at each such generic point. Over an affine open set $V = \mathrm{Spec}\, A$, trivializing

$\mathcal{L}_{|V}$, we can write $\sigma_{|V}$ as a fraction f/s, $f, s \in A$, and f, s non zero divisors. The Cartier divisor $D = \operatorname{div}(\sigma)$ is defined in such a way that the restriction $D_{|V}$ has for local equation f/s. Then $\mathcal{L} \cong \mathcal{O}(D)$.

We ask about a similar correspondence on an algebraic stack. Let $\mathcal{C}$ be a reduced algebraic stack, and let $p : U \to \mathcal{C}$ an (smooth) atlas, with U a reduced scheme.

Definition 3.80. By an **effective Cartier divisor** $D \subset \mathcal{C}$ we mean a closed substack, such that $D_U := p^{-1}(D) = U \times_{\mathcal{C}} D$ is a Cartier divisor of U.

Clearly the definition is independent of the atlas, since for any smooth surjective morphism $f : V \to U$, and an effective Cartier divisor D_U on U, $f^*(D_U)$ is an effective Cartier divisor. An effective Cartier divisor gives rise to a line bundle $\mathcal{O}(D)$ on $\mathcal{C}$, which comes from the line bundle $\mathcal{O}(D_U)$. Since $pd_0 \cong pd_1$, and D_U comes from D, it is obvious that $\mathcal{O}(D_U)$ is endowed with a descent datum, which in turn says it comes from a well-defined line bundle on $\mathcal{M}$. This is the expected $\mathcal{O}(D)$. Notice this sheaf comes together with a canonical global section $\sigma \in \Gamma(\mathcal{C}, \mathcal{L})$ (3.36), the image of 1 in $\Gamma(U, \mathcal{O}(D_U))$. Then D is the closed substack with "equation" $\{\sigma = 0\}$. Say a line bundle $\mathcal{L}$ comes from a Cartier divisor f we can find two effective Cartier divisors D, E such that $\mathcal{L} \cong \mathcal{O}(D) \otimes \mathcal{O}(E)^{-1}$.

Example 3.81. Let G be a finite constant group, and **BG** the associated stack. We know that a line bundle on this stack corresponds to a character of G. Let us denote $\mathcal{L}_\chi$ the line bundle associated to $\chi : G \to k^*$. Then a global section of $\mathcal{L}_\chi$ is a scalar fixed by χ, which in turn can be non-zero only if $\chi = 1$. So only the trivial line bundle comes from a divisor.

Example 3.82. Let us describe the Cartier divisors on the stacky projective line $\mathbb{P}^1(p, q)$ (Subsection 3.1.1). We can use the smooth atlas $\mathbb{A}^2 - \{(0, 0)\}$ to get immediately that the effective divisors correspond to the quasi-homogeneous plane curves, that is curves given by a quasi-homogeneous equation $F(x, y) = 0$ of a given degree d, i.e., $F(\lambda^p x, \lambda^q y) = \lambda^d F(x, y)$ ($\lambda \in k^*$). To such a curve we know there is an associated line bundle $\mathcal{O}F^{-1}$ on the punctured plane, which in turn yields the line bundle $\mathcal{O}(d)$ on $\mathbb{P}^1(p, q)$ (Proposition 3.79). The degree d belongs to $\mathbb{N}p + \mathbb{N}q$. Unless $p = 1$ or $q = 1$, $\mathbb{N}p + \mathbb{N}q \neq \mathbb{N}$. But if p and q are relatively prime, i.e., $1 \in \mathbb{Z}p + \mathbb{Z}q$, then we can find a homogeneous rational function of degree one. Therefore any line bundle comes from a Cartier divisor. If $\gcd(p, q) = d$ we see that the line bundles coming from a Cartier divisor is the subgroup of index d in $\operatorname{Pic}(\mathbb{P}^1(p, q)) = \mathbb{Z}$.

To close this section, we are going to define the two fundamental operations on sheaves: pushforward and pullback along a morphism, referring to ([50], Section 6), [44] for a wider approach and more details. Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of algebraic (DM) stacks. The goal is to define a pair of adjoint functors

$$\begin{tikzcd} \mathbf{Qcoh}_{\mathcal{X}} \arrow[r, bend left, "F_*"] & \mathbf{Qcoh}_{\mathcal{Y}} \arrow[l, bend left, "F^*"] \end{tikzcd} \tag{3.44}$$

where F^* is left adjoint to F_*. Recall that we need a quasi-compactness assumption of a morphism of schemes $f : X \to Y$ in order that f_* maps $\mathbf{Qcoh}_X$ to $\mathbf{Qcoh}_Y$. This functor has a left adjoint f^* ([33], Chap. II, Proposition 5.8). If furthermore X, Y are noetherian, then f^* maps a coherent sheaf to a coherent sheaf.

Now let F be a **quasi-compact** morphism of stacks. This means that for any morphism $y : Y \to \mathcal{Y}$, with V a quasi-compact scheme, then $\mathcal{X} \times_{\mathcal{Y}} Y$ is quasi-compact ($\exists$ an atlas which is quasi-compact). The definition of F_* mimics the known definition for schemes ([50], Lemma 6.5). Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$, we define the pushforward $F_*(\mathcal{F})$ as a sheaf on $\mathcal{Y}_{et}$ with sections over an étale open $y : V \to \mathcal{Y}$:

$$\Gamma(V, F_*(\mathcal{F})) = \Gamma(\mathcal{X} \times_{\mathcal{Y}} V, \mathcal{F}). \tag{3.45}$$

Since $p : \mathcal{X} \times_{\mathcal{Y}} V \to \mathcal{X}$ is étale, the definition of the restriction sheaf $p^*(\mathcal{F})$ is clear. The problem is to check that under the quasi-compactness assumption $F_*(\mathcal{F})$ is quasi-coherent. It is convenient to choose a presentation $R \rightrightarrows U \to \mathcal{X}$, and as in the previous discussion of global sections, one can see that $F_*(\mathcal{F})$ is given by

$$F_*(\mathcal{F}) = \ker\big(d_{0*} - d_{1*} : F_{U,*}(\mathcal{F}_U) \longrightarrow F_{R,*}(\mathcal{F}_R)\big).$$

This reduces the proof to the case where $\mathcal{X}$ is a scheme, i.e., F is representable. In this case the result follows easily from the case of schemes. A key feature of the functor F_* is that it commutes with an étale morphism $\mathcal{Y}' \to \mathcal{Y}$.

To define F^* on the level of quasi-coherent sheaves $\mathcal{G} \in \mathbf{Qcoh}_{\mathcal{Y}}$, we can either use an atlas, or Definition 3.69 (see [50], Section 6 for a more precise construction). Using Definition 3.69, let $x : X \to \mathcal{X}$ be a section over $X \in \mathbf{Sch}$, then we set

$$F^*(\mathcal{G})_{(X,x)} = \mathcal{G}_{(X,F(x))}. \tag{3.46}$$

It is straightforward to check that the definition is consistent. Alternatively we can also work with an atlas. Let $y : V \to \mathcal{Y}$ be an atlas, and let us choose $U \to \mathcal{X} \times_{\mathcal{Y},y} V$ an atlas of $\mathcal{X} \times_{\mathcal{Y},y} V$. The composition $x : U \to \mathcal{X} \times_{\mathcal{Y},y} V \to \mathcal{X}$ is étale and surjective, therefore is an atlas of $\mathcal{X}$. The 2-commutative square

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "F"] & \mathcal{Y} \\
U \arrow[u, "x"] \arrow[r, "f"] & V \arrow[u, "y"']
\end{tikzcd}$$

can be extended in a diagram (see (3.40))

$$\begin{tikzcd}
\mathcal{X} \arrow[r, "F"] & \mathcal{Y} \\
U \arrow[u, "x"] \arrow[r, "f"] & V \arrow[u, "y"'] \\
R \arrow[u, shift left, "d_0"] \arrow[u, shift right, "d_1"'] \arrow[r, "g"] & S \arrow[u, shift left, "p_0"] \arrow[u, shift right, "p_1"']
\end{tikzcd} \tag{3.47}$$

where the vertical arrows represent the associated groupoid schemes, and the lower square is commutative. Let $\psi : p_0^*\mathcal{G}_{(V,y)} \xrightarrow{\sim} p_1^*\mathcal{G}_{(V,y)}$ be the descent datum defining $\mathcal{G}$. By diagram chasing it is not difficult to prove that

$$g^*(\psi) : d_0^*(f^*(\mathcal{G}_{(V,y)})) \xrightarrow{\sim} d_1^*(f^*(\mathcal{G}_{V,y)}))$$

satisfies the cocycle condition (3.32), which in turn says the quasi-coherent sheaf $f^*(\mathcal{G}_{(V,y)})$ comes from a quasi-coherent sheaf on $\mathcal{X}$, we take it as $F^*(\mathcal{G})$.

Exercise 3.83. *Check that the two suggested definitions of the pullback $F^*(\mathcal{G})$ are identical.*

Exercise 3.84. *Let p, q be relatively prime. Show that the natural morphism $\mathbb{P}^1(p,q) \to \mathbb{P}^1$ to the coarse moduli space (Definition 3.37) yields an morphism $\mathrm{Pic}(\mathbb{P}^1) \hookrightarrow \mathrm{Pic}(\mathbb{P}^1(p,q)) \cong \mathbb{Z}$ (see Subsection 3.1.1) with image $pq\mathbb{Z}$. Therefore*

$$\mathrm{Pic}(\mathbb{P}^1) \otimes \mathbb{Q} \cong \mathrm{Pic}(\mathbb{P}^1(p,q)) \otimes \mathbb{Q}.$$

Exercise 3.85. *Let $F : \mathcal{X} \to \mathcal{Y}$ be a 1-morphism of (DM) stacks. Take $\mathcal{F} \in \mathbf{Qcoh}_{\mathcal{X}}, \mathcal{G} \in \mathbf{Qcoh}_{\mathcal{Y}}$. Define a "natural" isomorphism*

$$\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(F^*(\mathcal{G}), \mathcal{F}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}}(\mathcal{G}, F_*(\mathcal{F})).$$

4. Moduli stacks of curves and covers

4.1. Moduli stacks of curves

The construction that follows is essentially contained in Mumford's book [48], see also [19], or [31], [3] for details and much more information. There is a parallel construction dealing with moduli of principally polarized abelian varieties, see the introductory notes by M. Olsson [52].

Throughout, the underlying site is $\mathbf{Sch}_{\mathrm{et}}$. The objects we are interested in are the relative curves, or families of curves:

Definition 4.1. By a **curve of genus** g over $S \in \mathbf{Sch}$, we mean a smooth proper morphism $\pi : C \to S$ whose geometric fibers are connected curves of genus g.

The curves over $S \in \mathbf{Sch}$ are the objects of a groupoid, and varying S we get a fibered category in groupoids.

Definition 4.2. We define $\mathcal{M}_g$ as the fibered category over $\mathbf{Sch}$ whose objects over S are the curves of genus g over S, and with morphisms the cartesian diagrams

$$\begin{tikzcd} C \arrow[r, "\pi"] & S \\ C' \arrow[u, "g"] \arrow[r, "\pi'"] & S', \arrow[u, "f"'] \end{tikzcd} \tag{4.1}$$

that is, $(g, \pi') : C' \to C \times_S S'$ is an isomorphism.

We can see $\mathcal{M}_g$ as a presheaf in groupoids, with restriction functor

$$f^*(C \to S) = (C \times_S S' \to S').$$

Our aim is to prove that $\mathcal{M}_g$ is an algebraic Deligne-Mumford stack for $g \geq 2$. When the genus g is 0 or 1, in order to get a DM stack we need to rigidify the previous definition. These two cases will be treated separately. The proof consists in two steps.

1) $\mathcal{M}_g$ is a prestack.
2) $\mathcal{M}_g$ is an algebraic stack, indeed a DM stack.

Here is a very sketchy discussion. For more details, we refer to the foundational paper [15], or to [31] for a pedagogical explanation. The first step amounts to checking that the presheaf $\mathbf{Isom}_S(C, C')$ is an étale sheaf. Indeed this is the presheaf of points of a finite unramified scheme over S.

Proposition 4.3. *If $\pi : C \to S$ and $\pi' : C' \to S$ are curves over S, then the presheaf $\mathbf{Isom}_S(C, C')$ is a finite unramified S-scheme.*

Proof. First observe that a curve $\pi : C \to S$ of genus $g \geq 2$ is projective over S, meaning we can realize it as a closed subscheme of some relative projective space $\mathbb{P}^N \times S/S$. This follows from the fact (see below for details) that on a smooth curve C/k of genus $g \geq 2$, the canonical line bundle $\Omega^1_{C/k}$ is ample, indeed if we choose $k \geq 3$, the sheaf of k-fold differentials $(\Omega^1_{C/k})^{\otimes k}$ is very ample ([33], Chap. IV). Also it is generated by its global sections $\Gamma(C, (\Omega^1_{C/k})^{\otimes k}$, a vector space of dimension $(2k-1)(g-1)$ as given by the Riemann-Roch theorem (see [33], Chap. IV). An important consequence of the projectivity of (relative) curves is that for two such curves the Zariski sheaf

$$\mathbf{Hom}_S(C, C') : (T \to S) \mapsto \{T\text{-morphisms} C \times_S T \to C' \times_S T\}$$

is representable. The idea is to identify a morphism with its graph, a closed subscheme of a suitable fixed projective space, and then to use the machinery of the Hilbert scheme. We refer to ([48], preliminaries, no. 5) for a *resumé* and references. As a consequence the open subsheaf $\mathbf{Isom}_S(C, C')$ of the $\mathbf{Hom}$ scheme is representable by a scheme of finite type over S, better it is proper over S, a fact very specific to curves. To check the first half of this fact that is the finiteness condition, one must consider the graph of an isomorphism in $C \times C'$, and see that, relatively to a fixed pluricanonical polarization, only a finite number of Hilbert polynomials occur, see also the subsection below. For the second half, it suffices to apply the valuative criterion of properness ([33], Chap. II, Theorem 4.7).

Thus we take a discrete valuation ring A with fraction field K. Assume given two curves C, C' of genus g over $\operatorname{Spec} A$. Let us denote C_K, C'_K the corresponding generic fibers. Then we must prove that any isomorphism $\varphi : C_K \xrightarrow{\sim} C'_K$ extends (uniquely) to an isomorphism $C \xrightarrow{\sim} C'$. But as formulated this is a very special case of the uniqueness of the minimal models of regular curves, see M. Romagny's lectures ([54], Section 2.5). Since $g \geq 2$, a smooth proper curve of genus g has only

finitely many automorphisms ([33], Chap. 4, Exercise 2.5), therefore the morphism $\mathrm{Isom}_S(C, C')$ is quasi-finite, being proper and quasi-finite, a theorem of Chevalley: proper + quasi-finite = finite, yields the result. To prove the last point, i.e., the unramified assumption, we must check that the geometric fibers are reduced. But this amounts to showing that $\mathrm{Aut}_k(C)$ is a reduced finite group scheme for any curve of genus $g \geq 2$ over a field $k = \overline{k}$. The Lie algebra of this group scheme is easily identified with the regular vector fields on the curve, i.e., the global sections of the tangent sheaf T_C. This is a line bundle of degree $2 - 2g < 0$, therefore without non-zero section. This proves 1).

The proof of step 2) is the content of the subsection below. It relies on projective techniques introduced by Mumford [48].

4.1.1. Hilbert embedding of smooth curves. To begin with, we record some results about the pluricanonical embeddings of a curve of genus $g \geq 2$ defined over an algebraically closed field.

Lemma 4.4. *Let C be a smooth projective connected curve of genus $g \geq 2$ over an algebraically closed field k. Then the sheaf $\omega_C = \Omega^1_{C/k}$ is an invertible sheaf of degree $2g - 2$ generated by its global sections. For $\nu \geq 3$ the sheaf $\omega_C^{\otimes \nu}$ is very ample. Furthermore if $\nu \geq 2$ we have:*

$$\dim \mathrm{H}^1(C, \omega_C^{\otimes \nu}) = 0 \quad \textit{and} \quad \dim \mathrm{H}^0(C, \omega_C^{\otimes \nu}) = (2g-2)\nu - g + 1.$$

Proof. Use the Riemann-Roch theorem ([33], Chap. 4, Section 1). □

Let $\pi : C \to S$ be an S-curve of genus $g \geq 2$ with *relative dualizing sheaf* $\omega_{C/S} := \Omega^1_{C/S}$. Using the base change theorem in cohomology ([33], Chap. III, Theorem 12.11) one can check:

Lemma 4.5. *If $\nu \geq 1$, the $\mathcal{O}_S$ coherent module $\pi_*(\omega_{C/S})^{\otimes \nu}$ is locally free of rank $(2g-2)\nu - g + 1$, if $\nu \geq 2$ (g if $\nu = 1$), whose formation commutes with any base change, and $R^1\pi_*(\omega_{C/S}^{\otimes \nu}) = 0$ when $\nu \geq 2$. As a consequence for $\nu \geq 1$, and for any cartesian diagram*

$$\begin{tikzcd} C \arrow[r, "\pi"] & S \\ C' \arrow[u, "g"] \arrow[r, "\pi'"] & S' \arrow[u, "f"'] \end{tikzcd}$$

the canonical morphism $\pi'_*(g^*(\omega_{C/S})^{\otimes \nu})) \xrightarrow{\sim} f^*(\pi_*(\omega_{C/S})^{\otimes \nu})$ is an isomorphism.

Proof. (sketch) Suppose that $\nu \geq 2$. Then for any $s \in S$, we have from the previous lemma[38]

$$\mathrm{H}^1(\pi^{-1}(s), (\Omega^1_{C/S})^{\otimes \nu}_{|\pi^{-1}(s)}) = 0$$

[38] Since π has geometrically connected fibers it suffices to check the vanishing property over $\overline{k(s)}$, i.e., after the flat base change $\mathrm{Spec}\, k(s) \to \mathrm{Spec}\, \overline{k(s)}$.

therefore (loc. cit.) $\mathrm{R}^1\pi_*(\Omega^1_{C/S})^{\otimes\nu} = 0$. From the same reference we infer that $\pi_*(\Omega^1_{C/S})^{\otimes\nu}$ is a locally free sheaf on S whose formation commutes with any base change. If $\nu = 1$, Riemann-Roch theorem yields $\dim \mathrm{H}^1(C_s, \Omega^1_{C/S})_{|\pi^{-1}(s)}) = 1$ so we must change in part the proof. First we know by the same argument that for any $s \in S$, we have $\mathrm{R}^1(C_s, \omega_{C/S}) \otimes k(s) \cong \mathrm{H}^1(C_s, \omega_{C_s}) \cong k(s)$ and that the formation of this sheaf commutes with any base change. The last isomorphism is given by the trace map, see [33], Chap. III, Section 7. However, we cannot conclude that $\mathrm{R}^1(\omega_{C/S})$ is locally free of rank 1. But we know from deformation theory (see Proposition 4.8 below) that at least after étale localization, the curve $C \to S$ comes by a base change from a curve with smooth base. When the base is reduced we are able to conclude that $R^1(\omega_{C/S})$ is locally free of rank one. The theorem about cohomology and base change now applies and yields the claim, viz. the formation of $\pi_*(\omega_{C/S})$ commutes with arbitrary base change, and is a locally free sheaf of rank g. □

In particular if $\nu \geq 3$, the line bundle $(\omega_{C/S})^{\otimes\nu}$ is relatively very ample. Let us fix such ν. Then we can canonically embed C/S in the relative projective space $\mathbb{P}(\pi_*(\omega_{C/S})^{\otimes\nu})$. This is the **$\nu$-canonical embedding**.

We set $N = (2g-2)\nu - g$. We are going to see that the ν-canonical curves of genus g embedded in $\mathbb{P}^N$. This uses the Hilbert scheme Hilb_P with polynomial $P(x) = (g-1)(2t\nu - 1)$. The result is (see [48]):

Proposition 4.6. *There is a unique locally closed subscheme $H_\nu \subset \mathrm{Hilb}_P(\mathbb{P}^N)$ such that a morphism $f : S \to \mathrm{Hilb}_P(\mathbb{P}^N)$ factors through H_ν if and only if the three assumptions below are realized:*

i) *The pullback $\mathcal{C} \subset S \times \mathbb{P}^N$ of the universal family along f is a curve of genus g*
ii) *The invertible sheaf $p_2^*(\mathcal{O}(1))_{\mathcal{C}}$ is isomorphic to $\omega_{\mathcal{C}/S}^{\otimes\nu} \otimes p_1^*(\mathcal{L})$ for some line bundle $\mathcal{L}$ on S.*
iii) *For all $s \in S$, the curve $\mathcal{C}_s \subset \mathbb{P}^N$ is non degenerate, i.e., is not contained in a hyperplane.*

The subscheme H_ν is stable under the action of $G := \mathrm{PGL}(N+1)$.

We do not prove this rather technical result, and refer to [48], Chap. 7, [31], Chap. 2, Section C for details. We want to apply it to the proof of the main result, which not only provides a smooth atlas to $\mathcal{M}_g$ but also gives a very geometric description of the stack $\mathcal{M}_g$ when $g \geq 2$.

Theorem 4.7. *With our previous notations, we have:*

1. *The stack $\mathcal{M}_g$ is isomorphic to the quotient stack $[H_\nu/G]$. In particular, it is an algebraic stack, even better a GIT-stack (Subsection 3.1.4).*
2. *The stack $\mathcal{M}_g$ is a smooth DM stack over $\mathbb{Z}$, with connected geometric fibres, of relative dimension $3g-3$.*
3. *Its coarse moduli space is the GIT-quotient H_ν/G.*

Proof. (sketch) 1. Recall that a section over S of the quotient stack $[H_\nu/G]$ is a diagram

$$\begin{tikzcd} P \arrow[r, "\varphi"] \arrow[d, "f"'] & H_\nu \\ S & \end{tikzcd} \tag{4.2}$$

where f is a G-principal bundle,and φ is G-equivariant. The universal subscheme yields a universal curve $U_\nu \subset H_\nu \times \mathbb{P}^N$ with base H_ν. The group G acts diagonally on $H_\nu \times \mathbb{P}^N$ keeping U_ν invariant. The curve $\varphi^*(U_\nu \to H_\nu)$ is a curve over P together with a G-action which lifts the action of G on P. The quotient by G yields a S-curve of genus g.

Conversely, if we are given a curve $\mathcal{C} \to S$, the projective bundle

$$\mathbb{P}(\pi_*(\omega^\nu_{\mathcal{C}/S})) \to S$$

gives us a G-principal bundle $f : P \to S$ together with an isomorphism

$$P \times_S \mathbb{P}(\pi_*(\omega^\nu_{\mathcal{C}/S})) \xrightarrow{\sim} P \times \mathbb{P}^N. \tag{4.3}$$

The pullback $f^*(\pi : \mathcal{C} \to S)$ yields a ν-canonical curve into $P \times \mathbb{P}^N$. Therefore this yields a morphism $\varphi : P \to H_\nu$. It is readily seen that φ is G-equivariant. These two constructions put together allow us to identify the stacks $[H_\nu/G]$ and $\mathcal{M}_g$.

2. To prove that $\mathcal{M}_g$ is a DM stack amounts to checking that the diagonal is unramified, see Proposition 3.3, and the comment that follows. It is a general principle that this claim is equivalent to the fact that for two curves the S-scheme $\mathbf{Isom}_S(C_1, C_2)$ is unramified, equivalently for one curve $\mathrm{Aut}_S(C)$ is unramified over S, but this is precisely the content of Proposition 4.3. More details will be found in [15].

We know that the smoothness of an algebraic stack, say define over $\mathbf{Sch}_k$, amounts to studying the formal deformation theory of its points (Subsection 3.1.3). For $\mathcal{M}_g\,(g \geq 2)$ smoothness follows quickly from the fact that the deformation functor of a smooth projective curve is unobstructed, i.e., there is no obstruction to lift a curve $\pi : C \to \operatorname{Spec} A\,(A \in \mathbf{Art}_k)$ to $A' \in \mathbf{Art}_k$ if $A' \to A$ is a small surjection (see Definition 3.28). The precise result is:

Proposition 4.8. *Let C/k a smooth proper curve of genus $g \geq 2$. The tangent space to the deformation functor* Def_C *is naturally identified with* $\mathrm{H}^1(C, T_C)$ *while the obstructions to the infinitesimal lifting of deformations of C are in* $\mathrm{H}^2(C, T_C) = 0$. *In turn the (uni)versal deformation of C is parametrized by the (formal) spectrum of a power series ring in $N = 3g-3$ variables over k (resp. $W(k)$) if $\mathcal{M}_m$ is over* $\mathbf{Sch}$.

Proof. We refer to [15], [31], or [6] for more details. We just give an outline of this classical proof. The fact that we are working with a smooth curve appears only to ensure the vanishing of $\mathrm{H}^2(C, T_C)$. First take a covering by finitely many affine opens $C = \cup_i U_i$. It is not difficult to describe a deformation of C to $A \in \mathbf{Art}_k$. Let $\tilde{C} \to \operatorname{Spec} A$ be such a deformation. Definition of (formal) smoothness yields that

a smooth affine k-scheme is rigid (see [62], Definition 02H0 and Lemma 02H6). Therefore $\tilde{C}$ is obtained by gluing the affine schemes $U_i \times \operatorname{Spec} A$ along the pairwise intersections $U_i \cap U_j$. The gluing is provided by an automorphism of the trivial deformation $(U_i \cap U_j) \times \operatorname{Spec} A$, inducing the identity over the residual field k of A. When $A = k[\epsilon]$ ($\epsilon^2 = 0$), then these local automorphisms look like

$$id + \epsilon\partial_{ij}, \quad (\partial_{ij} \in \Gamma(U_i \cap U_j, T_C),$$

the cocycle condition taking the form $\partial_{ij} + \partial_{jk} = \partial_{ik}$ for all (i, j, k). We recognize a Čech cocycle. It is easy to deduce from this that the deformation classes are in one-to-one correspondence with the cohomology classes $\in \mathrm{H}^1(C, T_C)$. Indeed

$$\mathrm{Def}_C(k[\epsilon]) \cong \mathrm{H}^1(C, T_C).$$

Likewise we can check that the obstructions to the infinitesimal lifting of deformations classes are in $\mathrm{H}^2(C, T_C)$. Finally it is worth mentioning that the versal deformation of C is indeed universal (i.e., condition (H4) holds (Theorem 3.31), effective and algebraizable. These last two points mean that the universal deformation is really represented by a curve over the spectrum of a formal power series ring, and not only a projective system of curves over Artin rings, and better comes from a smooth curve defined over the local ring of a smooth algebraic k-variety at a point. See [33], Chap. II, Section 9 for an introduction to these formal aspects. □

To close the proof of Theorem 4.7, notice that the dimension of $\mathcal{M}_g$ (over $\mathbf{Sch}_k$) is given by the dimension of an étale atlas, and due to the smoothness, the dimension of the tangent space of the formal deformation space of an arbitrary point, that is $\dim \mathrm{H}^1(C, T_C) = \dim \mathrm{H}^0(C, \Omega_C^{\otimes 2}) = 3g - 3$. The connectedness is much more difficult; the first and main step is to build a compactification of $\mathcal{M}_g$, and for this we refer to [15]. □

Once the stack $\mathcal{M}_g$ is constructed, it is not difficult to define the stack $\mathcal{M}_{g,n}$ whose sections over S are curves of genus $g \geq 2$ (smooth, projective) $\pi : \mathcal{C} \to S$, together with a set of n ordered pairwise disjoint sections $x_i : S \to \mathcal{C}$. The morphisms are the morphisms of curves sending the jth section onto the jth section. Let $U_\nu \to H_\nu$ the previous universal family. Then the fiber product $pr_2 : U_\nu \times_{H_\nu} U_\nu \to U_\nu$ together with the diagonal $\Delta : U_\nu \to U_\nu \times_{H_\nu} U_\nu$ is the universal canonically embedded curve with a section. Therefore our previous reasoning yields

$$\mathcal{M}_{g,1} \cong [U_\nu/G]. \tag{4.4}$$

Consider the iterated fiber product

$$U_{\nu,n} := U_\nu \times_{H_\nu} U_\nu \times_{H_\nu} \times \cdots \times_{H_\nu} U_\nu.$$

This is a closed subscheme of U_ν^n which classifies the ν-canonically embedded curves of genus g, together with n labelled points, distinct or not. Let $V_{\nu,n}$ be the complementary subset of the "big" diagonal. This is an open stable subset of $U_{\nu,n}$. Then

Proposition 4.9. *$\mathcal{M}_{g,n} \cong [V_{\nu,n}/G]$ is a DM stack.*

One can show that the stack $\mathcal{M}_{g,n}$ ($g \geq 2$) is smooth and irreducible over $\operatorname{Spec}\mathbb{Z}$. Furthermore if $n >> g$ ($n \geq 2g+3$ precisely), then $\mathcal{M}_{g,n}$ is an algebraic space. This follows from the fact that a non-trivial automorphism cannot fix more than $2g+2$ distinct points, see Exercise 4.11. As a corollary of the GIT construction of the Hurwitz scheme, one can show that it is really a scheme. For $n \geq 3$, the stack $\mathcal{M}_{0,(n)}$, the classifying stack of n unordered points on a $\mathbb{P}^1$, is a DM stack, but not a scheme (see Definition 3.25).

Exercise 4.10. *Show that the morphism forgetting the points $\mathcal{M}_{g,n} \longrightarrow \mathcal{M}_g$ ($g \geq 2$) is representable.*

Exercise 4.11. *Assume given a smooth projective curve C, of genus g, defined over $k = \overline{k}$. Prove that a non-trivial automorphism of C cannot fix n distinct points of C if $n \geq 2g+3$.*

4.1.2. Moduli stack of elliptic curves. In the previous section we studied $\mathcal{M}_g$ with $g \geq 2$. In the present section we focus on the seminal example $\mathcal{M}_{1,1}$, the moduli stack of elliptic curves [30], [37]. Throughout we will work over $\mathbb{Z}[1/6]$, in order to drop the bad characteristics 2 and 3. Then a scheme is one in which 6 is invertible in its structural sheaf. Recall that $\mathcal{M}_{1,1}$ stands for the fibered category in groupoids with sections over S, the groupoid of smooth projective connected curves over S endowed with a section called the 0-section:

$$\begin{tikzcd} C \arrow[r, "\pi"] & S \arrow[l, bend left, "O"] \end{tikzcd} \tag{4.5}$$

the morphisms are given by the cartesian diagrams with an obvious compatibility with the sections. Recall that in the case $S = \operatorname{Spec} k$, the scheme C is canonically endowed with a commutative group law with zero the marked point O, and over a general base C is endowed of a structure of S-abelian group scheme. Classically to describe $\mathcal{M}_{1,1}$ as a DM stack is to work with the so-called Weierstrass equations. Before we take this road, it is worth recording some consequences of the Riemann-Roch theorem regarding curves of genus 1. Let (C, O) be an elliptic curve over k, thus O is rational over k.

Lemma 4.12.

1) *One has $\mathrm{H}^1(C, \mathcal{O}(kO)) = 0$ for $k > 0$, and $\dim \mathrm{H}^0(C, \mathcal{O}(kO)) = k$ for all $k \geq 0$.*
2) *The line bundle $\mathcal{O}(kO)$ is very ample for $k \geq 3$.*

Notice that the inclusion $\mathcal{O}_C \subset \mathcal{O}(O)$ yields $k = \mathrm{H}^0(C, \mathcal{O}_C) \cong \mathrm{H}^0(C, \mathcal{O}(O))$. Let us denote e the image of 1 in $\mathrm{H}^0(C, \mathcal{O}(O))$. Let z be a local parameter at O. Then we can choose a basis[39] $\{e^2, f\}$ of $\mathrm{H}^0(C, \mathcal{O}(2O))$ such that f has for polar part at 0, $z^{-2} + \cdots$. Likewise we can choose a basis $\{e^3, ef, g\}$ of $\mathrm{H}^0(C, \mathcal{O}(3O))$ such that the leading term of the polar part of g at O is z^{-3}. In the 6-dimensional vector space $\mathrm{H}^0(C, \mathcal{O}(6O))$ the sections $e^6, e^4f, e^2f^2, f^3, e^3g, efg, g^2$ must be linearly dependent. It is readily seen that we can normalize further our choice of f

[39] *Product* means *tensor product.*

and g so that this relation reads

$$g^2 = f^3 + ae^4 f + be^6 \quad (a, b \in k). \tag{4.6}$$

The non singularity of C forces the discriminant of the right-hand term $\delta = 4a^3 + 27b^2$ to be $\neq 0$. More generally it is not too difficult to describe the important graded ring ([33], Chap. IV, Exercise 4.1):

$$R = \oplus_{k \geq 0} \mathrm{H}^0(C, \mathcal{O}(kO)).$$

Lemma 4.13. *One has $R = k[e, f, g]/(g^2 - f^3 - ae^4 f - be^6)$ where the respective degrees of e, f, g are $1, 2, 3$.*

It is a general fact that $\mathcal{O}(O)$ being an ample line bundle on C, then $C = \mathrm{Proj}(R)$, which in turn describes C as a curve of degree 6 in the weighted projective space $\mathbb{P}^2(1, 2, 3)$. It is more convenient to use the linear system $|\mathcal{O}(3O)|$ to embed C in the ordinary projective plane $\mathbb{P}^2$. Using the basis (e^3, ef, g) of $\mathrm{H}^0(C, \mathcal{O}(3O))$ we easily check that C embeds into $\mathbb{P}^2$ as a cubic curve with equation $ZY^2 = X^3 + aXZ^2 + bZ^3$, where $Z = e^3, X = ef, Y = g$. This is the so-called Weierstrass form of C. In this description the only choice we must fix is that of z. Another choice $z' = \lambda z + \cdots$ leads to $f' = \lambda^{-2} f, g' = \lambda^{-3} g$.

This construction extends to a curve $\pi : C \to S$ over an arbitrary base, and section $O : S \hookrightarrow C$. Using Lemma 4.12 together with tools about variation of cohomology similar to those used in Lemma 4.5, one can check that $\pi_*(\mathcal{O}(kO))$ is a locally free sheaf on S of rank k. In particular

$$\mathcal{O}_S = \pi_*(\mathcal{O}_C) \cong \pi_*(\mathcal{O}(O)) \subset \pi_*(\mathcal{O}(2O)) \subset \pi_*(\mathcal{O}(3O)). \tag{4.7}$$

Let $\mathcal{L}$ be the normal line bundle along the section O. Then the exact sequence $0 \to \mathcal{O}((k-1)O) \to \mathcal{O}(kO) \to \mathcal{L}^{\otimes k} \to 0$ yields for $k > 1$,

$$\pi_*(\mathcal{O}(kO))/\pi_*(\mathcal{O}((k-1)O)) \cong \mathcal{L}^{\otimes k}.$$

Shrinking S if necessary, we may assume that $\mathcal{L}$ is trivial, say $\mathcal{L} = \mathcal{O}t$. Then $\pi_*(\mathcal{O}(kO))$ is free of rank k. Then the same reasoning as before says that we can choose a basis (e^2, f) of $\pi_*(\mathcal{O}(2O))$ with $f \mapsto t^2$ in $\pi_*(\mathcal{O}(2O))/\pi_*(\mathcal{O}(O)) = \mathcal{L}^{\otimes 2}$, and likewise a basis (e^3, ef, g) of $\pi_*(\mathcal{O}(3O))$ such that $g \mapsto t^3$. In $\pi_*(\mathcal{O}(6O))$ normalizing further, it turns out that the following relation holds:

$$g^2 = f^3 + ae^4 f + be^6 \tag{4.8}$$

for some $a, b \in \Gamma(S, \mathcal{O}_S)$, and $\delta = 4a^3 + 27b^2 \in \Gamma(S, \mathcal{O}_S)^*$. If we change t to $t' = \lambda t$, $\lambda \in \Gamma(S, \mathcal{O}_S)^*$, then a, b move to $a' = \lambda^4 a, b' = \lambda^6 b$. This shows that at^{-4} and bt^{-6} are section of respectively $\mathcal{L}^{-4}$ and $\mathcal{L}^{-6}$. Finally the curve $C \to S$ can be embedded into the relative projective plane $\mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ as the relative curve with equation of Weierstrass type $ZY^2 = X^3 + aXZ^2 + bZ^3$

$$\begin{tikzcd} C \arrow[r, hook] \arrow[dr, "\pi"'] & \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3) \arrow[d] \\ & S. \end{tikzcd}$$

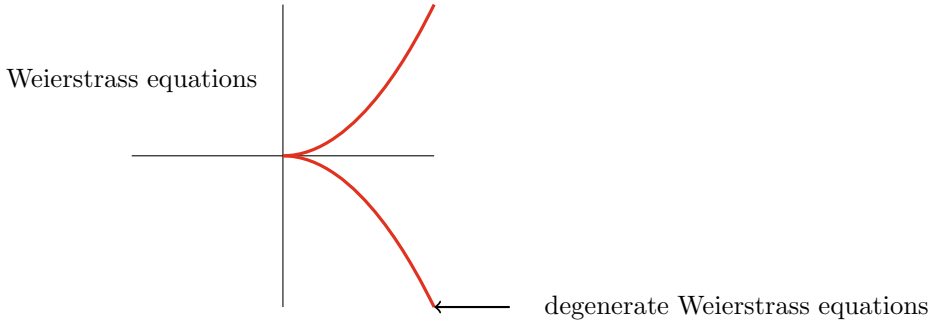


FIGURE 1. The space of Weierstrass equations

The meaning of this global equation is clear at least locally. As seen above the choice of a, b makes that the local construction glue together. Notice that the smoothness of C/S yields about the coefficients (a, b),

$$\mathcal{L}^{-\otimes 12} = (4a^{\otimes 3} + 27b^{\otimes 2})\mathcal{O}_S. \tag{4.9}$$

This is clear since it holds fiberwise. This suggests the definition:

Definition 4.14. By a **Weierstrass equation** with coefficients in a line bundle $\mathcal{L}$ over S we mean the following datum: a pair of sections $a \in \Gamma(S, \mathcal{L}^{-4}), b \in \Gamma(S, \mathcal{L}^{-6})$ such that (4.9) holds, i.e.,

$$\delta := 4a^{\otimes 3} + 27b^{\otimes 6} \in \Gamma(S, \mathcal{L}^{-12}) \tag{4.10}$$

has no zero, i.e., $\mathcal{L}^{-12} = \mathcal{O}_S\delta$.

The Weierstrass equation over S together with the obvious isomorphisms between two of them define a groupoid, and varying S, we get a fibered category in groupoids $\mathcal{M}_W$. But viewing $\mathcal{L}$ as defining a $\mathbf{G}_m$-torsor over S, namely $P = \mathrm{Spec}(\oplus_{n\in\mathbb{Z}}\mathcal{L}^n)$, we see the pair (a, b) yields a morphism $P \to \mathrm{Spec}(\mathbb{Z}[1/6][A, B])$. This morphism becomes is $\mathbf{G}_m$-equivariant if the variables A, B are affected with the weights $4, 6$ respectively. The non vanishing condition (4.9) says the morphism factors through the open $\mathbf{G}_m$-invariant subset $\delta(A, B) \neq 0$.

The following is by now clear[40]

Proposition 4.15. *$\mathcal{M}_W$ is a DM stack, indeed*

$$\mathcal{M}_W \cong [\mathrm{Spec}\,(\mathbb{Z}[1/6][A, B]) - \{\delta = 0\}/\,\mathbf{G}_m]\,.$$

[40]If we drop the condition $6 \neq 0$ in the ground ring, then the story is somewhat different. It is a know fact that over an arbitrary ground field, an elliptic curve can be put in a generalized Weierstrass form $ZY^2 + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3$ [61]. The change of coordinates takes here a more complicated form, but we can build a groupoid to encapsulate these transformations equivalently the isomorphisms between elliptic curves in Weierstrass form. The problem due to the primes 2 and 3 is that this groupoid is only flat, not étale, so no longer defines an étale stack. Despite this, one can prove that the stack $\mathcal{M}_{1,1}$ is really a DM stack.

Remark 4.16. The construction gives $\mathcal{M}_W$ as an open substack of

$$[\mathbb{A}^2 - \{0,0\}/\,\mathbf{G}_m] = \mathbb{P}^1(4,6).$$

The difference with our previous example of stacky projective line (Subsection 3.1.1) is the fact that here the weights are not coprime. The subgroup $\mu_2 = \{\pm 1\}$ acts trivially on $\mathbb{A}^2$, a fact equivalent to the assertion that an arbitrary elliptic curve has a permanent involutive automorphism. In a Weierstrass form this is $(x,y) \mapsto (x,-y)$.

The curve $\delta = 0$ in the punctured plane is an orbit of the $\mathbf{G}_m$-action. Thus $\mathcal{M}_W = \mathbb{P}^1(4,6) - \infty$, where ∞ is the punctual closed substack image of this exceptional orbit.

Finally the relationship between $\mathcal{M}_W$ and $\mathcal{M}_{1,1}$ is:

Theorem 4.17. *We have $\mathcal{M}_{1,1} \cong \mathcal{M}_W$.*

Proof. There is a natural morphism $\mathcal{M}_W \longrightarrow \mathcal{M}_{1,1}$ which assigns to a Weierstrass equation $(\mathcal{L}, a, b) \in \mathcal{M}_W(S)$ the elliptic curve $C \subset \mathbb{P}(\mathcal{O}_S \oplus \mathcal{L}^2 \oplus \mathcal{L}^3)$ given by the global Weierstrass equation $ZY^2 = X^3 + aZ^2X + bZ^3$. This morphism is clearly an epimorphism, due to the fact that every elliptic curve over S is isomorphic to one defined by a global Weierstrass equation. The morphism is also fully faithful. This amounts to checking that the isomorphisms between two elliptic curves over S associated to two Weierstrass equation are the same as the isomorphisms between these equations. Indeed, an isomorphism $f : S' \to S$ over S, taking O' onto O, induces first a natural isomorphism $\mathcal{O}(kO') \cong f^*(\mathcal{O}(kO)$, and an isomorphism $\varphi : \mathcal{L}' \cong \mathcal{L}$ of line bundles on S. It is readily seen that φ defines an isomorphism between the Weierstrass equations

$$\varphi : (\mathcal{L}', a', b') \xrightarrow{\sim} (\mathcal{L}, a, b)$$

and conversely. This proves out claim. □

Even if the description of $\mathcal{M}_{1,1}$ via a groupoid scheme is satisfactory, it would be interesting to describe the versal deformation space, i.e., a local chart, at some bad point, for example the point corresponding to the curve $y^2 = x^3 - x$. We know that it suffices to find a local slice at the point $(1,0) \in \mathbb{A}^2 - \{\delta = 0\}$, we can take the vertical line $a = 1$. This means that the one parameter family of curves $y^2 = x^3 - x + \lambda,\ (27\lambda^2 \neq 4)$ yields a local chart, that is the morphism

$$\operatorname{Spec} k[\lambda, \frac{1}{27\lambda^2 - 4}] \longrightarrow \mathcal{M}_{1,1} \tag{4.11}$$

is étale. Observe $j(\lambda) = 1728\frac{4}{27\lambda^2-4}$ is ramified with order two at the point $\lambda = 0$.

It is a classical but important fact that the coarse moduli space of $\mathcal{M}_{1,1}$ is the j-line, meaning that elliptic curves over an algebraically closed field are classified by the j-invariant ([33], Chap. 4.1, Theorem 4.1). Our previous discussion of the stacky projective line (Subsection 3.1.1) yields this result:

Proposition 4.18. *The coarse moduli space of $\mathcal{M}_{1,1}$ is the affine line $\mathbb{A}^1$, more specifically the canonical morphism $\mathcal{M}_{1,1}$ is given by the j-invariant*

$$j(C) = 1728\frac{4a^3}{4a^3 + 27b^2} \tag{4.12}$$

Proof. Proposition 3.45 gives us the fact that coarse moduli space of the stack[41] $\mathbb{P}^1(4,6)$ is the projective line $\mathbb{P}^1$ with coordinate $t = \frac{a^3}{b^2}$, equivalently $j = \frac{4a^3}{\delta}$. Now the coarse moduli space of the open substack $\delta \neq 0$ is the image $j \neq \infty \subset \mathbb{P}^1$. This shows that the coarse moduli space is the affine line $\mathbb{P}^1(j) - \{\infty\} = \operatorname{Spec} \mathbb{Z}[1/6][j]$. The factor 1728 is classical. □

Remark 4.19. One can ask if the Legendre form of an elliptic curve helps to describe $\mathcal{M}_{1,1}$. Recall that the Legendre form amounts to working with the three distinct roots of the polynomial $x^3 + ax + b$, so we write formally $x^3 + ax + b = (x - e_1)(x - e_2)(x - e_3)$, and we take the e_i's as new coefficients. Notice $a = \sum_{i<j} e_i e_j$, $b = -e_1 e_2 e_3$. The $\mathbf{G}_m$-action lifts to the affine space $\operatorname{Spec} k[e_1, e_2, e_3]$, the weight of the e_i's is then equal to 2. Let S stands for the open subset

$$S = \{(e_1, e_2, e_3),\ e_i \neq e_j,\ \sum_i e_i = 0 \tag{4.13}$$

There is an obvious action of the semi-direct product $\mathbf{G}_m \rtimes \mathbf{S}_3$ of $\mathbf{G}_m$ by the symmetric group on S. Then one can show that $\mathcal{M}_{1,1} = [S/\,\mathbf{G}_m \rtimes \mathbf{S}_3]$. We cannot drop the factor $\mathbf{G}_m$ to get a quotient by a finite group, since $\pm 1 \subset \mathbf{G}_m$ acts trivially, otherwise the canonical involution would collapse.

Remark 4.20. *The residual gerbe at a point of the j-line.* Let k be a perfect field, and $j_0 \in k$ a point (the affine j-line). Let $\mathcal{G}$ be the residual gerbe at $j_0 \in M_{1,1}$ (see Proposition 3.47). Then $\mathcal{G}$ is neutral with band $\operatorname{Aut}_k(E)$ where E is an elliptic curve with j-invariant j_0 (see exercise below). Then one can identify the sections of $\mathcal{G}$ over k, with elements of the Galois cohomology group $\mathrm{H}^1(\mathrm{Gal}_{\overline{k}/k}, \operatorname{Aut}_k(E))$, see Silverman [61].

Exercise 4.21. *Consider the elliptic curve $\pi : C \to \mathbb{A}^1 - \{0, 1728\}$ given by the Weierstrass equation*

$$y^2 = x^3 + \frac{27}{4} \cdot \frac{j}{1728 - j}(x + 1)$$

and $\pi(x, y, j) = j$.

1. *Show that the fiber over j has j-invariant j (4.12).*
2. *Show that this family cannot be extended smoothly across the whole affine j-line.*

Exercise 4.22. *Show that the one-parameter family $C_\mu : y^2 = x^3 + \mu x^2 - (1 + \mu)x$ yields a versal deformation of C_0.*

Exercise 4.23. *Show that $S \cong \mathbf{G}_m \times \mathbb{A}^1 - (0, 1)$ equivariantly, and identify the action on the right-hand side. Check Remark 4.19.*

[41] The fact that the weights are not coprime does not alter the result.

4.1.3. Stable curves and the compactification of $\mathcal{M}_{g,n}$. The moduli stack $\mathcal{M}_{1,1}$ as shown previously (Proposition 4.15) comes as a quotient stack from the action of $\mathbf{G}_m$ on $\mathbb{A}^2 = \operatorname{Spec} \mathbb{Z}[1/6][x, y]$, with respective weights $4, 6$, i.e.,

$$\mathbf{G}_m \times (\mathbb{A}^2 - \{\delta = 0\}) \rightrightarrows (\mathbb{A}^2 - \{\delta = 0\}) \tag{4.14}$$

where d_0 is the first projection, and d_1 the action. If instead of $\mathbb{A}^2 - \{\delta = 0\}$ we work with the punctured plane $\mathbb{A}^2 - \{0\}$, then the corresponding quotient stack is the stacky projective line $\mathbb{P}^1(4, 6)$, with coarse moduli space $\mathbb{P}^1$ (Proposition 3.45. A Weierstrass equation lying over a point with $\delta = 0$ reduces to

$$y^2 = (x - \alpha)^2(x + 2\alpha) \quad (\alpha \neq 0). \tag{4.15}$$

The curve $\{\delta = 0\}$ minus $(0, 0)$ is the $\mathbf{G}_m$-orbit of $(-3, 2)$, which in turn says that these curves are pairwise isomorphic. These curves are all irreducible, but singular, with a node at $(\alpha, 0)$.

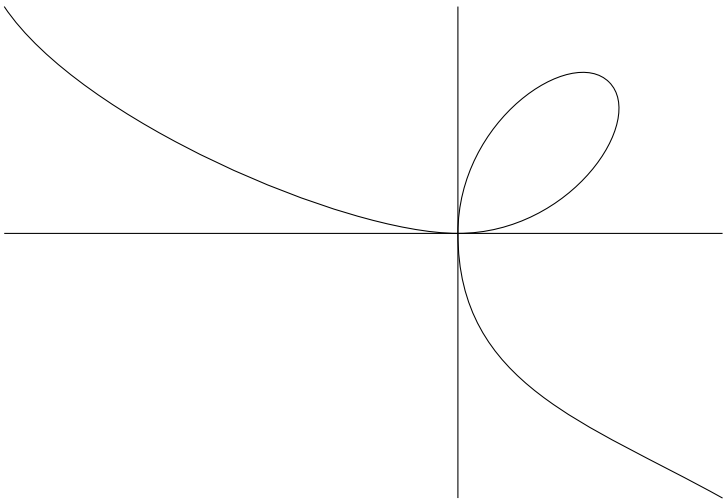


FIGURE 2. A nodal elliptic curve

The lesson which follows from this construction is that

$$\overline{\mathcal{M}}_{1,1} = [\mathbb{A}^2 - \{0\}/\, \mathbf{G}_m] \tag{4.16}$$

is a proper smooth stack over $\mathbb{Z}[1/6]$. There is an *a priori* reason to explain the completeness of this stack, this is the **semi-stable reduction theorem** ([61], Chap. VII). Assume we are given an elliptic curve over the fraction field K of a complete discrete valuation ring R, with residue field k. Then we can adjust the equation to get the so-called minimal equation, i.e., a Weierstrass equation with coefficients in R, and discriminant Δ with minimum valuation. We can reduce the equation

equation to k. The resulting curve can be not only singular, but also with a bad singular point (a cusp). The reduction is **good** if it is non singular, and **semi-stable** if the singular point is a node. A key result ([61], Chap. VII, Proposition 5.4) asserts that we can find a finite (ramified) extension K'/K, such that after pullback to K', the curve has either good or semi-stable reduction[42]. Since $\overline{\mathcal{M}}_{1,1}$ is of finite type over $\operatorname{Spec}\mathbb{Z}[1/6]$, this fact translated in stack language means precisely that $\overline{\mathcal{M}}_{1,1}$ is proper over $\operatorname{Spec}\mathbb{Z}[1/6]$. Even if this result is satisfactory, what is lacking is a modular interpretation of the added point, i.e., of the whole stack $\overline{\mathcal{M}}_{1,1}$. The "boundary" of $\overline{\mathcal{M}}_{1,1}$ is the stack

$$[\{\delta = 0\} - (0,0)/\,\mathbf{G}_m] \cong \mathbf{B}(\mathbb{Z}/2\mathbb{Z})$$

which "represents" the nodal cubic curve, according to the definition:

Definition 4.24.

i) Let $P \in C$ be a k-point of an algebraic curve (i.e., a 1-dimensional k-scheme). It is called a **node** if the completed local ring is isomorphic to $\widehat{\mathcal{O}}_P \cong k[[x,y]]/(xy)$.
ii) A **nodal curve** over an algebraically closed field k is a reduced connected proper one-dimensional k-scheme whose singularities are at worse nodal points.

Thus one can expect the following description of the fibered category $\overline{\mathcal{M}}_{1,1}$.

Proposition 4.25. *A section of the stack $\overline{\mathcal{M}}_{1,1}$ over $S \in \mathbf{Sch}/\mathbb{Z}[1/6]$ is a relative curve $\pi : C \to S$ (proper flat map) together with section $O : S \to C$, such that an arbitrary geometric fiber is either a smooth connected curve of genus 1, or a rational connected curve with one node, and the point O is a smooth point in any fiber.*

Proof. It is a good exercise to check directly that the fibered category as described above, denoted $\overline{\mathcal{M}}^*_{1,1}$ for the proof, is a DM stack. We are going to proceed indirectly, showing that we can build an isomorphism of fibered categories

$$\overline{\mathcal{M}}^*_{1,1} \xrightarrow{\sim} \overline{\mathcal{M}}_{1,1} = [(\mathbb{A}^2 - \{0\}/\,\mathbf{G}_m] \tag{4.17}$$

The proof is completely analogous to the smooth case, once we can check that the line bundle $\mathcal{O}_C(O)$ on C, i.e., of the Cartier divisor defined by the image of the section O, enjoys similar cohomological properties than in the case of smooth elliptic curves, namely the content and the proof of both Lemmas 4.12, 4.13 remain valid. Thus the proof of Proposition 4.15 extends to the nodal case. This proof yields first a morphism of fibered categories and then shows it is an isomorphism. □

Remark 4.26. It is worth mentioning that one can give a modular interpretation to the smooth atlas $\mathbb{A}^2 - \{\delta = 0\} \to \mathcal{M}_{1,1}$, respectively of $\mathbb{A}^2 - \{\delta = 0\}$ (see [30],

[42] Having good or semi-stable reduction is really a "stable" property, see [61], Chap. VII, Proposition 5.4, b).

Section 8). Let us define a new fibered category $\mathcal{M}_{1,\vec{1}}$ over $\mathbf{Sch}/\mathbb{Z}[1/6]$. The sections over S are the elliptic curves

$$\begin{tikzcd} C \arrow[r, "\pi"] & S \arrow[l, bend left, "O"] \end{tikzcd}$$

together with a global section $\omega \in \Gamma(C, \Omega^1_{C/S})$ which is nonzero in the fibers. Morphisms are defined in an obvious manner. One can check with exactly the same the proof of Lemma 4.5 that the coherent sheaf $\pi_*(\Omega^1_{C/S})$ is a line bundle, its formation commutes with an arbitrary base change. This defines the so-called **Hodge line bundle** on $\mathcal{M}_{1,1}$. Then the choice of ω is a trivialization of this line bundle. It is easy to see that these new objects do not have an automorphism $\neq 1$. The proof of Theorem 4.17 shows that this stack is indeed a scheme, viz. $\mathbb{A}^2 - \{\delta = 0\}$. Then the same arguments as those used in the proof of Proposition 4.25, yields that $\mathcal{M}_{1,\vec{1}}$ is not only a DM stack, but an algebraic space. One sees easily that the points have no non-trivial automorphisms. Then using a Weierstrass form analysis as in (4.16), one check that

$$\mathcal{M}_{1,\vec{1}} \cong [(\mathbb{A}^2 - \{\delta = 0\}) \times \mathbb{A}^1 - \{0\} \,/\, \mathbf{G}_m] = \mathbb{A}^2 - \{\delta = 0\}.$$

This closes the remark.

The geometry of a connected nodal curve is very similar to the smooth case, due to the fact that such a curve is locally a complete intersection, which implies that there is a locally free rank 1 *dualizing sheaf* ω_C [15], [31]. Substituting this sheaf in place of Ω^1_C, the Serre duality theorem holds true (see [31], Chap. 3, Section A) for details and references:

$$\mathrm{Ext}^j(\mathcal{F}, \omega_C) \xrightarrow{\sim} \mathrm{H}^{1-j}(C, \mathcal{F})^* \tag{4.18}$$

for any coherent sheaf $\mathcal{F}$. The genus of C is $g = \dim \mathrm{H}^0(C, \omega_C) = \dim \mathrm{H}^1(C, \mathcal{O}_C)$. This also implies the Riemann-Roch theorem: for all invertible module $\mathcal{L}$, we have

$$\chi(C, \mathcal{L}) = \dim \mathrm{H}^0(C, \mathcal{L}) - \dim \mathrm{H}^1(C, \mathcal{L}) = \deg \mathcal{L} + 1 - g. \tag{4.19}$$

The canonical sheaf ω_C can be described in simple terms. Let $\rho : \tilde{C} \to C$ the normalization. Each node P of C comes from the identification of two distinct points $P', P'' \in \tilde{C}$, the "branches" at P. Then ω_C is the subsheaf of the direct image of the sheaf of rational differentials ω on $\tilde{C}$, whose sections have at worst first-order poles only at the branches, and such that for all nodes $P \in C$

$$\mathrm{Res}_{P'}(\omega) + \mathrm{Res}_{P''}(\omega) = 0. \tag{4.20}$$

The sheaf ω_C is distinct from Ω^1_C, unless C is smooth. Let $\pi : C \to S$ be a nodal curve. Then the previous construction of the sheaf ω_C globalizes, this yields an invertible sheaf $\omega_{C/S}$, called the dualizing sheaf[43] of C/S. Its formation is compatible with any base change.

[43] There are more compact definitions. One is $\omega_C = \det \Omega^1_C$ [41].

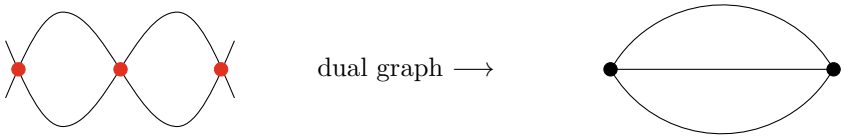


FIGURE 3. Nodal genus 2 curve

In order to enlarge the stacks $\mathcal{M}_g$ by singular curves, we must limit ourselves to singular curves with a finite automorphism group. The key definition is [15], Definition 1.1:

Definition 4.27. A **stable curve** over an algebraically closed field k is a proper connected nodal curve C/k such that $\mathrm{Aut}(C)$ is finite.

Assume given a nodal curve C with irreducible components $C_1, \ldots, C_r$ and nodes $P_1, \ldots, P_d$. Denote g_i the genus of the normalization $\tilde{C}_i$ of C_i. The normalization of C is

$$p : \tilde{C} = \coprod_i \tilde{C}_i \to C.$$

The preimage of P_i in $\tilde{C}$ is a pair of points of $\tilde{C}$, called the branches at P_i. Finally let ℓ_i be the number of branches which belong to $\tilde{C}_i$. There is a well-known formula for the genus $g = \dim \mathrm{H}^1(C, \mathcal{O}_C)$ of C:

$$g = \sum_i g_i + d - r + 1. \tag{4.21}$$

This follows easily from the following exact sequence:

$$0 \to \mathcal{O}_C \to p_*(\mathcal{O}_{\tilde{C}}) = \oplus_{i=1}^r \mathcal{O}_{\tilde{C}_i} \xrightarrow{\varphi} k^d \to 0 \tag{4.22}$$

where $\varphi = \oplus_j \varphi_j$, and φ_j is the map sending a section (f_i) near the two branches (P'_j, P''_j) of P_j to $f_{i_1}(P'_j) - f_{i_2}(P''_j)$ if $P'_j \in \tilde{C}_{i_1}$, $P''_j \in \tilde{C}_{i_2}$.

Lemma 4.28. *A connected nodal curve C is stable if and only if for all $i = 1, \ldots, r$, the following holds:*

$$2g_i - 2 + \ell_i > 0. \tag{4.23}$$

Exercise 4.29. *Prove the exactness of the sequence* (4.22)*, after that the genus formula* (4.21)*.*

Exercise 4.30. *Prove that on a stable curve there is no non zero global regular vector field, i.e.,* $\mathrm{Hom}_{\mathcal{O}_C}(\Omega^1_{C/k}, \mathcal{O}_C) = 0$.

It is convenient to encode the topological structure of a nodal curve into a graph, the so-called **dual graph**. The vertices are the irreducible components, and the arrows are in one to correspondence with the nodes. A node Q has for end points the two components[44] containing Q.

[44] An arrow is a loop if the node is a point of self-intersection of a component.

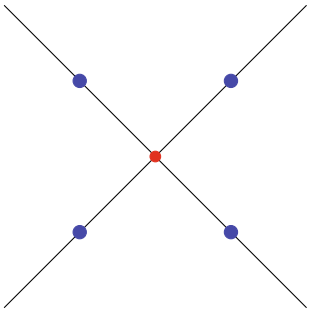


FIGURE 4. Stable curve of genus 0 curve with 4 marked points

We can also add marked points to a nodal curve, to relax somewhat the stability condition. A **marked (or pointed)** curve is a nodal connected curve together with a collection of n distinct labelled[45] *smooth* points $Q_1, \ldots, Q_n$. A nodal marked curve is called **stable** if the group of automorphisms of C preserving the n marked points is finite. This is equivalent to the condition (4.23) modified in the following way:

$$2g_i - 2 + \ell_i + m_i > 0 \tag{4.24}$$

where m_i stands for the number of marked points which belong to the component C_i. In the dual graph a marked point pictured by a monovalent arrow (a leg). Clearly if a stable curve of genus g with n marked point exists then either $g \geq 2$, or $g = 0$, $n \geq 3$, or $g = 1$, $n \geq 1$. The curve pictured below (Figure 1) is a genus 2 stable curve with two rational components meeting at three points with its dual graph.

One can notice that $r - d + 1 = \dim \mathrm{H}^1(\Gamma)$ is the number of cycles of the dual graph Γ of the curve C. When $g = 0$, then this number is 0, thus C is a tree of $\mathbb{P}^1$, the stability being the result of the marked point. Below a stable marked curve with $g = 0, n = 4$.

Exercise 4.31. *Prove that there are only finitely many graphs that occur as dual graphs of stable curves of genus g with n marked points* $(3g - 3 + n > 0)$.

With the definition of a stable curve in hand, we are ready to define the fibered category in groupoids whose objects are the stable curves of fixed genus g, with n marked points:

Definition 4.32. Let $S \in \mathbf{Sch}$. A stable curve (resp. a stable n-marked curve) of genus g over S, is a proper flat morphism $\pi : C \to S$, such that the geometric fibers $C_s = \pi^{-1}(s)$ are connected stable nodal curves with genus g, respectively together with n labelled sections $Q_i : S \to C$, such that the geometric fibers are stable with respect to the induced marking.

[45] We can also work with n unlabelled points.

The morphisms are the cartesian diagrams exactly as in the smooth case. In the presence of marked points we must add an obvious compatibility with these sections:

$$\begin{tikzcd} C' \arrow[r] \arrow[d] & C \arrow[d] \\ S' \arrow[r, "f"] \arrow[u, bend left, "Q'_j"] & S. \arrow[u, bend left, "Q_j"] \end{tikzcd}$$

If $\pi : C \to S$ is a stable curve of genus g with n marked points, there is a canonical rank 1 locally free sheaf $\omega_{C/S}$ on C called the **relative dualizing sheaf**, such that for all $s \in S$, we have $\omega_{C/S} \otimes k(s) = \omega_{C_s/k(s)}$. The formation of $\omega_{C/S}$ commutes with an arbitrary base change ([15], Section 1). It is not difficult to show that the stability condition is also equivalent to the fact that $\omega_{C/S}(\sum_{i=1}^n P_i)^{\otimes 3}$ is very ample, see [31].

Prior to the study of the definition and study of the stack $\overline{\mathcal{M}}_{g,n}$, we need some results about the deformations functor of a node, and of a stable marked curve. Our next goal is to show that a similar treatment of curves of genus $g \geq 1$ is possible. We need some preliminary results about the deformation functor of a node and of a nodal curve.

Roughly, one can say that a node has a very good deformation theory in Schlessinger's sense[46]. This means that a node $\mathcal{O} = k[[x, y]]/(xy)$ admits a **versal deformation** with parameter space the (formal) spectrum of $R_{\text{ver}} = k[[t]]$ (a formal disk), recall that we are working over $\mathbf{Sch}_k$. The versal effective deformation is explicitly known, given by

$$\operatorname{Spec} k[[x, y, t]]/(xy - t) \longrightarrow \operatorname{Spec} k[[t]]. \tag{4.25}$$

Clearly the tangent space to the versal deformation is 1-dimensional. An closer inspection of the deformation functor yields a natural identification between this tangent space and $\operatorname{Ext}^1_{\mathcal{O}}(\hat{\Omega}^1_{\mathcal{O}/k}, \mathcal{O})$, see [6].

To check that (4.25) is a versal deformation amounts to showing that if we are given a deformation R of the nodal algebra over $A \in \mathbf{Art}_k$, i.e., R is a flat A-algebra and $k[[t]]/(xy) \cong R \otimes_A k$, then one can find an isomorphism of A-algebras, but not a unique one

$$A[[x, y]]/(xy - a) \xrightarrow{\sim} R$$

for some a in the maximal ideal of A. If $x \mapsto \xi$, $y \mapsto \nu$, the pair (ξ, ν) with $\xi\nu = a$ is called a **formal system of coordinates** of the node. The ideals $R\xi$ and $R\nu$, up to a permutation, are independent of the choice of local coordinates, they define the **branches** of the node. This can be checked directly without appealing to general results about deformation theory of singularities of hypersurfaces [65]. The same description works over any complete noetherian local ring A, and yields the formal structure of a curve near a node:

[46] One can analyse more generally the deformation of a singular point of an hypersurface [6].

Proposition 4.33. *Let $\pi : C \to S$ be a proper flat nodal curve (the geometric fibers are connected curves with only nodes as singularities). Let P be a node of the fiber $C_s = \pi^{-1}(s)$. Then the complete local ring of C at P has the precise form*

$$\widehat{\mathcal{O}}_{C,P} \cong \widehat{\mathcal{O}}_{S,s}[[x,y]]/(xy - a) \tag{4.26}$$

for some a in the maximal ideal of $\widehat{\mathcal{O}}_{S,s}$. This description remains valid over a suitable étale neighborhood of s.

Proof. Usually this result is proved by using deep results such as Artin's algebraization theorem, see for example [6]. The reader will find an elementary proof in [65], Proposition 2.2.2. □

Exercise 4.34. *Show that a system of local coordinates for a node is unique up to a transformation $(\xi, \nu) \mapsto (u\xi, v\nu)$, $uv = \gamma \in A^*$ (see [65]).*

The next thing to do is to study the deformation functor of a a stable marked curve. This study fits into the general framework initiated in Subsection 3.1.3. Let us recall where we are going on. Suppose that $(C, (P_i)_{1 \le i \le n}$ is an n-marked stable curve over k.

Definition 4.35.

i) A **lift** of C to $A \in \mathbf{Art}_k$ (or $A \in \widehat{\mathbf{Art}_k}$) is a stable marked curve

$$\begin{tikzcd} \mathcal{C} \arrow[r] & \operatorname{Spec} A \arrow[l, bend right, "P_i"'] \end{tikzcd}$$

together with an isomorphism $C \xrightarrow{\sim} \mathcal{C} \otimes_A k$. Two lifts $\mathcal{C}_j \to \operatorname{Spec} A$ for $j = 1, 2$ are **equivalent** (or **isomorphic**) if there is a commutative diagram:

$$\begin{tikzcd} \mathcal{C}_1 \arrow[rr, "\sim"] & & \mathcal{C}_2 \\ & C. \arrow[ul, "\sim"] \arrow[ur, "\sim"'] & \end{tikzcd}$$

ii) A deformation of $(C, (P_i))$ to A is an equivalence class of lifts.

Denote $\operatorname{Def}_C(A)$ the set of deformations of C to A. This defines a covariant functor, the morphisms being induced by base change $\mathbf{Art}_k \longrightarrow \mathbf{Set}$. The tangent space to Def_C is the set $\operatorname{Def}_C(k[\epsilon])$, $\epsilon^2 = 0$. Schlessinger's theory (Theorem 3.31) works perfectly, and yields (see [15] for the case $n = 0$):

Theorem 4.36. *The deformation functor of a stable marked curve is pro-representable and smooth, i.e., there is a* **universal** *deformation with base the (formal) spectrum of a power series ring in $N = 3g - 3 + n$ variables. The tangent space is naturally identified with $\operatorname{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C(-\sum_{i=1}^n P_i))$.*

Proposition 4.37. *Assume given a connected nodal curve C/k with nodes $P_1, \ldots, P_d$. Then*

i) $\mathrm{Ext}^2_C(\Omega^1_C, \mathcal{O}_C) = 0$.

ii) *The natural global-to-local map*

$$\mathrm{Ext}^1_C(\Omega^1_C, \mathcal{O}_C) \longrightarrow \prod_{i=1}^{d} \mathrm{Ext}^1_{\hat{\mathcal{O}}_{P_i}}(\hat{\Omega}_{P_i}, \hat{\mathcal{O}}_{P_i}) \tag{4.27}$$

is surjective.

Proof. We refer to ([15], Proposition 1.5) for the proof when $n = 0$. The proof extends verbatim to the general case. Notice iii) shows that there is no non-zero regular vector field on C with a zero at each P_i. Consequently there is no non-trivial infinitesimal automorphism in a deformation, which in turn says that the versal deformation of the marked curve is *universal*. The vector space $\mathrm{Ext}^1_C(\Omega^1_C, \mathcal{O}_C)$ is the tangent space of the formal deformation ring, its dimension is $3g-3$. The right-hand side of (4.27) measures the contribution to the first-order deformations of C of the nodes. The surjectivity means that each node contributes to one parameter in a versal deformation of C. □

We are ready to show that the stable n-marked curves of fixed genus are parameterized by a smooth Deligne-Mumford stack, the so-called Knudsen-Mumford stack $\overline{\mathcal{M}}_{g,n}$. Theorem 4.7 holds true almost verbatim with stable curves instead of smooth curves. The result is:

Theorem 4.38. *The fibered category in groupoids whose objects are the stable curves of genus g and n marked points, is a smooth DM stack denoted $\overline{\mathcal{M}}_{g,n}$ of dimension*[47] $3g - 3 + n$ $(3g - 3 + n \geq 0)$. *The stack $\mathcal{M}_{g,n}$ is an open substack of $\overline{\mathcal{M}}_{g,n}$. There is a divisor with only normal crossings (the boundary) with support $\overline{\mathcal{M}}_{g,n} - \mathcal{M}_{g,n}$.*

Proof. We refer to [15] for details. Part of the first assertion follows from the structure of the sheaf $\mathrm{Isom}_S(C_1, C_2)$. One must prove that this sheaf is representable, more precisely is finite unramified over S. The representability is a special case of the existence of the Hilbert scheme, taking into account that if $\pi : C \to S$ is a stable curve, then π is projective. If $C_1 = C_2$ the group scheme $\mathrm{Aut}_S(C)$ which represents $\mathbf{Isom}_S(C, C)$ has a trivial Lie algebra. Indeed the tangent space at id of $\mathrm{Aut}_S(C)$ is canonically identified with the space of global regular vector fields on C. It is a trivial matter to check due to the stability condition, that there is no non-zero regular vector field. One can also prove that $\mathrm{Aut}_S(C) \to S$ is proper, this follows from the valuative criterion [33], then being quasi-finite, it is finite over S by the Chevalley theorem (loc. cit.), see ([15], Theorem 1.11) for details.

The last assertion follows from Proposition 4.37, ii). Indeed this says that a local chart, i.e., an étale neighborhood of a stable curve with d nodes is an open subset of an affine space with $3g-3+n = \dim \mathrm{Ext}^1(\Omega^1_C, \mathcal{O}_C(\sum_{i=1}^n Q_i))$ parameters

[47] The dimension of a noetherian DM stack is the dimension of an arbitrary atlas.

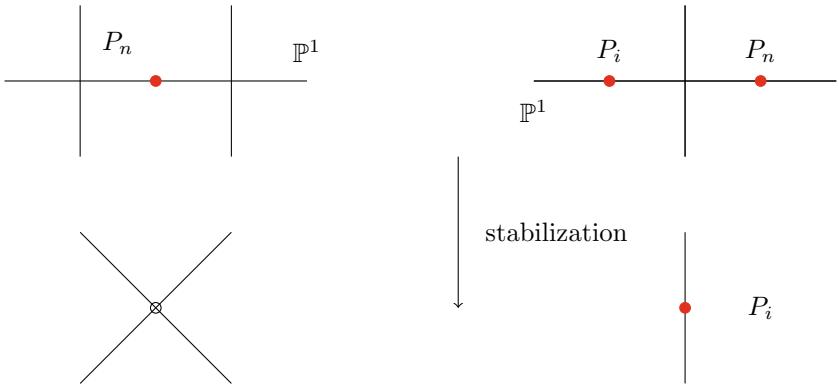


FIGURE 5. Stabilization

$t_1, \ldots, t_{3g-3+n}$, each node contributes for one parameter, says $t_1, \ldots, t_d$. The local equation of the boundary divisor is $t_1, \ldots, t_d = 0$. This shows the irreducible components of the boundary divisor are the closure of the different loci of stable marked point with only node. $\square$

As for the case of $\overline{\mathcal{M}}_{1,1}$, one can show that the stack $\overline{\mathcal{M}}_{g,n}$ is proper over $\operatorname{Spec}\mathbb{Z}$. This follows from a key result, extending the stable reduction theorem for elliptic curves, the so-called stable reduction theorem for curves, which is discussed in Romagny's talk [54]. The result is as follows:

Theorem 4.39. *Let R be a discrete valuation ring with fraction field K and residue field k. Let C/K be a smooth (stable) curve marked by n points. Then there is a finite extension K'/K, and a stable marked curve $\mathcal{C}'$ over the normalization R' of R in K', such that $\mathcal{C}' \otimes K' \cong C \otimes_K K'$.*

When marked points are concerned, there is an important morphism called **forgetting a marked point** of Knudsen ([41], Definition 1.3). Let $(C, (P_i)_{1 \leq i \leq n}) \in \operatorname{Ob}(\overline{\mathcal{M}}_{g,n})$. If we forget the point P_n, then we can lost the stability. This occurs when P_n is on a smooth rational component meeting the others components in exactly two points, or if there is some $i \in [1, n-1]$ such that P_i and P_n are the only marked points on a smooth rational component meeting the others in one point.

Once P_n is forgotten, we can contract the component $\mathbb{P}^1$ containing P_n to a point, the result is a stable curve with $n-1$ marked points, the images of $P_1, \ldots, P_{n-1}$. The key point is that this stabilization process works in family, thus gives rise to a 1-morphism of stacks (loc. cit.)

Theorem 4.40. *Forgetting the last point yields a 1-morphism*

$$\overline{\mathcal{M}}_{g,n} \longrightarrow \overline{\mathcal{M}}_{g,n-1} \quad (g + n \geq 4). \tag{4.28}$$

Proof. See Knudsen [41], Theorem 2.4. □

Exercise 4.41. *Prove that there is a locally free sheaf* $\mathbb{E}_g$ *of rank* g *on* $\mathcal{M}_g$ $(g \geq 2)$ *with "fiber" at the section* $(C \to \operatorname{Spec} k) \in \overline{\mathcal{M}}_g$ *the vector space* $\Gamma(C, \Omega^1_C)$*. This is the so-called Hodge bundle. Show that this vector bundle extends to* $\overline{\mathcal{M}}_g$*. See* [30]*, Section* 5.4 *for the case* $g = 1$*, i.e.,* $\mathcal{M}_{1,1}$*.*

4.2. Hurwitz stacks

4.2.1. Hurwitz stacks: smooth covers. Hurwitz stacks parameterize covers between smooth more generally stable curves, with fixed genus, and fixed ramification datum. Our goal is to focus on the geometric aspects of Hurwitz stacks. The arithmetic questions are the subject of Dèbes' lectures [12].

To begin with, the ingredients for the construction of Hurwitz stacks are a DM stack $\mathcal{M}$, and a finite constant group G. Throughout, we work over the site $(\mathbf{Sch}_k)_{et}$ of schemes over a fixed ground field k. It will be assumed that $|G| \neq 0 \in k$, i.e., G is reductive[48]. The first step is the construction of an auxiliary stack $\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})$.

Define $\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})(S)$ as the groupoid $\mathrm{Hom}(\mathbf{BG} \times S, \mathcal{M} \times S)$ whose objects are the 1-morphisms, and the (iso)morphisms are the 2-isomorphisms. It is clear how to define the "pullback" of a section by a morphism $S' \to S$ of $\mathbf{Sch}_k$, this is simply a base change $f^*(F) = F \times_S S'$. The notation $\mathcal{M} \times S$ stands for the stack $\mathcal{M} \times_{\mathcal{S}/S} S$ over $\mathbf{Sch}/S$ (Exercise 2.10). We have $\mathbf{BG} \times S = B(G \times S/S)$.

There is a general existence theorem for Hom-stacks due to Olsson [49], which in this very special case asserts that $\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})$ is a DM stack. This can be seen rather easily once the stack $\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})$ reinterpreted[49].

If we think of $\mathbf{BG} = [\operatorname{Spec} k/G]$ as a quotient, one can expect that a section over S of $\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})$ is the same thing that a morphism $\operatorname{Spec} k \to \mathcal{M}$ which is "invariant" by G ([54], Theorem 3.3). This can be readily seen. Suppose that $F : \mathbf{BG} \times S \longrightarrow \mathcal{M} \times S$ is a 1-morphism. Then $F(S \times G \to S) = x \in \mathcal{M}(S)$. The group G acts on the trivial bundle $S \times G \to S$ by left translations. The functor F converts this action into a morphism $\rho : G \to \mathrm{Aut}(x)$. Conversely if we are given such datum $(x \in \mathcal{M}(S), \rho : G \to \mathrm{Aut}(x))$, it is not difficult to extend it to a morphism $F : \mathbf{BG} \times S \to \mathcal{M} \times S$, thus providing an inverse functor to the previous one. Indeed let $P \to T$ be a section of $\mathbf{BG}$ over $T \in \mathbf{Sch}_S$. Let us describe this bundle by a cocycle of gluing functions $g_{ij} : T_{ij} \to G$ relatively to an étale covering $(T_i \to T)_i$. Let $x' = (x_i)_i$ be the pullback of x to $T' = \coprod_i T_i$. Restricting to T_{ij} we have two canonical isomorphisms, i.e., a canonical descent datum

$$x_{i|T_{ij}} \xrightarrow{\sim} x_{|T_{ij}} \xleftarrow{\sim} x_{j|T_{ij}}. \tag{4.29}$$

We can twist (4.29) composing with $\rho(g_{ij}) : T_{ij} \to \mathrm{Aut}(x_{|T_{ij}})$, this yields a new descent datum on x', in turn a new object $x^P = F(P \to T) \in \mathcal{M}(T)$. This construction is analogous to the construction of the twist quotient $P \times^G F$ (see Section 2.2). Thus the objects of $\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})$ are the pairs $(x, \rho : G \to \mathrm{Aut}(x))$,

[48] More generally, we can take as ground ring $\mathbb{Z}[1/|G|]$.

[49] Another interpretation is $\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M}) = \mathcal{M}^G$ the stack of fixed points where $\mathcal{M}$ is viewed as a "G-stack", the action of G being trivial !, see [54] Definition 2.1 and Corollary 3.11.

the morphisms $(x,\rho) \to (x',\rho')$ over $f : S \to S'$ being the morphisms $x \to x'$ over f which are G-equivariant in an obvious sense.

Let $p : G \to G'$ be a morphism of groups. There is an obvious 1-morphism

$$\underline{\mathrm{Hom}}(\mathbf{BG}', \mathcal{M}) \longrightarrow \underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M}) \tag{4.30}$$

It is given by the composition $\mathbf{BG} \longrightarrow \mathbf{BG}' \longrightarrow \mathcal{M}$ (2.18). On the other hand it maps (x,ρ') to $(x, \rho = \rho'.p)$. Finally this discussion extends Example 2.24.

Lemma 4.42. *Suppose that p is a surjection with kernel H, then* (4.30) *is a closed immersion.*

Proof. Let there be given $(x, \rho : G \to \mathrm{Aut}(x))$ an object of $\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})(S)$. A section over $f : T \to S$ of the 2-fiber product $\underline{\mathrm{Hom}}(\mathbf{BG}', \mathcal{M}) \times_{\underline{\mathrm{Hom}}(\mathbf{BG},\mathcal{M}),(x,\rho)} S$ is a datum $(y, (\sigma_{g'})) \in \underline{\mathrm{Hom}}(\mathbf{BG}', \mathcal{M})(T)$ together with a G- isomorphism $\varphi : \xrightarrow{\sim} f^*(x)$. In other words this is equivalent to the datum of $f : T \to S$, together with the constraint $f^*(\rho_g) = 1$ for all $g \in H$. This is best understood with the diagram

$$\begin{tikzcd} & \mathrm{Aut}_S(s) \arrow[d, bend left] \\ T \arrow[r, "f"] & S. \arrow[u, bend left, "\rho_g"'] \end{tikzcd}$$

Since $\mathrm{Aut}_S(x)$ is an S-algebraic group, this functor is clearly represented by a closed subscheme of S, precisely f must factors through the largest closed subscheme on which the equality $\rho_h = 1$ holds for all $h \in H$. $\square$

The stack we are interested in is the substack of $\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})$ whose sections are the $(x, \rho : G \to \mathrm{Aut}(x))$ with ρ injective, i.e., ρ yields a faithful action of G. Due to Lemma 4.42 this is the open substack

$$\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M}) - \bigcup_{1 \neq H \triangleleft G} \underline{\mathrm{Hom}}(\mathbf{B}(G/H), \mathcal{M}) \tag{4.31}$$

the union being taken over the normal proper subgroups.

Definition 4.43. The Hurwitz stack $\mathcal{M}(G)$ classifying the objects of $\mathcal{M}$ equipped with a faithful G-action, is the open substack (possibly empty) given by (4.31).

The stack $\underline{\mathrm{Hom}}(BG, \mathcal{M})$ is equipped with a natural morphism

$$\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M}) \to \mathcal{M}$$

given by forgetting G, viz. $(x, \rho) \mapsto x$:

$$\begin{tikzcd} \underline{\mathrm{Hom}}(BG, \mathcal{M}) \arrow[r] & \mathcal{M} \\ \mathcal{M}(G). \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

Proposition 4.44. *Under the previous assumptions,*

1. *The stack* $\underline{\mathrm{Hom}}(BG, \mathcal{M})$ *is a DM stack and the morphism* $\underline{\mathrm{Hom}}(BG, \mathcal{M}) \to \mathcal{M}$ *is representable, finite if* $\mathcal{M}$ *has a finite diagonal.*
2. *Assume that* $\mathcal{M}$ *is proper, with finite diagonal* (*i.e., separated*)*, then* $\underline{\mathrm{Hom}}(BG, \mathcal{M})$ *is proper.*
3. *The stack* $\mathcal{M}(G)$ *is a DM stack and the morphism* $\mathcal{M}(G) \longrightarrow \mathcal{M}$ *is representable, finite* (*unramified*) *if* $\mathcal{M}$ *has a finite diagonal.*

Proof. Let us prove 1). Given $(x, (\rho_g))$ and $(x', (\rho'_g))$ two sections over S of $\underline{\mathrm{Hom}}(BG, \mathcal{M})$, the sheaf $\mathbf{Isom}_S(x, (\rho_g)), (x', (\rho'_g)))$ is indeed the subsheaf with sections over T, the $\xi \in \mathbf{Isom}_S(x, x')(T)$, such that $\xi\rho_g = \rho'_g\xi$ for all $g \in G$. Clearly this is a closed subscheme.

Next we need to exhibit an étale atlas. Suppose that $R \underset{d_1}{\overset{d_0}{\rightrightarrows}} U \xrightarrow{p} \mathcal{M}$ is an étale presentation of $\mathcal{M}$. Let us introduce the subscheme of $U \times R^G$ ($R^G = R \times G = R \times \cdots \times R$) whose T-points are the tuples $(y, (\rho_g))$, where $y \in U(T), \rho_g \in R(T)$ and for all $g \in G$, $d_0(\rho_g) = d_1(\rho_g) = y$, for all $g, h \in G$, $\rho_g \circ \rho_h = \rho_{gh}$ (composition in the groupoid), and finally $\rho_1 = 1_y$, the unity at y. There is a natural morphism $V \to \underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})$ sending $(u, (\rho_g))$ to $(x = p(u), (\rho_g))$. We want to check this morphism is an étale epimorphism. Let $(x, (\sigma_g)) \in \underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})(S)$, and let (h, f) be a T-point of the fiber product $V \times_{\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})(S)} S$, that is a commutative square up isomorphism

$$\begin{tikzcd} V \arrow[r, "p"] & \underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})(S) \\ T \arrow[u, "h"] \arrow[r, "f"] & S. \arrow[u, "(x,(\sigma_g))"'] \end{tikzcd}$$

Let $\theta : p(y) \xrightarrow{\sim} f^*(x)$ the equivariant isomorphism, part of the datum. Then with the isomorphism θ alone, we can recover the ρ_g's, indeed $\rho_g = \theta^{-1} f^*(\sigma_g)\theta : T \to R = U \times_{\mathcal{M}} U$. Thus

$$V \times_{\underline{\mathrm{Hom}}(\mathbf{BG}, \mathcal{M})(S)} S \cong U \times_{\mathcal{M}} S.$$

This shows that V is an atlas, thereby proving 1).

We are going to check that $\underline{\mathrm{Hom}}(BG, \mathcal{M}) \to \mathcal{M}$ is representable. Notice that this indirectly implies the first assertion. Take $x \in \underline{\mathrm{Hom}}(BG, \mathcal{M})(S)$, and perform the fiber product

$$\begin{tikzcd} \underline{\mathrm{Hom}}(BG, \mathcal{M}) \arrow[r] & \mathcal{M} \\ \mathcal{M}(G) \times_{\mathcal{M}} S \arrow[u] \arrow[r] & S. \arrow[u, "x"'] \end{tikzcd}$$

A section over $f : T \to S$ of this 2-fiber product is given by $(y, \rho : G \hookrightarrow \mathrm{Aut}(y))$ together with an isomorphism $\theta : y \cong f^*(x)$. It is readily seen that this fiber product is equivalent to the fibered category whose groupoid of sections over $f : T \to S$ is $\mathrm{Hom}(G, \mathrm{Aut}(x) \times_S T)$. The sheaf $\mathrm{Aut}(x)$ is an algebraic group of finite

type over S, then it is a simple exercise to prove that the presheaf $T \mapsto \mathrm{Hom}(G \times T, \mathrm{Aut}(x) \times_S T)$ is a scheme. If $\mathrm{Aut}(x)$ is a finite group scheme, it is also finite.

2) If $\mathcal{M}$ is of finite type the proof of 1) yields that $\underline{\mathrm{Hom}}(BG, \mathcal{M})$ is of finite type. The assertion 2) amounts to checking the valuative criterion of properness (Definition 3.66). Let R be a discrete valuation ring with fraction field K, and residue field k. Let $(x, (\rho_g))$ be a section of $\underline{\mathrm{Hom}}(BG, \mathcal{M})$ over K. Since $\mathcal{M}$ is proper, after a suitable finite extension K'/K, the section x extends to the normalization R' of R in K'. Thus we may assume that x is a section defined over R. Then the R group scheme $\mathrm{Aut}(x)$ is by assumption finite unramified, thus the sections ρ_g over K extend uniquely to the whole R, which in turn says that $(x, (\rho_g))$ extends to R.

3) All follows readily from 1) and 2), unlike the fact that $\mathcal{M}(G) \to \mathcal{M}$ is finite. We know that $\mathcal{M}(G)$ is open in $\underline{\mathrm{Hom}}(BG, \mathcal{M})$, but assuming that $\mathcal{M}$ is a DM stack of finite type over $\mathbf{Sch}_k$, in particular with an unramified diagonal (Proposition 3.3), we infer that $\mathcal{M}(G)$ is also closed in $\underline{\mathrm{Hom}}(BG, \mathcal{M})$. Let $(x, (\rho_s)_{s \in G})$ be a section of $\underline{\mathrm{Hom}}(BG, \mathcal{M})$ over S, with S connected, then our claim amounts to checking that if two automorphisms ρ_{s_i}, $i = 1, 2$ coincide schematically at some point $s \in S$, they are equal. This is a key property of unramified morphisms, which follows quickly from the fact that the diagonal of an unramified morphism is open ([62], Lemma 02GE). □

The stacks $\mathcal{M}(G)$ have interesting functorial properties with respect to G. Let $G_1 \to G_2$ be a morphism, which in turn yields a 1-morphism $\mathbf{BG}_1 \to \mathbf{BG}_2$ (Exercise 2.22). Composing with this morphism yields a 1-morphism $\mathcal{M}(G_2) \longrightarrow \mathcal{M}(G_1)$. Assuming $G_1 \to G_2$ surjective with kernel H, we would like a morphism going in the opposite direction. We must for this kill the automorphisms $\rho(h) \in \mathrm{Aut}(x)$, $h \in H$. This will be possible with covers.

Exercise 4.45. *Show a* 1*-morphism* $F : \mathbf{BG} \to \mathcal{M}$ *represented by* (c, ρ) *is representable if and only if* ρ *is injective* (*compare with Exercise* 2.31).

The application we have in mind is to $\mathcal{M} = \mathcal{M}_g$ $(g \geq 2)$. Let G be a finite group with order $|G|$. To avoid future complications with wild group actions, it will be safer to assume from now that $\mathcal{M}_g$ is a stack over $\mathbb{Z}[\frac{1}{|G|}]$. An object over S of the DM stack $\mathcal{M}_g(G)$ is a pair

$$(p : C \to S, \rho : G \hookrightarrow \mathrm{Aut}_S(C))$$

where ρ is an embedding. Call such a pair a G-curve of genus g. A morphism of G-curves is a cartesian diagram

$$\begin{array}{ccc} C' & \xrightarrow{\phi} & C \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p} \\ S' & \xrightarrow{f} & S \end{array} \tag{4.32}$$

where the upper horizontal arrow ϕ is required to be G-equivariant. When $S' = S$, $f = 1$, an isomorphism is a G-equivariant isomorphism $C \xrightarrow{\sim} C'$.

Definition 4.46. The DM stack $\mathcal{M}_g(G)$ will be called the Hurwitz stack parameterizing the smooth genus g curves together with a faithful G-action. It will be denoted $\mathcal{H}_{g,G}$.

Notice that the 1-morphism $\mathcal{H}_{g,G} \to \mathcal{M}_g$, forgetting the group G, is representable, finite (Proposition 4.44). There is a useful variant of Definition 4.46. We can enrich the pair (C, ρ) by adding a reduced G-invariant divisor $D = \sum_{i=1}^{n} P_i$, i.e., $g(D) = D\ \forall g \in G$. Namely

Definition 4.47. An object of the stack $\mathcal{M}_{g,(n)}(G)$ over S (if $3g - 3 + n > 0$), is a triple $(\pi : C \to S)$ a smooth curve of genus g, together with a relative Cartier divisor $D \subset C$ étale over S with degree n, and a faithful G-action on C preserving D.

The morphisms of $\mathcal{M}_{g,(n)}(G)$ are clear. In the cartesian diagram (4.32) the morphism ϕ is required to maps D' onto D. It is straightforward to check that this fibered category in groupoids is a DM stack. This stack parameterizes the smooth curves of genus g equipped with a faithful action of G, together with a G-invariant collection of n **unordered** points, i.e., marked G-curves. The marked points are permuted by the G-action, therefore cannot be labeled.

In order to study families of G-Galois covers, it is important to manage the quotient by the finite group G in families. Let $p : C \to S$ be an object of $\mathcal{M}_g(G)$. The projectivity of p ensures that the quotient of C by G makes sense (Proposition 3.40). It is however not clear if $D = C/G$ is again a flat family of curves with the commutation rule $D_s = C_s/G$. In general this is a rather subtle problem, see the discussion in ([37], Appendix to Chap. 7), and ([7], Theorem 3.10). A key assumption is the fact that G acts freely at the generic points of the geometric fibers. For a family of smooth (labelled or not) curves the fact that the automorphisms group scheme is unramified ensures this condition, thus providing us with a smooth curve $D = C/G \to S$, and a canonical morphism $\pi : C \to D$. We can state (without proof) a key technical result:

Proposition 4.48. *Under the preceding conditions, the quotient $C/G \to S$ is a flat family of curves, furthermore this quotient commutes with an arbitrary base change, namely $(C \times_S S')/G \cong (C/G) \times_S S'$ canonically.*

The provocative remark that explains the result is if a cyclic group G of order N acts faithfully on $A[T]$ by $T \mapsto \zeta T$ for some root of the unity, then $A[T]^G = A[N(T)]$, where $N(T) = T^N$ is the norm of T. The commutation with any base change in this toy example is clear.

Returning to our setting, suppose that (f, ϕ) is a morphism as in (4.32), then it gives rise to a commutative diagram

$$\begin{tikzcd} C' \arrow[r, "\phi"] \arrow[d, "\pi'"'] & C \arrow[d, "\pi"] \\ D' \arrow[r, "h"] & D. \end{tikzcd} \tag{4.33}$$

This shows that we can think of the objects of $\mathcal{M}_g(G)$ as G-Galois covers

$$\pi : C \to D \cong C/G \tag{4.34}$$

with the faithful action of G on C being part of the datum, and the morphism π identifying D with C/G. In this framework morphisms are given by the diagrams (4.33). This is the **reduced** variant of [12], Section 1.6.

It is natural to fix the genus of $D = C/G$. Indeed in a flat family of smooth projective curves, the genus, i.e., the Euler characteristic $\chi(C_s, \mathcal{O}_s)$, is locally constant. When $S = \operatorname{Spec} k\,(k = \overline{k})$, we know the genus of C and that of $D = C/G$ are related by the Riemann-Hurwitz formula

$$2g_C - 2 = |G|(2g_D - 2) + \deg(R) \tag{4.35}$$

where R denote the ramification divisor. If $e(P)$ stands for the ramification index at a point P, i.e., $e(P) = |G_P|$, recall that P is called a ramification point if $e(P) > 1$. Then we set

$$R = \sum_{P \in C} (e(P) - 1)P$$

In the relative situation R makes sense as a relative Cartier divisor defined by the equality, the ramification formula

$$\Omega^1_C \otimes \pi^*({\Omega^1_D}^{-1}) = \mathcal{O}_C(R). \tag{4.36}$$

One can use Lemma 4.50, i). In a more sophisticated form (see [40], [48]):

$$R = \det(\pi^*(\Omega^1_D \to \Omega^1_C).$$

The divisor $B = \pi_*(R)$ is the branching divisor. The multiplicities involved in R can be readily seen locally constant along the geometric fibers, which in turn says they are constant if S is connected. This suggests that if you want to limit the size of the Hurwitz stack, it will be convenient to fix the **ramification datum**.

Definition 4.49. Let $\pi : C \to D = C/G$ be a Galois cover defined over the algebraically closed field k. The **local monodromy** at a branch point $Q \in D$ is the conjugacy class of the pair (H, χ) where $H = G_P$ is the stabilizer of $P \in \pi^{-1}(Q)$, and $\chi : G_P \to k^*$ is the character of the 1-dimensional faithful representation of H afforded by the cotangent space at P. Then H is cyclic, and the order of the character χ is $e = e(P) = |H|$, the ramification index.

It will be convenient to label the branch point $Q_1, \ldots, Q_b$, and then to denote $[H_i, \chi_i]$ the local monodromy at Q_i. The brackets mean the pair is considered up to conjugacy. We say that the pairs (H, χ), (H', χ') are **conjugate** if for some $s \in G$, we have $H' = sHs^{-1}$ and $\chi'(t) = \chi(s^{-1}ts)$ for all $t \in H'$.

Suppose given a coherent systems of N-roots of the unity, where $N = |G|$. Then it is readily seen that a conjugacy class $[H, \chi]$ can be identified with the conjugacy class $C = [g] \subset G$ where $H = \langle g \rangle$ and $\chi(g) = \zeta_N^{N/e}$. The monodromy type of the cover $\pi : C \to D$ or the **Hurwitz (or ramification) datum** is the

collection of labelled conjugacy classes $\xi = \{[H_i, \chi_i]\}$, equivalently an ordered collection of conjugacy classes $(C_1, \ldots, C_b)$ of G. It is important to be able to work with families of G-curves or covers, i.e.,

$$\begin{tikzcd} C \arrow[rr, "\pi"] \arrow[dr, "p"'] & & D = C/G \arrow[dl, "q"] \\ & S. & \end{tikzcd}$$

We now focus on families of Galois covers. We begin by collecting two elementary but very useful remarks. Recall that if a finite group G acts on a scheme X, that the fixed points subscheme X^G is the closed subscheme such that G acts trivially on it, and any equivariant morphism $f : T \to X$, where G acts trivially on T factors through X^G. The sheaf of ideals of X^G is locally generated by the sections $g(f) - f$, for all $g \in G$ and all sections f of $\mathcal{O}_X$.

Lemma 4.50. *Assume that $\pi : C \to D$ is a smooth*[50] *G-cover over a connected base scheme S.*

i) *Let H be a cyclic subgroup of G. The fixed points subscheme C^H is a relative Cartier divisor (over S).*
ii) *The Hurwitz datum is constant along the geometric fibers.*

Proof. See [8], Proposition 3.1.1 and Lemme 3.1.3 for more details. We just check briefly i). If $x \in C$ is a fixed point with $\pi(x) = s$, then due to the tameness of the action of G, we can at least formally, linearized the action at x, i.e., after a faithfully flat extension assume that H acts on $\widehat{\mathcal{O}}_x = \widehat{\mathcal{O}}_s[[t]]$ by $t \mapsto \sigma(t) = \zeta t$, where ζ is root of the unity of order $e = |H|$, and $H = \langle \sigma \rangle$. Then the equation of C^H at x is $(\sigma(t) - t = (\zeta - 1)t = 0$. This proves i). Finally ii) can be deduced from i). □

Exercise 4.51. *Let H be a cyclic subgroup of G. Show one can define a locally closed subscheme Δ_H whose points are the points with exact isotropy H.*

The previous remark shows that it will be convenient to fix the Hurwitz datum when dealing with a moduli problem of covers. Before we define the Hurwitz stack it is time to discuss one point of terminology about the classification of covers.

First we use the letter ξ to denote the Hurwitz datum. Recall that we are working with a G-curve (smooth for the moment) that is a curve equipped with a faithful action of a fixed finite group G, i.e., a section of $\mathcal{M}_g(G)$. We can think of this stack as the classifying stack of G-cover $\pi : C \to D = C/G$, where G-cover means that the action of G on C is taken into account. Important is the description

[50] The curve C (therefore D) is smooth over S.

of morphisms. A morphism of $\mathcal{M}_g(G)$ over $f : S' \to S$, viz. a diagram

$$\begin{tikzcd} C' \arrow[r, "\phi"] \arrow[d, "p'"'] & C \arrow[d, "p"] \\ S' \arrow[r, "f"] & S \end{tikzcd}$$

with ϕ G-equivariant, will be seen as a morphism of G-covers

$$\begin{tikzcd} C' \arrow[rr, "\phi"] \arrow[dr, "\pi'"] \arrow[dd, "p'"'] & & C \arrow[dr, "\pi"] \arrow[dd, "p"] & \\ & D' \arrow[rr, "h"] \arrow[dl] & & D \arrow[dl] \\ S' \arrow[rr, "f"] & & S. & \end{tikzcd} \tag{4.37}$$

Since h is uniquely provided by ϕ, we see these two definitions yields equivalent stacks, i.e., forgetting D is the equivalence. When D is of genus 0, this should be compared with a (slightly) different stack, with sections over S the G-covers $\pi : C \to \mathbb{P}^1_S$ but for which a morphism is a diagram (4.37) in which $h : \mathbb{P}^1_{S'} \to \mathbb{P}^1_S$ is the canonical morphism. Compare with the definitions in [12], Section 1.1. In this stack an automorphism of the G-cover $\pi : C \to D = \mathbb{P}^1$ defined over $k = \overline{k}$ is an element of $Z(G)$ the center of G, which in turn shows that $\mathcal{M}_g(G)$ is an algebraic space[51] if $Z(G) = 1$.

Suppose now that our moduli problem deals with marked curves, i.e., C is marked by a G-invariant reduced divisor $\sum_{i=1}^n P_i$ (Definition 4.47). It will be convenient to assume that this divisor contains the ramification divisor R. For this reason we can write it R, even if R is larger than the ramification divisor. As a sum of G-orbits, we can define the extended Hurwitz datum ξ or R. The (extended) Hurwitz datum is the old Hurwitz datum plus the number of free orbits. We can see this as a sum

$$\xi = \sum_{i=1}^{b} [H_i, \chi_i], \tag{4.38}$$

i.e., a collection of unlabelled conjugacy classes of pairs $[H, \chi]$. Obviously a free orbit contributes by the trivial class $H = 1$. The image of a G-orbit contained in R will be called a branch point, even if the orbit is free. The genus of C and g' of D are related by the Riemann-Hurwitz formula:

$$2g - 2 = |G| \left(2g' - 2) + \sum_{i=1}^{b} (1 - \frac{1}{|H_i|}) \right). \tag{4.39}$$

[51] It is a scheme.

Clearly one can change the picture, considering these G-orbits, equivalently the branch points, as labelled. We will try to make the distinction between these two settings clear.

Over a general base S, the marked points are unlabelled sections $P_i : S \to C$. We can argue as in Lemma 4.50 to check that the (extended) Hurwitz datum ξ is invariant along the geometric fibers in a family of covers over a connected base. In conclusion, a reasonable definition of the Hurwitz stack of G-covers marked by a divisor, étale over the base, of fixed Hurwitz datum ξ is:

Definition 4.52. The Hurwitz stack $\mathcal{H}_{g,G,\xi}$ is the stack parameterizing the G-covers $\pi : C \to D$ where C and D are smooth projective curves, and g is the genus of C, with Hurwitz datum ξ. We have two moduli stacks, in one the branch points are labelled, for the other they are not.

The morphisms of the fibered category $\mathcal{H}_{g,G,\xi}$ are those described by the diagrams (4.37), but preserving the marking, i.e., the divisor. In a cover $\pi : C \to D$ over a field k, the ramification points are always in some sense distinguished. Recall we are assuming that the marked points contain the ramification points. If $\pi : C \to D$ is such a G-cover marked by an invariant divisor R, then G acts freely on C minus R. In this setting the genus of $D = C/G$ is known, and given by the Riemann-Hurwitz formula (4.39). In the same way we prove that $\mathcal{M}_{g,n}$ is a DM stack, we can check:

Proposition 4.53. *The stack $\mathcal{H}_{g,G,\xi}$ (with branch points labelled or not) is a Deligne-Mumford stack.*

Caution: the DM stack $\mathcal{H}_{g,G,\xi}$ is not necessarily connected. It appears as the union of a selected set of connected components of the bigger stack $\mathcal{M}_{g,n}(G)$, and

$$\mathcal{M}_{g,n}(G) = \coprod_{\xi} \mathcal{H}_{g,G,\xi} \tag{4.40}$$

the disjoint union running over all admissible types ξ, τ. Let $p : C \to S$, $P_i : S \to C$ be an object of $\mathcal{H}_{g,G,\xi}$. Let Q_j $(1 \leq j \leq b)$ be the distinct images of the P_i's. Recall that this lead to two moduli problem according to the fact that the branch points are labelled, or unlabelled. In the sequel, without further specification, the branch points will be **labelled**. Therefore the curve $D = C/G$ marked by the "branch points" Q_j's is a section of $\mathcal{M}_{g',b}$, where g' is the genus given from ξ by the Riemann-Hurwitz formula (4.39). We get in this way a very important 1-morphism

$$\delta : \mathcal{H}_{g,G,\xi} \longrightarrow \mathcal{M}_{g',b} \tag{4.41}$$

called the **discriminant morphism**. This morphism plays a fundamental role in the understanding of $\mathcal{H}_{g,G,\xi}$. It will be proved in the next section that δ is proper quasi-finite, but not representable in general. Despite this δ has a well-defined degree, in a stacky sense, which is called the **Hurwitz number**. If we forget the group G we get a (finite) morphism $\mathcal{H}_{g,G,\xi} \longrightarrow \mathcal{M}_{g,(n)}$.

As a consequence it is expected that $\dim \mathcal{H}_{g,G,\xi} = 3g' - 3 + b$. Notice that the automorphism group of a geometric point $\pi : C \to D$ is the center $Z(G)$ of

G, therefore if $Z(G) = 1$, $\mathcal{H}_{g,G,\xi}$ is an algebraic space. This favorable fact will no longer be true if we pass to the stable compactification (Subsection 4.2.2). If we forget the group G we get a (finite) morphism $\mathcal{H}_{g,G,\xi} \longrightarrow \mathcal{M}_{g,(n)}$.

Finally the Hurwitz scheme will be seen as a correspondence between two stacks of marked curves

$$\begin{tikzcd} & \mathcal{M}_{g,(n)} \\ \mathcal{H}_{g,G,\xi} \arrow[ur] \arrow[dr] & \\ & \mathcal{M}_{g',b}. \end{tikzcd}$$

Despite the fact that the stack $\mathcal{H}_{g,G,\xi}$ is generally not connected, it will be proved below that it is smooth, therefore the connected components are the same as the irreducible components. The number of these connected components is the so-called **Nielsen number**. This number is topological in nature, and has an expression in terms of a Hurwitz braid group action on the Nielsen classes (see [24] or [12], Section 1.3). Interesting examples and methods to separate the orbits have been produced by Fried, Serre and others, see [26], and below for a brief introduction to the **spin invariant**.

Let us now focus on some examples.

Example 4.54. *Elliptic curves revisited.* The slogan is that the modular elliptic curves (as stacks) are Hurwitz stacks for suitable groups and Hurwitz data. We will illustrate this with two examples.

Let us try to describe the Hurwitz stack that parametrizes the pairs (C, σ) where C is a smooth curve of genus 1, and τ is an involutive automorphism with 4 fixed labelled points, assuming that the ground field k has odd characteristic. Notice that once σ has a fixed point, then there are exactly 4 fixed points. Let $p : C \to S$, $P_i : S \to C$ be a section over S. Pick the first point $P_1 = O$ as origin to see C as an elliptic curve, therefore an S-abelian scheme ([37], Chap. 2). Then the P_i's are the points of order 2 of the S-group scheme $C \to S$. Therefore our moduli problem is the same as the choice of a group isomorphism

$$(\mathbb{Z}/2\mathbb{Z})^2 \xrightarrow{\sim} C[2] \tag{4.42}$$

that is of a so-called 2-level structure. This is the moduli problem known as the Legendre normal form of an elliptic curve, briefly discussed in Remark 4.19. The result is $\mathcal{H}_{1,\mathbb{Z}/2\mathbb{Z},4} \cong [S/\,\mathbf{G}_m]$ with

$$S = \left\{ (e_1, e_2, e_3) \in \mathbb{A}^3,\ e_i \neq e_j, \sum_i e_i = 0 \right\}$$

and the weight of the e_i's equal to 2.

It is interesting to extend this example to the study of the Hurwitz stack of cyclic covers of the line $\mathbb{P}^1$, i.e., $G = \mathbb{Z}/d\mathbb{Z}$, with 4 distinct branch points. The Hurwitz datum is encoded into 4 numbers (a_1, a_2, a_3, a_4) such that $\sum_1^4 a_i \equiv$

$0 \pmod d$. The ramification index at Q_i is $e_i = \frac{d}{(d,a_i)}$. Denote $\mathcal{H}_{a_1,a_2,a_3,a_4}$ the corresponding Hurwitz stack. The discriminant $\delta : \mathcal{H}_{a_1,a_2,a_3,a_4} \longrightarrow \mathcal{M}_{0,4}$ is an isomorphism at the coarse moduli space level, but not at the stacks level. See [22], [10] for nice variations along these lines.

Exercise 4.55. *Prove that* $\mathcal{M}_{1,1} \cong [S/\,\mathbf{G}_m \times S_3] \cong [\mathbf{G}_m \times \mathbb{A}^1/\,\mathbf{G}_m \times S_3]$, *where* S *is as before*

Remark 4.56. Besides the moduli stack $\mathcal{H}_{g,G,\xi}$ denoted $\mathcal{H}^{in}_{g,G,\xi}$ in Fried's notations, it is worth recalling that Fried suggested a variant in which the Galois group of the cover is not identified to G, see [24] or [25]. Equivalently, the morphisms are no longer G-equivariant. This defines a new moduli stack $\mathcal{H}^{ab}_{g,G,\xi}$, the so-called "absolute" moduli stack. Clearly this definition takes place in the general setting $\mathcal{M}(G)$ (4.31). The distinction between the *in* and *ab* moduli stacks is the same as between the modular elliptic curves Y_0 and Y_1 ([61], Appendix C, § 13).

The objects of the category $\mathcal{H}^{ab}_{g,G,\xi}$ are the Galois covers $\pi : C \to D$ with Galois group isomorphic to G, as considered previously, but now we relax the isomorphism between G and the Galois group. On the other hand if we think of the ramification datum as a collection of labelled conjugacy classes $C_1, \ldots, C_b$, denoting $\mathrm{Aut}_\xi(G)$ the subgroup of automorphisms of G preserving the conjugacy classes $C_1, \ldots, C_b$, there is an obvious "action" of $\mathrm{Aut}_\xi(G)$ on the moduli stack $\mathcal{H}_{g,G,\xi}$ given by twisting the action of G. Assuming the center of G equal to 1, this action factors through the group of outer automorphisms $\mathrm{Out}_\xi(G)$.

Let $\pi : C \to D$ denote a section over S, then the action of $\sigma \in \mathrm{Out}_\xi(G)$ maps this cover to the same cover but with the action of G twisted by σ, i.e., $(g, x) \mapsto \sigma(g)x$. Even if a precise definition of an action of a group on a stack is not given in these notes (one can read [54] for a complete definition), we will speak freely of the natural action of $\mathrm{Out}_\xi(G)$ on $\mathcal{H}_{g,G,\xi}$. The result is, assuming $Z(G) = 1$, in which case $\mathcal{H}_{g,G,\xi}$ is a scheme:

Proposition 4.57. *We have* $\mathcal{H}^{ab}_{g,G,\xi} \xrightarrow{\sim} [\mathcal{H}_{g,G,\xi}/\,\mathrm{Out}_\xi(G)]$, *where the brackets indicate a quotient stack.* □

Example 4.58. *Fried's dihedral toy.* It seems useful to see how Fried's toy model of the **dihedral tower** fits into the framework of Hurwitz stacks (see [25] and the references therein). Let q be an odd integer. In this example it will be assumed that a scheme is a $\mathbb{Z}[\frac{1}{2q}]$-scheme. Recall the dihedral group $\mathbb{D}_q$ of order $2q$, is the group with presentation $\mathbb{D}_q = \langle s, t \,|\, s^2 = (st)^2 = t^q = 1\rangle$. One has $st^j s = t^{q-j}$ therefore the "reflections", i.e., the elements of order 2 form one conjugacy class C_2. In our example, the "dihedral toy", we are concerned with the moduli stack of G-covers of $\mathbb{P}^1$ with $G = \mathbb{D}_q$, and with ramification datum $4C_2 = \{C_2, C_2, C_2, C_2\}$. Consider such a cover $\pi : C \to \mathbb{P}^1$. The (labelled) branch points are $(Q_i)_{1\le i\le 4}$. The cyclic group $\langle t\rangle$ of order q acts freely and transitively on $\pi^{-1}(Q_i)$, since the cardinal of the fiber is q. The Riemann-Hurwitz formula yields that C is of genus 1. The conjugacy class of s contains q elements, therefore s has exactly one fix

point P_i over Q_i $(1 \leq i \leq 4)$. Over a base S, P_i becomes a section $P_i : S \to C$ of π (Lemma 4.50). We can take P_1 as origin, viz.

$$\begin{tikzcd} C \arrow[r, "\pi"] & S \arrow[l, bend right, "P_1"'] \end{tikzcd} \tag{4.43}$$

and therefore see $C \to S$ as an elliptic curve (4.5), i.e., endowed with a group law with $-1_C = s$. The automorphism t has no fixed point, and is of order q, therefore t is the translation be a point $\omega \in C[q]$ of exact order q. In this picture the 4 points P_i are the points of order two. The ramification divisor is

$$R = \sum_{i=1}^{4} \sum_{j=0}^{q-1} t^j(P_i).$$

We can understand the datum $(C, (P_i), \omega)$ as a $(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/q\mathbb{Z}$**-level structure** on the elliptic curve $(C, P_1 = O)$. In order to find a relationship with the modular curve $Y_1(q)$, first recall (see [37], Chap. 3 or [61], Appendix C, § 13):

Definition 4.59. A $\Gamma_1(q)$-structure on an elliptic curve $\pi : C \to S$, $O : S \to C$ is an injective morphism[52] $(\mathbb{Z}/q\mathbb{Z}) \hookrightarrow (C, +)$. This is equivalent to giving an S-point of C of exact order q along the fibers of $C \to S$.

It is easy to define the moduli stack $\mathcal{Y}_1(q)$ whose sections over S are the elliptic curve together with a $\Gamma_1(q)$-level structure. There is an obvious 1-morphism

$$F : \mathcal{H}_{1,\mathbb{D}_q,(4C_2)} \longrightarrow \mathcal{Y}_1(q) \tag{4.44}$$

A $\mathbb{D}_q$-cover $C \to D \cong \mathbb{P}^1$ maps to $(C, O = P_1, \omega)$. On the Hurwitz side there is an extra structure, viz. the labelling of the three points P_j $(2 \leq j \leq 4)$. The morphism F forgets the labelling. Let $\mathbf{S}_3$ stand for the permutation group on 3 letters. This group acts by relabelling the P_j's $(2 \leq j \leq 4)$. The claim is that (4.44) is an $\mathbf{S}_3$-torsor.

This means the following: let there be given a section $S \to \mathcal{Y}_1(q)$. Then the 2-fiber product $\mathcal{H}_{1,\mathbb{D}_q,(4C_2)} \times_{\mathcal{Y}_1(q)} S$ is an $\mathbf{S}_3$-Galois cover. Indeed assume the section $S \to \mathcal{Y}_1(q)$ given by the pair (E, ω). The subgroup $E[2] \subset E$ of fixed points of -1_E is a relative divisor étale of degree 4 over S. Therefore we can find an étale covering $S' \to S$ such that $(E \times_S S')[2]$ is split, which in turn yields

$$\left(\mathcal{H}_{1,\mathbb{D}_q,(4C_2)} \times_{\mathcal{Y}_1(q)} S\right) \times_S S' \xrightarrow{\sim} S' \times \mathbf{S}_3 \,.$$

Let $\tilde{\mathcal{H}}_{1,\mathbb{D}_q,(4C_2)}$ be the Hurwitz stack, the branch points unlabelled, i.e., the "quotient" of $\mathcal{H}_{1,\mathbb{D}_q,(4C_2)}$ by the $\mathbf{S}_4$-action. We have the picture

$$\begin{tikzcd} \mathcal{H}_{1,\mathbb{D}_q,(4C_2)} \arrow[r, "F"] \arrow[d] & \mathcal{Y}_1(q) \\ \tilde{\mathcal{H}}_{1,\mathbb{D}_q,(4C_2)} & \end{tikzcd}$$

[52] By $(C, +)$ we mean the abelian group of S-points of $C \to S$.

Finally notice that the discriminant morphism $\delta : \mathcal{H}_{1,\mathbb{D}_q,(4C_2)} \to \mathcal{M}_{0,4}$ yields a 1-morphism $\mathcal{H}_{1,\mathbb{D}_q,(4C_2)}/\mathbf{S}_3 \to \mathcal{M}_{0,1,(3)}$ to the moduli stack of 4 distinct points on a line, one labelled, and three unlabelled.

Exercise 4.60. *Prove that* $\mathcal{M}_{0,1,(3)} = \mathbb{A}^1$.

Exercise 4.61. *Show that the stacks* $\mathcal{Y}_1(q)$ *and* $\tilde{\mathcal{H}}_{1,\mathbb{D}_q,(4C_2)}$ *have the same coarse moduli space.*

4.2.2. Compactified Hurwitz stacks. In this subsection we keep the same notations as before, in particular ξ denotes an (extended) Hurwitz datum. Recall that the branch points are labelled. Our goal is to "compactify" a Hurwitz stack, i.e., makes it proper, in such a way that the discriminant morphism (4.41) extends to this compactification. The resulting picture will be a correspondence

$$\begin{tikzcd} \overline{\mathcal{H}}_{g,G,\xi} \arrow[r, "\delta"] & \overline{\mathcal{M}}_{g',b} \\ \mathcal{H}_{g,G,\xi} \arrow[u, hook] \arrow[r, "\delta"] & \mathcal{M}_{g',b.} \arrow[u, hook] \end{tikzcd}$$

Since $\mathcal{H}_{g,G,\xi}$ is a substack of the larger and proper stack $\overline{\mathcal{M}}_{g,n}(G)$, an obvious answer would be to take the closure in it. The problem is to describe intrinsically the curves which belong to this closure, that is the sections of $\overline{\mathcal{M}}_{g,n}(G)$ which are degeneration of smooth G-curves. The answer is given by the equivariant deformation theory of a nodal G-curve:

Theorem 4.62. *Let* $C \in \overline{\mathcal{M}}_{g,n}$ *be a stable curve with* n *labelled points* (P_i)*. Assume that the group* G *acts faithfully on* C*, the set of marked points being fixed. Then we can deform equivariantly* $(C, \sum_i P_i)$ *to a smooth curve if and only if the following holds: for any node* $P \in C$ *fixed by some* $1 \neq g \in G$*, with stabilizer* $H = G_P$*, one of the following two conditions is satisfied:*

1) *the subgroup* H *is cyclic, say of order* $e > 1$*, the branches at* P *are fixed by* H*, and the local monodromies along the two branches are opposite*[53]*.*
2) *the subgroup* H *is dihedral of order* $2e$*,* $e \geq 1$*, and the rotations of* H *preserve the branches, and acts as in* 1)*, whereas the reflections of* H *exchange the branches.*

Proof. This follows from an analysis of the induced G-action on the base of the formal universal deformation of the stable curve C. One must avoid that the subscheme of G-fixed points be a subscheme of the discriminant of the universal deformation. Localizing at a branch point, this restriction yields 1) and 2). For details, see [8], Section 5.1 and notably Théorème 5.1.1. □

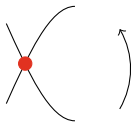
[53]Suppose that the node is $xy = 0$, and H acts via a faithful character χ_x, resp. χ_y on the x branch (resp. y branch) then $\chi_x\chi_y = 1$. The complete local ring of the image of the node in C/G is $k[[u,v]]/(uv)$ where $x = u^e, v = y^e$. The image is therefore a node.

In this definition, a dihedral group[54] of order $2e$ is a semi-direct product $\mathbb{D}_e = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}/e\mathbb{Z}$. The elements of $\mathbb{Z}/e\mathbb{Z}$ are the rotations, the others the reflections (order 2). In the dihedral case, we can choose formal coordinates x, y along the branches such that the stabilizer is the dihedral group $\mathbb{D}_e = \langle \sigma, \rho \rangle$ with two generators, and the relations $\sigma^2 = \rho^e = (\sigma\rho)^2 = 1$, with the action

$$\rho(x) = \zeta^e x, \rho(y) = \zeta^{-e} y, \rho(x) = y$$

for some root of the unity ζ of order e.

Definition 4.63. A faithful action of a finite group G on a stable curve (marked or not) is called stable if Theorem 4.62 is satisfied at each node.



exchanged branches

Suppose that the dihedral case 2) occurs at a node P, then in the quotient $\pi : C \to D = C/G$ the point $\pi(P)$ becomes regular. This is easily seen using formal coordinates, indeed $(k[[x, y]]/(xy))^H = k[[t]]$ with $t = x^e + y^e$. The nodes of type 2) are also responsible of the coalescence of the ramification points. This is explain by the following result:

Lemma 4.64. *Let $\pi : C \to S$ be a nodal G-curve, with or without marked points, over a connected base. Assume that the action of G stable. Let C_s be a geometric fiber, then $b'(s)$ stands for the number of G-orbits of smooth points with stabilizer $\neq 1$, and $b''(s)$ stands for the number of G-orbits of nodes with dihedral stabilizer $\mathbb{D}_e$ $(e \geq 1)$. Then the number $b(s) = b'(s) + 2b''(s)$ is constant along the geometric fibers. If there is a smooth fiber, then b is the number of branch points.*

Proof. See [8], Proposition 4.3.2. □

Example 4.65. *An example in genus* 2. In this example, we take $R = k[[t]]$ with fraction field K, and $S = \operatorname{Spec} R$. Let C_K be the genus 2 curve over K given by $y^2 = x^2(x^2 - 1)^2 - t^2$. The group G is the group of order two generated by the hyperelliptic involution $x \mapsto x, y \mapsto -y$. On can sees easily that the reduction stable of C_K to k is the nodal curve given by two copies of $\mathbb{P}^1$ intersecting in three nodes. Indeed the six Weierstrass points of C_K collapse pairwise on the three nodes, as shown by Figure 6 reproduced on top of the next page.

We see in this example that we cannot extend the discriminant map to the degenerated curve, since some branch points collapse. To forbid this rather unpleasant situation, it is necessary to work with G-curves marked by a ramification divisor as in Definition 4.52. This means the curves are now marked by a G-invariant divisor $\sum_i P_i$, unlabelled points, but labelled orbits, and G acting freely on $C - \{P_i\}$, recall that the ramification points are among the P_i's. With this

[54]The case $e = 1$ is accepted.

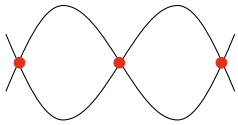


FIGURE 6. Collision of ramification points

assumption the nodes are all of type 1). Case 2) can not happen, and nodes of C yield nodes in the curve $D = C/G$. See [8], Chap. 4, for a complete discussion.

Our last definition is that of **stable Galois covers**. Let us fix a Galois group G, an extended ramification Hurwitz datum ξ associated to G.

Definition 4.66. A stable Galois cover of group G, ramification (Hurwitz) type ξ, is given by a stable curve of genus g, together with a stable action of G, such that the combinatorial datum attached to the action and the divisor of marked points is given by ξ. Denote $\overline{\mathcal{H}}_{g,G,\xi}$ the fibered category whose sections are the stable Galois G-covers of the indicated type.

Then, as expected:

Theorem 4.67. *The category fibered in groupoids $\overline{\mathcal{H}}_{g,G,\xi}$ is a DM-smooth and proper stack over $\mathbf{Sch}_k$ of dimension $3g'-3+b$. The discriminant* (4.41) *extends to $\overline{\mathcal{H}}_{g,G,\xi}$, defining a morphism $\delta : \overline{\mathcal{H}}_{g,G,\xi} \longrightarrow \overline{\mathcal{M}}_{g',b}$, in general not representable, even if $Z(G) = 1$.*

We will not give the details, but only some focus on the main ingredients of the proof. That the definition yields a DM stack is not difficult, and mimics previous proofs. The second claim is the smoothness. This amounts to checking the formal deformation space of a stable Galois cover is formally smooth, i.e., the completed local ring of the corresponding point of a given atlas is a formal power series ring. This follows a more precise result indicating how such a Galois cover deforms.

Assume given a stable Galois cover $\pi : C \to D = C/G$. Let $Q_1, \ldots, Q_b$ be the (labelled) branch points. Let $R_1, \ldots, R_d$ be the nodes of D, and let $e_i \geq 1$ the order of the cyclic stabilizer of any node of C above R_i. It is not difficult to see that $\pi : C \to D$, extends to the formal deformations spaces of respectively the stable G-cover, and the stable branched curve D. This extension is the local form of the discriminant δ.

Theorem 4.68. *One can choose formal coordinates $(t_1, \ldots, t_d, \ldots, t_{3g'-3+b})$ and $(u_1, \ldots, u_d, \ldots, u_{3g'-3+b})$ for the versal deformations of the G-cover $\pi : C \to D$, respectively the marked curve $(D, \{Q_j\})$ such that extension of π to the versal deformations spaces takes the form*

$$\pi^* : W(k)[[u_1, \ldots, u_d, \ldots, u_{3g'-3+b}]] \longrightarrow W(k)[[t_1, \ldots, t_d, \ldots, t_{3g'-3+b}]] \quad (4.45)$$

with $\pi^*(u_i) = t^{e_i}$ when $1 \leq i \leq d$, and $\pi^*(u_i) = t_i$ otherwise, and $W(k)$ stands for the Witt ring of k.

This result is a natural extension of the one concerning the deformation theory of stable curves ([15], Proposition 1.5). One has to take into account the action of G on the base of the universal deformation of the curve C, with respect to the parameters associated to the double points on one hand, and the parameters of deformations of the base $D = C/G$ on the other hand, see [8], Théorème 5.1.5.

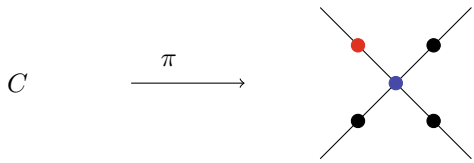
As a consequence of this deformation-theoretic result, we see that the discriminant map δ is étale on the open substack $\mathcal{H}_{g,G,\xi}$, said differently, the deformation functor of a "smooth cover", is isomorphic to the deformation functor of the base curve marked by the branch points. For nodal (stable) curves, this is no longer true, δ is generally ramified along the "boundary". Another corollary of these computations is that $\delta : \overline{\mathcal{H}}_{g,G,\xi} \longrightarrow \overline{\mathcal{M}}_{g',b}$ is everywhere flat.

It remains to check that $\overline{\mathcal{H}}_{g,G,\xi}$ is proper. Fortunately this is a rather direct consequence of either the construction of the stack as a closed substack of $\mathcal{M}_g(G)$, or more directly from the stable reduction theorem [54]. Indeed given a cover $C_K \to D_K$ defined over the generic point of a discrete valuation ring, the action of G extends to the stable model C of C_K. Then it is easy to check that the quotient curve $D = C/G$ is stable marked by the images of the branch points. □

Example 4.69. *The cusps of the modular curve* $Y_1(q)$. In Example 4.58 the stack $\mathcal{Y}_1(q)$ was identified with a Hurwitz stack of dihedral covers of $\mathbb{P}^1$. We would like to see how this identification reads at the boundary, i.e., at the **cusps**. Recall we have the discriminant map

$$\delta : \overline{\mathcal{H}}_{1,\mathbb{D}_q,4C_2}/\,\mathbf{S}_3 = \overline{\mathcal{Y}}_1(q) \longrightarrow \overline{\mathcal{M}}_{0,1,(3)} = \mathbb{P}^1.$$

We would like to describe the covers lying over the point at infinity.



Let us choose a double point of $P \in C$ lying over the double point of D. Denote C_1, C_2 the components of C intersecting at P. It is easy to check that the stabilizer of C_i in $G = \mathbb{D}_q$ is $G_i = \mathbb{D}_l$, where l divides q, the stabilizer of P being $H = G_1 \cap G_2$, a cyclic group of order $l \geq 1$. The curves C_i are ramified covers of $\mathbb{P}^1$ with dihedral Galois group, and three branch points, two with ramification index 2 and the third with index l. Therefore $C_i \cong \mathbb{P}^1$. It is readily seen that C is an n-gon of $\mathbb{P}^1$'s where $n = q/l$, as expected from the known description of the cusps of the modular curves ([37], **8.6**).

The three cusps of cyclic covers of $\mathbb{P}^1$ with 4 branch points play an important role in the computations of [10], [22].

Finally there is an alternative presentation of the stack of G-stable covers with fixed ramification, see Abramovich, Corti and Vistoli [1]. Let $\pi : C \to D = C/G$ be a stable cover over a base S. Consider the S-stack $\mathcal{C} = [C/G]$. We know

that $p : C \to \mathcal{C}$ is a principal G-bundle, therefore is classified by a morphism $q : \mathcal{C} \to \mathbf{BG} \times S$. In turn π factors as

$$\pi : C \xrightarrow{p} [C/G] \xrightarrow{q} D = C/G.$$

The S-stack $\mathcal{C} = [C/G]$ is Deligne-Mumford with coarse moduli space $q : \mathcal{C} \to D$.

Lemma 4.70. *The morphism q is representable, and its formation commutes with any base change.*

Proof. Let $P \to T$ be a G-bundle $T \in \mathbf{Sch}_S$. A section over $U \to T$ of the associated 2-fiber product $[C/G] \times_{\mathbf{BG}} T$ can be identified to a G-morphism $P \times_T U \longrightarrow C \times_S U$, therefore the fiber stack is equivalent to the scheme $\mathrm{Hom}_G(P, C \times_S T)$. The second claim comes from two facts. The first is that the quotient stack $[C/G]$ is compatible with any base change, i.e., if $T \to S$ is a morphism, one has $[C \times_S T/G] \cong [C/G] \times_S T$ canonically, this is easy to check due to the 2-universal property of the quotient (see [55] for details). The ordinary quotient, equivalently the coarse moduli space does not commute in general with an arbitrary base change, but here, since the action of G is assumed tamely ramified, it is easy to check this is indeed the case [37]. □

The ramification datum of the G-cover π is encoded in the stack $\mathcal{C}$ in the following way. As explained before, the fiber of $[C/G]$ over a geometric point $s \in S$ is $[C_y/G]$. Thus we can assume that $S = \operatorname{Spec} k$ with $k = \overline{k}$. Let Q be a closed point of $D = C/G$, which is a branch point of π. Choose $P \in C$ over Q, and set $H = G_P$. It is know that we can find an H-invariant étale neighborhood of P, of the form $\mathbb{A}^1 \to C$, $0 \mapsto P$, the action of H on the line given by the cotangent character χ_P. Therefore $[\mathbb{A}^1/H]$ is a local chart of $\mathcal{C}$ around Q. Now if Q is a node, choose a node P lying over Q. The deformation theory of a node tells us that we can find an étale neighborhood of P of the form $\operatorname{Spec} k[x,y]/(xy) \to C$, where H acts through the character χ_P on the x-branch, and χ_P^{-1} on the y-branch. In turn this yields a local chart of $\mathcal{C}$ at P of the form $[\operatorname{Spec}(k[x,y]/(xy))/H] \to \mathcal{C}$.

Finally we are able to recover the old cover $C \to D$, i.e., C, from the 2-commutative square

$$\begin{tikzcd} \mathcal{C} \arrow[r, "q"] & \mathbf{BG} \times S \\ C \arrow[u, "p"] \arrow[r] & S. \arrow[u] \end{tikzcd}$$

The moral of this construction is that we can think about a stable G-cover over S in terms of a single representable morphism

$$q : \mathcal{C} \to \mathbf{BG}$$

but where $\mathcal{C}$ is a twisted stable curve (over S) with stacky structure governed by the ramification datum. This is the point of view of Abramovich, Corti and Vistoli [1].

Exercise 4.71. *Suppose that $k = \overline{k}$ is of characteristic $p > 0$. Consider the Artin-Schreier curve with equation $y^{p-1} = x^p - x$ as a cyclic cover of degree $p-1$ of $\mathbb{P}^1$ given by $(x, y) \mapsto x$.*

i) *Show that the branch locus is $\mathbb{P}^1(\mathbb{F}_p)$.*
ii) *Prove that the universal equivariant deformation of this cover is*

$$y^{p-1} = x^p + u_1 x^{p-1} + \cdots + u_{p-2} x^2 + (-1 - \sum_{i=1}^{p-2} u_i)x$$

over the spectrum of $W[[u_1, \ldots, u_{p-2}]]$ (W = Witt ring of k).

4.3. Mere covers versus Galois covers

4.3.1. Galois closure. Until now covers were Galois covers. Obviously one can ask about the construction of Hurwitz stacks parameterizing arbitrary (mere) covers $\pi : C \to D$ between smooth connected projective curves of fixed genus, and with prescribed "ramification datum".

Let us first assume that the ground field is $\mathbb{C}$. Denote by $Q_1, \ldots, Q_b \in D$ the branch points, and let $\star \in D - \{Q_i\}$ be a base point. Let us choose a labeling of the points of C lying over $\star$. Suppose that $\deg(\pi) = n$. We know how the monodromy action (on the right) of $\pi = \pi_1(D - \{Q_i\})$ on $\pi^{-1}(\star) = \{P_1, \ldots, P_n\}$ is defined. If $[\alpha]$ is the homotopy class of a loop based at $\star$, then choose a lift $\tilde{\alpha}$ starting at P_i, then

$$P_i.[\alpha] = \tilde{\alpha}(1).$$

Denote by G the monodromy group, i.e., the image of π in S_n, the permutation group of the P_i's. The group G is a transitive subgroup of S_n, well defined up to conjugacy since relabelling the P_i's changes G into a conjugate subgroup. Let γ_i be a small loop encircling Q_i. The image σ_i of γ_i in G lies in a well-defined conjugacy class, say C_i. Then the tuple $C_1, \ldots, C_b$ is called the **ramification (or monodromy) datum** of the cover (compare Definition 4.49). Recall the well-known topological fact that the points lying over Q_i are in one-to-one correspondence with the disjoint cycles of the permutation σ_i. The ramification index at such a point is the length of the corresponding cycle.

We know that the topological cover $\pi : C \to D$ admits a Galois closure $\tilde{\pi} : \tilde{C} \to C \to D$ such that G can be identified with its Galois group, i.e., $\mathrm{Aut}(\tilde{C}/D)$. The topological surface $\tilde{C}$ has a well-defined structure of compact Riemann surface (algebraic curve). It is also known that the ramification datum $\{C_1, \ldots, C_b\}$ described above yields the ramification datum ξ of the Galois closure as defined in a previous section. Let H be the stabilizer of one of the P_i's, say P_1, then

$$\cap_{s \in G} sHs^{-1} = 1. \tag{4.46}$$

It is clear how to recover $\pi : C \to D$ from $\tilde{\pi} : \tilde{C} \to D$: we have

$$C = \tilde{C}/H \to D = \tilde{C}/G. \tag{4.47}$$

The condition (4.46) implies that G acts faithfully on G/H with in turn allows us to identify G with a permutation subgroup of the set $G/H = \pi^{-1}(Q_1)$.

This motivates the following definition:

Definition 4.72.

i) A **monodromy (or ramification) datum** for mere covers is a triple $m = (G, H, \xi)$, where H is a subgroup of the finite group G, with condition (4.46), and m is ramification (Hurwitz) datum associated to G. We identify $m = (G, H, \xi)$ and the conjugate (G, sHs^{-1}, ξ).

ii) By an m**-Galois closure** with monodromy $m = (G, H, \xi)$ of a cover $\pi : C \to D$, we mean a G-Galois cover $\tilde{\pi} : \tilde{C} \to D$ with ramification datum ξ, together with a factorization of $\tilde{\pi}$ through C such that $\mathrm{Aut}(\tilde{C} \to C) = H$.

$$\begin{tikzcd} \tilde{C} \arrow[r, "h"] \arrow[dr, "\tilde{\pi}"'] & C \arrow[d, "\pi"] \\ & D. \end{tikzcd} \tag{4.48}$$

If we think of ξ as a tuple $(C_1, \ldots, C_b)$ of conjugacy classes of G, then any $\sigma \in C_i$ defines a permutation of G/H. The lengths of the disjoint cycles of this permutation yield the ramification indices over Q_i.

The choice of a Galois closure is somewhat ambiguous, therefore we must clarify the relationship between a cover and its Galois closures. Clearly if we start with a G-Galois cover $\tilde{\pi} : \tilde{C} \to D$ with monodromy ξ, then $\tilde{\pi} : \tilde{C} \to D$ is an m-Galois closure of $\tilde{C}/H \to D$. The correspondence

$$m\textbf{-Galois covers} \Longleftrightarrow \textbf{covers with monodromy } m$$

is generally not one-to-one. Let us consider two m-Galois covers $\tilde{\pi}_i : \tilde{C}_i \to D$ of the cover $\pi : C \to D$. Galois theory tells us that there is an isomorphism $f : \tilde{C}_1 \xrightarrow{\sim} \tilde{C}_2$ making the diagram

$$\begin{tikzcd} \tilde{C}_1 \arrow[dr, "h_1"] \arrow[drr, bend left, "\tilde{\pi}_1"] \arrow[dd, "f"', "\wr"] & & \\ & C \arrow[r, "\pi"] & D \\ \tilde{C}_2 \arrow[ur, "h_2"'] \arrow[urr, bend right, "\tilde{\pi}_2"'] & & \end{tikzcd} \tag{4.49}$$

commutative [11]. Clearly $g \mapsto fgf^{-1}$ maps G onto G, and H onto H. Denote $\theta \in \mathrm{Aut}(G)$ the automorphism $\theta(g) = fgf^{-1}$. It belongs to the group

$$\mathrm{Aut}_m(G) = \{\theta \in \mathrm{Aut}(G),\ \theta(G) = G, \theta(H) = H, \theta(\xi) = \xi\}. \tag{4.50}$$

Therefore the isomorphism f becomes a G-isomorphism between the Galois $\tilde{C}_1 \to D$ and $\tilde{C}_2^{\theta} \to D$ with G-action twisted by θ.

Exercise 4.73. *The ground field is $\mathbb{C}$. Let $\pi : C \to D$ e a ramified cover between smooth curves, of degree n. Let σ be the monodromy permutation of the fiber $\pi^{-1}(\star) = \{P_1, \ldots, P_n\}$ induced by a small loop surrounding a branch point Q_i. Prove that the points lying over*

Q_i are in one-to-one correspondence with the disjoint cycles of σ, in such a way that the ramification index coincides with the length of the cycle.

Exercise 4.74. *Under condition* (4.46) *check that the adjoint action embeds H into* $\mathrm{Aut}(G)$.

We are going to see that all this material extends immediately to covers over an arbitrary base. Let $m = (G, H, \xi)$ be a monodromy type fixed. Let $\pi : C \to D$ be a cover between smooth projective curves over a base scheme S. We limit ourselves to $S \in \mathbf{Sch}_k$, schemes over an algebraically ground field k, and $|G| \neq 0 \in k$. Say $\pi : C \to D$ has monodromy m if the monodromy along the geometric fibers of π is fixed equal to m. The monodromy is a discrete invariant, therefore fixed over a connected S.

Definition 4.75. A **Galois closure** of $\pi : C \to D$ is a pair composed of a G-Galois cover $\tilde{\pi} : \tilde{C} \to D$ with monodromy m and an isomorphism

$$\begin{tikzcd} \tilde{C}/H \arrow[r] \arrow[d, "\varphi"', "\wr"] & \tilde{C}/G \arrow[d, "\psi"', "\wr"] \\ C \arrow[r, "\pi"] & D. \end{tikzcd} \tag{4.51}$$

In simple words, a Galois closure of $\pi : C \to D$ is a G-cover $\tilde{\pi} : \tilde{C} \to D$ such that $\tilde{\pi}$ factors through π. A Galois closure as defined need not exist, but one can prove the following:

Proposition 4.76. *Locally for the étale topology on S there is a Galois closure, and étale locally two Galois closures are isomorphic. In other terms, the Galois closures of a fixed cover over a base S form a gerbe over S.*

Proof. The first assertion follows readily from the specialization theory of the tame fundamental group as developed in [57], see also ([15], § 5). Alternatively we can use the deformation theory of tamely ramified covers, see the details in [8], Theorem 6.6.6). Let $C_s \to D_s$ be a geometric fiber of π. The main theorem of the deformation theory of tamely ramified cover between smooth curves says that the deformation theory of the cover is the same as the deformation theory of the base curve marked by the branch points, i.e., the deformations rings are isomorphic, and formally smooth. Let $\tilde{C}_s \to D_s$ be an m-Galois cover of $C_s \to D_s$. Let $\tilde{\mathcal{C}}_s \to \mathcal{D}_s$ be the universal deformation[55] of this Galois cover. Then we can identify the cover $\tilde{\mathcal{C}}_s/H \to \mathcal{D}_s$ with the universal formal deformation of $\pi : C_s \to D_s$. Our assumptions allow to algebraize this formal deformation meaning it can be defined over an étale extension of $\mathrm{Spec}\,\mathcal{O}_s$. This gives a sketchy proof that étale-locally a Galois closure does exist.

It remains to prove that two Galois closures $\tilde{C}_i \to C \to D$ of $C \to D$ are étale locally isomorphic. There is no loss in assuming S connected. For any

[55] Universal since the objects under consideration have no infinitesimal automorphisms [34].

$\theta \in \mathrm{Aut}_m(G)$ (4.50), let $\mathrm{Isom}_G(\tilde{C}_1, \tilde{C}_2^\theta)$ be the corresponding scheme, known to be finite and unramified over S, which in turn gives that the closed subscheme

$$Z^\theta = \{f \in \mathrm{Isom}_G(\tilde{C}_1, \tilde{C}_2^\theta)\,,\ h_2 f = h_1\}$$

is finite unramified over S, possibly empty. If θ runs over $\mathrm{Aut}_m(G)/H$ the image in S of these subschemes are pairwise disjoint, thus there must be a θ with $Z^\theta \longrightarrow S$ onto. At the level of geometric points, this morphism is bijective. Indeed assuming S to be the spectrum of an algebraically closed field, two isomorphisms belonging to Z^θ must differ by an element of $H \cap Z(G) = 1$, due to (4.46). The conclusion follows easily from the fact that a finite, unramified morphism, bijective on the geometric points is an isomorphism. This proves the proposition. □

4.3.2. Hurwitz stacks of mere covers. The Galois closure construction has a nice interpretation in terms of moduli stacks. Let us form $\mathcal{H}_m$ the fibered category in groupoids over $\mathbf{Sch}_k$ with sections over S the S-covers $\pi : C \to D$, with fixed monodromy m. The number of branch points is fixed, say equal to b, and D/S is therefore marked by b sections $Q_i : S \to D$. The isomorphisms in the groupoid $\mathcal{H}_m(S)$ are the commutative diagrams

$$\begin{tikzcd} C_1 \arrow[r, "\pi_1"] \arrow[d, "\varphi"'] & D_1 \arrow[d, "\psi"] \\ C_2 \arrow[r, "\pi_2"] & D_2. \end{tikzcd} \tag{4.52}$$

It is an easy exercise to check directly that $\mathcal{H}_m$ is a DM stack. The construction that follows will give the result as a byproduct. Let us denote by $\mathcal{H}_{G,m}$ the DM stack denoted previously $\mathcal{H}_{G,g,\xi}$ where $m = (G, H, g, \xi)$. There is an obvious functor

$$\gamma : \mathcal{H}_{G,m} \longrightarrow \mathcal{H}_m \tag{4.53}$$

which on the objects is $(\tilde{C} \to D) \mapsto (C = \tilde{C}/H \to D)$. Introduce the group $\Delta_m = \mathrm{Aut}_m(G)/H$, see (4.50). We can be a bit more precise:

Lemma 4.77. *The morphism γ is representable. More precisely, it is a Δ_m-torsor.*

Proof. Let $\pi : C \to D$ be a section of $\mathcal{H}_m$ over S. The proof amounts to checking that the 2-fibered product $\mathcal{H}_{G,m} \times_{\mathcal{H}_m} S$ is Δ_m-torsor over S, in particular a scheme. Since all stacks involved in the construction are Deligne-Mumford, it is a DM stack. To check this is a scheme, it suffices to prove that the objects have no non-trivial automorphisms. Such an object is given by a G-cover $\tilde{\pi} : \tilde{C} \to \tilde{D}$ together with an isomorphism $(\tilde{C}/H \to \tilde{D}) \cong (C \to D)$. Therefore an automorphism of this object is a G-automorphism of $\tilde{C}$ inducing the identity on $\tilde{C}/H$, thus given by an element of $Z(G) \cap H = 1$ (4.46). This proves the representability of γ. There is an obvious action of $\mathrm{Aut}_m(G)$ on this scheme. At the category level, we twist the G-action by $\theta \in \mathrm{Aut}_m(G)$. It is easy to check that the subgroup $H \subset \mathrm{Aut}_m(G)$ acts trivially. Indeed if $h \in H$, the canonical isomorphism $h : \tilde{C} \xrightarrow{\sim} \tilde{C}^{int(h)}$ shows that $\tilde{C} \to \tilde{D}$

is a fixed point[56]. As a consequence the action of $\mathrm{Aut}_m(G)$ factors through Δ_m. This yields a simply transitive action of Δ_m. The result follows. $\square$

The construction of a quotient stack $[X/G]$ given in Section 2.1 extends to the case when X is replaced by a DM stack (see [55], Definition 2.3(ii) and Theorem 4.1 for more details). In conclusion we get[57]:

Corollary 4.78. *The stack $\mathcal{H}_m$ is a quotient stack $\mathcal{H}_m = [\mathcal{H}_{G,m}/\Delta_m]$.*

This result suggests a way to get a nice compactification of the moduli stack $\mathcal{H}_m$: we simply take the quotient stack of the stable compactification of the Hurwitz stack $\overline{\mathcal{H}}_{G,m}$ by the (non strict) action of Δ_m. The result is a smooth proper DM stack:

$$\overline{\mathcal{H}}_m = [\overline{\mathcal{H}}_{G,m}/\Delta_m]. \tag{4.54}$$

On the level of coarse moduli spaces we have likewise $\overline{H}_m = \overline{H}_{G,m}/\Delta_m$, see [55].

Not only did we produce a compactification $\overline{\mathcal{H}}_m$ of $\mathcal{H}_m$, but the added boundary points have a clear description as the *degenerate covers*, that is the covers of the form $C = \tilde{C}/H \to D = \tilde{D} = \tilde{C}/G$ obtained by factoring out the stable G-covers with given monodromy $m = (G, H, \xi)$. The Galois cover $\tilde{C} \to \tilde{D}$ can be seen as a Galois closure of the stable cover $C \to D$. Notice that the ramification points (resp. the branch points) are always smooth points. In [32] a slightly different compactification was produced, the Harris-Mumford moduli stack of admissible covers. There is a clear relationship between these two, possibly different, stacks. Our construction is the normalization of their.

Exercise 4.79. *Let $\pi : C \to D$ be a stable cover, with ramification points (smooth) P_i, and ramification index e_i at P_i. Let $R = \sum_i (e_i - 1)P_i$ be the ramification divisor. Prove that $\omega_C \cong \pi^*(\omega_D) \otimes \mathcal{O}(R)$ (see Subsection 4.1.3 for the definition of ω_C).*

Exercise 4.80. *Let $\pi : C \to \mathbb{P}^1$ be a dihedral $\mathbb{D} - q$ Galois cover (Example 4.58). We set $H = \{1, s\}$. Check that $C/H \cong \mathbb{P}^1$. Show that the monodromy datum of the cover $\mathbb{P}^1 \cong C/H \to \mathbb{P}^1$ over any one of the 4 branch points is $(1, 2, \ldots, 2)$, that is $\frac{q-1}{2}$ points of ramification index 2.*

Example 4.81. *The classical Hurwitz stack.* This is the foundational example, which goes back to Hurwitz. A complete algebraic treatment is due to Fulton [27]. We are concerned with the "generic covers" of given degree d of $\mathbb{P}^1$, with b branch points. Notice that such a cover cannot be Galois if $d > 2$, and that b is even. Simple ramification means that over each branch point there is only one ramification point, then with index two. In topological terms the local monodromy at each branch point is a transposition. Therefore the monodromy group, i.e., the Galois group of a Galois closure, is the symmetric group S_d, where d is the degree of the cover. The Galois closure of such a simple cover lies in the Hurwitz stack $\mathcal{H}_{g,G,\xi,\tau}$

[56] This should be compared with the definition of a fixed point under a finite group action on a stack given in [55]

[57] Some subtlety appears because the action of Δ_m is not strict, in the sense that the associativity conditions are valid only up to 2-isomorphisms.

where $G = S_d$, and ξ denotes the conjugacy class of transpositions counted b times, i.e., $m = (S_d, S_{d-1}, (12)^b)$. The genus is given by Riemann-Hurwitz formula $2g - 2 = d!(\frac{b}{2} - 2)$. A classical result of Lüroth says that in this case the Hurwitz stack is connected, and indeed is a scheme (see [27]). Denote $\mathcal{H}_d$ the classical Hurwitz stack. With our previous definition, at least if $d \geq 4$, one has $\Delta_m = 1$. Indeed $\mathrm{Aut}(S_d) = \mathrm{Int}(S_d)$ if $d \geq 5, d \neq 6$. If $d = 6$, an automorphism of S_6 preserving the conjugacy class of non transitive subgroups of index 6 must be inner, some the same conclusion holds true. Thus there is no difference between the Hurwitz stack $\mathcal{H}_d$ and its Galois partner, and likewise for the compactified stack $\overline{\mathcal{H}}_d$.

In general the monodromy invariants are not sufficient to separate the connected components[58] of a Hurwitz stack. It is an interesting problem to exhibit finer discrete invariants. Assuming the ramification indices odd, then there is the well-known *spin* invariant of Fried and Serre [26]. Let $\pi : C \to D$ be a degree d cover between smooth curves, with ramification points $(P_i)_{1 \leq i \leq r} \in C$. Assume that for all i, the ramification index e_i of P_i is odd. This makes sense to the divisor, half of the ramification divisor

$$\frac{R}{2} = \sum_i \left(\frac{e_i - 1}{2} \right) P_i. \tag{4.55}$$

The coherent sheaf $E_\pi = \pi_*(\mathcal{O}(\frac{R}{2}))$ is locally free of rank d. Denote $\mathrm{T}r : k(C) \to k(D)$ the trace form, viz. $\mathrm{T}r(f, g) = \mathrm{Tr}_{k(C)/k(D)}(fg)$. We can use $\mathrm{T}r$ to define a bilinear form $E_\pi \times E_\pi \to \mathcal{O}_D$. We have the following result regarding the vector bundle E_π:

Proposition 4.82. *The trace form* $\mathrm{T}r : E_\pi \times E_\pi \to \mathcal{O}_D$ *is non degenerate, i.e., induces an isomorphism* $E_\pi \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_D}(E_\pi, \mathcal{O}_D)$.

Proof. This is a Zariski-local problem on D, therefore we are reduced to checking the non degeneracy property in the following framework: let A be a Dedekind ring with fraction field K, and B the normalization of A in a finite separable tamely ramified extension L/K Let $\mathcal{C} = \{b \in L, \mathrm{T}r(bB) \subset A\} = \mathcal{O}(R)$ be the inverse different $\mathcal{D}^{-1}$, that is $\mathcal{C} = \prod_{\mathcal{P}} \mathcal{P}^{-(e-1)}$ where the product goes over the primes of A, and e stands for the ramification index. We set $\sqrt{\mathcal{D}} = \prod \mathcal{P}^{\frac{e-1}{2}}$, likewise for $\sqrt{\mathcal{C}}$. The result amounts to checking that the trace yields a perfect pairing

$$\sqrt{\mathcal{C}} \times \sqrt{\mathcal{C}} \longrightarrow A. \tag{4.56}$$

There is no loss in assuming A is a local complete discrete valuation ring, which in turn implies that B is a product of finitely many complete discrete valuation rings. It is readily seen that we can further assume that B is local, let t denotes an uniformizing parameter of B. In this case $d = e - 1$ the exponent of the different. It is sufficient to check that (4.56) is surjective. Let $\varphi : \sqrt{\mathcal{C}} \to A$ be a linear form.

[58] Which are the same as the irreducible components.

Extended as a linear form $L \to K$, we know there is $x \in L$ such that $\varphi(y) = \mathrm{T}r(xy)$. Then $\mathrm{T}r(x\sqrt{\mathcal{C}}) \subset A$ which in turn yields $x\sqrt{\mathcal{C}} \subset \mathcal{C}$, therefore $x \in \sqrt{\mathcal{C}}$. $\square$

We can see this result in a way that fits in the framework of the duality for the finite flat morphism $\pi : C \to D$. The functor π_* has a right adjoint $\pi^\flat$ given by

$$\pi^\flat(G) = \overline{\pi}^*(\mathrm{Hom}(\pi_*(\mathcal{O}_C, G))$$

the overline means that a module over the sheaf $\pi_*(\mathcal{O}_C)$ is viewed as an $\mathcal{O}_C$-module. Indeed the definition yields $\pi^\flat(\mathcal{O}_D) = \mathcal{O}(R)$, therefore the duality theorem takes the form

$$\pi_*(\mathrm{Hom}_{\mathcal{O}_C}(F, \mathcal{O}(R)) \xrightarrow{\sim} (\mathrm{Hom}_{\mathcal{O}_D}(\pi_*(F), \mathcal{O}_D)$$

for F a vector bundle on C. When $F = \mathcal{O}_C(R/2)$, we recover (Proposition 4.82)

$$\pi_*(\sqrt{\mathcal{R}}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_D}(\pi_*(\sqrt{\mathcal{R}}, \mathcal{O}_D). \tag{4.57}$$

Indeed this construction of a quadratic form on the locally free sheaf E_π makes sense at the boundary points of the moduli stack. Let $\pi : C \to D$ be a stable cover. One can check as in the smooth case that $\mathcal{O}(R)$ (see Exercise 4.79) is isomorphic to $\pi^\flat$. Thus the previous duality argument continues to hold, which in turn yields the fact that $E_\pi = \pi_*(\mathcal{O}(R/2)$ is again a quadratic bundle even if π is not flat.

The "quadratic bundle" E_π leads to interesting discrete invariants (see [26] and the references therein). For example $\wedge^n E_\pi$ is a quadratic line bundle, therefore $(\wedge^n E_\pi)^{\otimes 2} \cong \mathcal{O}_D$, i.e., $\wedge^n E_\pi$ is a line bundle of order at most two. One can extract from E_π the so-called Spin invariant which helps to separate the connected component of the Hurwitz stacks in interesting example [26].

Exercise 4.83. *Let a stable cover $\pi : \mathbb{P}^1 \to \mathbb{P}^1$ with odd ramification indices and degree n. Using the fact that any coherent locally free sheaf on $\mathbb{P}^1$ is a direct sum of line bundles, check that $E_\pi \cong \mathcal{O}_D^n$, where n is the degree of π.*

4.4. Covers of the projective line

When the base curve of a cover is a projective line, one may expect the Hurwitz stacks to be more tractable. In this case the "moduli" are given by the branch points, since a projective line is rigid. A different approach is to think a cover $f : C \to \mathbb{P}^1$ as a map to $\mathbb{P}^1$, or as a rational function on the smooth genus g curve C. However we need to deviate slightly from our previous definition of the Hurwitz stack. In the present setting, the objects are unchanged, but the morphisms between $f : C \to \mathbb{P}^1$ and $f' : C' \to \mathbb{P}^1$ are the **equivalences** of [12], § 1.1, that is, the isomorphisms $\phi : C \to C'$ fitting in a commutative triangle:

$$\begin{tikzcd} C \arrow[rr, "\phi", "\sim"'] \arrow[dr, "f"'] & & C' \arrow[dl, "f'"] \\ & \mathbb{P}^1. & \end{tikzcd}$$

The previous definition of the Hurwitz stack refers to the PGL_2-reduced equivalence of [12]. Let us fix one branch point, put at $\infty \in \mathbb{P}^1$, and let us identify the ramification over ∞ with the sequence $k_1, \ldots, k_n$ of ramification orders taken along the preimages $P_1, \ldots, P_n$ of the branch point ∞. The P_i's are labelled. In this setting the P_i's are the **poles** of f. We need an additional information about the poles to recover the function f. The Duality part of the Riemann-Roch theorem yields the answer:

Definition 4.84. The polar part of $f \in k(C)$ at a pole P is the image of f in $\mathcal{P}_k(P) = \mathcal{M}_P^{-k}/\mathcal{O}_{C,P}$ where k is the order of the pole.

With a local parameter z at P, the polar part takes the concrete form

$$\frac{a_0}{z^k} + \cdots + \frac{a_{k-1}}{z} \tag{4.58}$$

If we ignore the branch points other than ∞ then we can almost recover the cover $f : C \to \mathbb{P}^1$, i.e., the rational function f, with the pair $(C, \{\varphi_i\})$, where $\{\varphi_i\}$ is the n-tuple of polar parts. This affirmation is correct in the sense that f can be recovered up to an additive constant, if we take into account that the φ_i's must satisfy g linear equations:

Proposition 4.85. *With the previous notations, for any regular* 1*-form* ω *on* C*, we have the following equation:*

$$\sum_{i=1}^{n} \mathrm{Res}_{P_i}(\varphi_i \omega) = 0. \tag{4.59}$$

Furthermore if we are given a n*-tuple of polar parts* (φ_i)*, solution of the previous equations, then these polar parts come from a rational function* f*, unique up to an additive constant.*

Proof. This follows easily from the duality theorem, where Res means the residue operator ([33], chap. III, theorem 7.14.2), Indeed we have the exact sequence

$$0 \to \mathcal{O}_C \to \mathcal{O}_C\left(\sum_{i=1}^{n} k_i P_i\right) \to \oplus_{i=1}^{n} \mathcal{P}_{k_i}(P_i) \to 0$$

from which we infer the exact sequence

$$0 \to k = \Gamma(C, \mathcal{O}_C) \to \Gamma\left(C, \mathcal{O}_C\left(\sum P_i\right)\right) \to \oplus_i \mathcal{P}_{k_i}(P_i) \xrightarrow{\delta} \mathrm{H}^1(C, \mathcal{O}_C).$$

Therefore an n-tuple of polar parts $(\varphi_i)_i$ comes from a rational function on C if and only if $\delta((\varphi_i)) = 0$. The residue theorem yields a canonical isomorphism

$$\mathrm{H}^1(C, \mathcal{O}_C) \xrightarrow{\sim} \mathrm{H}^0(C, \Omega_C^1)^*$$

taking into account this identification, we get (4.59). $\square$

It should be noted that Proposition 4.85 remains valid if C is a nodal curve [21], indeed with the same proof. Therefore we can work also at the boundary with rational functions on stable curves with preassigned polar parts at the poles. We can understand (4.59) as a defining set of "equations" of the Hurwitz stack as closed substack $\mathcal{Z}$ of the Deligne-Mumford stack parameterizing the pairs $(C, \{\varphi_i\})$, C is a smooth projective curve of genus g, marked by n points P_i, together with at each P_i, a polar part of exact order k_i. It is not difficult to check this defines a stack, indeed a cone over $\overline{\mathcal{M}}_{g,n}$. Denote it $\overline{\mathcal{M}}_{g,\vec{k}}$.

What makes this construction interesting, is the fact that it extends to the boundary, i.e., to degenerate covers. There is however one subtlety. The construction forces us to incorporate into the picture non stable marked curves, precisely to add marked curves with "tails". A tail is a smooth rational component, i.e., $\mathbb{P}^1$ intersection the rest of the curve in one point, and containing only one of the P_i's, therefore an unstable component.

Equivalently we enlarge the definition of the stack $\mathcal{M}_{g,\vec{k}}$ by allowing nodal curves marked by a n-tuple of polar parts according to the definition:

Definition 4.86. A nodal curve $(C, (\varphi_i)_{1\leq i\leq n})$ marked by a collection of polar parts located at smooth points is stable if the group $\mathrm{Aut}(C, \{\varphi_i\})$ is finite.

If P_i is the location of φ_i, Definition 4.86 does not say that $(C, (P_i))$ is stable, due to the presence of "tails". For example $(\mathbb{P}^1, \frac{1}{z^2})$ is stable in the sense of Definition 4.86.

Let $\pi : C \to D$ be a stable cover with base D a stable marked curve of genus 0. Recall that among the branch points, we forget all but one called the infinity Q_∞. As a consequence we forget all points lying over the $Q_i \neq Q_\infty$, and keep only the preimages $P_1, \ldots, P_n$ of Q_∞. Then we extract the polar part φ_i of $\pi : C \to D$ at P_i, notice this makes sense. The result is a not necessarily stable nodal curve marked by n polar parts. Stabilizing if necessary we get a point of $\overline{\mathcal{M}}_{g,\vec{k}}$. In turn this yields a 1-morphism (for a suitable ramification datum m)

$$\overline{\mathcal{H}}_m \longrightarrow \overline{\mathcal{M}}_{g,\vec{k}} \tag{4.60}$$

that factors through the substack $\overline{\mathcal{Z}}$. We can check that the model[59] $\overline{\mathcal{Z}}$ of the Hurwitz stack we get in this way is the correct one if the branch points except the ∞ point are all simple branch points. This construction yields a beautiful formula for the Hurwitz number as a Hodge integral, see [21].

Exercise 4.87. *Prove the fibered category $\overline{\mathcal{M}}_{g,\vec{k}}$ is really a DM stack. Prove the morphism $\overline{\mathcal{M}}_{g,\vec{k}} \to \overline{\mathcal{M}}_{g,n}$, which drops the polar part is representable, indeed makes $\overline{\mathcal{M}}_{g,\vec{k}}$ a cone over the base.*

[59] To be precise, the Hurwitz stack is the component of the locus (4.59) containing the smooth covers.

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Models of Curves

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Abstract. The main aim of these lectures is to present the stable reduction theorem with the point of view of Deligne and Mumford. We introduce the basic material needed to manipulate models of curves, including intersection theory on regular arithmetic surfaces, blow-ups and blow-downs, and the structure of the jacobian of a singular curve. The proof of stable reduction in characteristic 0 is given, while the proof in the general case is explained and important parts are proved. We give applications to the moduli of curves and covers of curves.

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1. Introduction

The problem of resolution of singularities over a field has a cousin of more arithmetic flavor known as *semistable reduction*. Given a field K, complete with respect to a discrete valuation v, and a proper smooth K-variety X, its concern is to find a regular scheme $\mathcal{X}$, proper and flat over the ring of integers of v, with generic fibre isomorphic to X and with special fibre a reduced normal crossings divisor in $\mathcal{X}$. Such a scheme $\mathcal{X}$ is called a *semistable model*. In general, one can not expect K-varieties to have smooth models, and semistable models are a very nice substitute; they are in fact certainly the best one can hope. Their occurrence in arithmetic geometry is ubiquitous for the study of ℓ-adic or p-adic cohomology, and of Galois representations. They are useful for the study of general models $\mathcal{X}'$, but also if one is interested in X in the first place. Let us give just one example showing some of the geometry of X revealed by its semistable models. If X is a curve, then Berkovich proved that the dual graph Γ of the special fibre of any semistable model has a natural embedding in the analytic space X^{an} (in the sense of Berkovich) associated to X and that this analytic space deformation retracts to Γ. (See [Be], Chapter 4.) In other words, the homotopy type of the analytic space

X^{an}, which is just a transcendental incarnation of X, is encoded in the special fibres of semistable models.

It is believed that semistable reduction is always possible after a finite extension of K. It is known only in the case of curves, where a refinement called *stable reduction* leads to the construction of a smooth compactification of the moduli stack of curves. The objective of the present text is to give a quick introduction to the original proof of these facts, following Deligne and Mumford's paper [DM]. Other subsequent proofs from Artin and Winters [AW], Bosch and Lütkebohmert [BL] or Saito [Sa] are not at all mentioned. (Note that apart from the original papers, some nice expositions such as [Ra2], [De], [Ab] are available.)

The exposition follows quite faithfully the plan of the lectures given by the author at the GAMSC summer school held in Istanbul in June 2008. Here is now a more detailed description of the contents of the article. When the residue characteristic is 0, the theorem is a simple computation of normalisation. Otherwise, the proof uses more material than could reasonably be covered within the lectures. I took for granted the semistable reduction theorem for abelian varieties proven by Grothendieck, as well as Raynaud's results on the Picard functor; this is consistent with the development in [DM]. Section 2 focuses on the manipulations on models: blow-ups and contractions, existence of (minimal) regular models. In Section 3, the description of the Picard functor of a singular curve is explained, and it is then used to make the link between semistable reduction of a curve and semistable reduction of its jacobian. This is the path to the proof of Deligne and Mumford. Finally, in Section 4, we translate these results to prove that moduli spaces (or moduli stacks) of stable curves, or covers of stables curves, are proper.

The main references are Deligne and Mumford [DM], Lichtenbaum [Lic], Liu's book [Liu] together with other sources which the reader will find in the bibliography in the end of this paper. I wish to thank the students and colleagues who attended the Istanbul summer school for their questions and comments during, and after, the lectures. Also, I wish to thank the referee for valuable comments leading to several clarifications.

2. Models of curves

In all the text, a *curve* over a base field is a proper scheme over that field, of pure dimension 1. Starting in Subsection 2.2, we fix a complete discrete valuation ring R with fraction field K and algebraically closed residue field k.

2.1. Definitions: normal, regular, semistable models

If K is a field equipped with a discrete valuation v and C is a smooth curve over K, then a natural question in arithmetic is to ask about the reduction of C modulo v. This implies looking for flat models of C over the ring of v-integers $R \subset K$ with the mildest possible singularities. If there exists a model with smooth special fibre over the residue field k of R, we say that C has *good reduction* at v (and otherwise we say that C has *bad reduction* at v).

It is known that there exist curves which do *not* have good reduction, and there are at least two reasons for this deficiency. The first reason is arithmetic: sometimes, the smooth special fibre (if it existed) must have rational points and this imposes some constraints on C. For example, consider the smooth projective conic C over the field $K = \mathbb{Q}_2$ of 2-adic numbers given by the equation $x^2+y^2+z^2 = 0$. If C had a smooth model X over $R = \mathbb{Z}_2$, then the special fibre X_k would have a rational point by the Chevalley-Warning theorem (as in [Se], Chap. 1) and hence X would have a $\mathbb{Z}_2$-integral point by the henselian property of $\mathbb{Z}_2$. However, it is easy to see by looking modulo 4 that C has no $\mathbb{Q}_2$-rational point. (One can easily cook up similar examples with curves of higher genus over a field K with algebraically closed residue field.) The second reason is geometric. Assuming a little familiarity with the moduli space of curves $\mathcal{M}_g$, it can be explained as follows: the "direction" in the *nonproper* space $\mathcal{M}_g$ determined by the path $\mathrm{Spec}(R)\backslash\{\text{closed point}\} \to \mathcal{M}_g$ corresponding to the curve C points to the boundary at infinity. For a simple example of this, consider the field of Laurent series $K = k((\lambda))$ which is complete for the λ-adic topology, and the Legendre elliptic curve E/K with equation $y^2 = x(x-1)(x-\lambda)$. Its j-invariant $j(\lambda) = 2^8(\lambda^2-\lambda+1)^3/(\lambda^2(\lambda-1)^2)$ determines the point corresponding to E in the moduli space of elliptic curves. Since $j(\lambda) \notin R = k[[\lambda]]$, the curve E has bad reduction (see [Si], Chap. VII, Prop. 5.5).

The arithmetic problem is not so serious, and we usually allow a finite extension K'/K before testing if the curve admits good reduction. However, the geometric problem is more considerable.

So, we have to consider other kinds of models. The mildest curve singularity is a *node*, also called *ordinary double point*, that is to say a rational point $x \in C$ such that the completed local ring $\widehat{\mathcal{O}}_{C,x}$ is isomorphic to $k[[u,v]]/(uv)$.

This leads to:

Definition 2.1.1. A *stable* (resp. *semistable*) curve over an algebraically closed field k is a curve which is reduced, connected, has only nodal singularities, all of whose irreducible components isomorphic to $\mathbb{P}^1_k$ meet the other components in at least 3 points (resp. 2 points).

A proper flat morphism of schemes $X \to S$ is called a *stable* (resp. *semi-stable*) curve if it has *stable* (resp. *semi-stable*) geometric fibres. In particular, given a smooth curve C over a discretely valued field K, a stable (resp. semi-stable) curve $X \to S = \mathrm{Spec}(R)$ with a specified isomorphism $X_K \simeq C$ is called a *stable* (resp. *semi-stable*) *model* of C over R.

One can also understand the expression *the mildest possible singularities* in an absolute meaning. For example, one can look for *normal* or *regular* models of the K-curve C, by which we mean a curve $X \to S = \mathrm{Spec}(R)$ whose total space is normal, or regular. By normalization, one may always find normal models. Regular models will be extremely important, firstly because they are somehow easier to produce than stable models, secondly because it is possible to do intersection theory on them, and thirdly because they are essential to the construction of stable models. We emphasize that in contrast with the notions of stable and semistable models,

the notions of normal and regular models are not relative over S, in particular such models have in general singular, possibly nonreduced, special fibres.

For simplicity we shall call *arithmetic surface* a proper, flat scheme relatively of pure dimension 1 over R with smooth geometrically connected generic fibre. We will specify each time if we speak about a normal arithmetic surface, or a regular arithmetic surface, etc.

2.2. Existence of regular models

From this point until the end of the notes, we consider a complete discrete valuation ring R with fraction field K and algebraically closed residue field k.

For two-dimensional schemes, the problem of resolution of singularities has a satisfactory solution, with a strong form. Before we state the result, recall that a divisor D in a regular scheme X has *normal crossings* if for every point $x \in D$ there is an étale morphism of pointed schemes $p : (U, u) \to (X, x)$ such that p^*D is defined by an equation $a_1 \dots a_n = 0$ where $a_1, \dots, a_n$ are part of a regular system of parameters at u.

Theorem 2.2.1. *For every excellent, reduced, noetherian two-dimensional scheme X, there exists a proper birational morphism $X' \to X$ where X' is a regular scheme. Furthermore, we may choose X' such that its reduced special fibre is a normal crossings divisor.*

In fact, following Lipman [Lip2], one may successively blow up the singular locus and normalize, producing a sequence

$$\cdots \to X_n \to \cdots \to X_1 \to X_0 = X$$

that is eventually stationary at some regular X^*. Then one can find a composition of a finite number of blow-ups $X' \to X^*$ so that the reduced special fibre of X' is a normal crossings divisor. For details on this point, see [Liu], Section 9.2.4 (note that in *loc. cit.* the definition of a normal crossings divisor is different from ours, since it allows the divisor to be nonreduced).

2.3. Intersection theory on regular arithmetic surfaces

The intersection theory on an arithmetic surface, provided it can be defined, is determined by the intersection numbers of 1-cycles or Weil divisors. The prime cycles fall into two types: *horizontal* divisors are finite flat over R, and *vertical* divisors are curves over the residue field k of R. Let $\mathrm{Div}(X)$ be the free abelian group generated by all prime divisors of X, and $\mathrm{Div}_k(X)$ be the subgroup generated by vertical divisors.

In classical intersection theory, as exposed for example in Fulton's book [Ful], the possibility to define an intersection product $E \cdot F$ for *arbitrary* cycles E, F in a variety V requires the assumption that V is smooth. It would be too strong an assumption to require our surfaces to be smooth over R, but as we saw in the previous subsection, we can work with regular models. As it turns out, for them one can define at least a bilinear map $\mathrm{Div}_k(X) \times \mathrm{Div}(X) \to \mathbb{Z}$.

More precisely, let X be a regular arithmetic surface over R, let $i : E \hookrightarrow X$ be a prime vertical divisor and $j : F \hookrightarrow X$ an arbitrary effective divisor. By regularity, Weil divisors are the same as Cartier divisors, so the ideal sheaf $\mathcal{I}$ of F is invertible. Since E is a curve over the residue field k there is a usual notion of *degree* for line bundles, and we may define an intersection number by the formula

$$E \cdot F := \deg_E(i^*\mathcal{I}^{-1}) \ .$$

It follows from this definition that if $E \neq F$, then $E \cdot F$ is at least equal to the number of points in the support of $E \cap F$, in particular it is nonnegative. It is easy to see also that if E and F intersect transversally at all points, then $E \cdot F$ is exactly the number of points in the support of $E \cap F$ (the assumption that k is algebraically closed allows not to care about the degrees of the residue fields extensions). The intersection product extends by bilinearity to a map $\mathrm{Div}_k(X) \times \mathrm{Div}(X) \to \mathbb{Z}$ satisfying the following properties:

Proposition 2.3.1. *Let E, F be divisors on a regular arithmetic surface X with E vertical. Then one has:*

(1) *if F is a vertical divisor then $E \cdot F = F \cdot E$,*
(2) *if E is prime then $E \cdot F = \deg_E(\mathcal{O}(F) \otimes \mathcal{O}_E)$,*
(3) *if F is principal then $E \cdot F = 0$.*

Proof. Cf. [Lic], Part I, Section 1. □

Here are the most important consequences concerning intersection with vertical divisors.

Theorem 2.3.2. *Let X be a regular arithmetic surface and let $E_1, \ldots, E_r$ be the irreducible components of X_k. Then:*

(1) *$X_k \cdot F = 0$ for all vertical divisors F,*
(2) *$E_i \cdot E_j \geq 0$ if $i \neq j$ and $E_i^2 < 0$,*
(3) *the bilinear form given by the intersection product on $\mathrm{Div}_k(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is negative semi-definite, with isotropic cone equal to the line generated by X_k.*

Proof. (1) The special fibre X_k is the pullback of the closed point of $\mathrm{Spec}(R)$, a principal Cartier divisor, so it is a principal Cartier divisor in X. Hence $X_k \cdot F = 0$ for all vertical divisors F, by 2.3.1(3).

(2) If $i \neq j$, we have $E_i \cdot E_j \geq \#|E_i \cap E_j| \geq 0$. From this together with point (1) and the fact that the special fibre is connected, we deduce that

$$E_i^2 = (E_i - X_k) \cdot E_i = -\sum_{j \neq i} E_j \cdot E_i < 0 \ .$$

(3) Let d_i be the multiplicity of E_i, $a_{ij} = E_i \cdot E_j$, $b_{ij} = d_i d_j a_{ij}$. Let $v = \sum v_i E_i$ be a vector in $\mathrm{Div}_k(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $w_i = v_i/d_i$. We have $\sum_i b_{ij} = X_k \cdot (d_j F_j) = 0$ by point (1), and $\sum_j b_{ij} = 0$ by symmetry, so

$$v \cdot v = \sum_{i,j} a_{ij} v_i v_j = \sum_{i,j} b_{ij} w_i w_j = -\frac{1}{2} \sum_{i \neq j} b_{ij} (w_i - w_j)^2 \leq 0 \ .$$

Hence the intersection product on $\mathrm{Div}_k(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is negative semi-definite. Finally if $v \cdot v = 0$, then $b_{ij} \neq 0$ implies $w_i = w_j$. Since X_k is connected, we obtain that all the w_i are equal and hence $v = w_1 X_s$. Thus the isotropic cone is included in the line generated by X_k, and the opposite inclusion has already been proved. $\square$

Example 2.3.3. Let X be a regular arithmetic surface whose special fibre is reduced, with nodal singularities. Let $E_1, \ldots, E_r$ be the irreducible components of X_k. Then $E_i \cdot E_j$ is the number of intersection points of E_i and E_j if $i \neq j$, and $(E_i)^2$ is the opposite of the number of points where E_i meets another component, by point (1) of the theorem. Hence X_k is stable (resp. semi-stable) if and only it does not contain a projective line with self-intersection -2 (resp. with self-intersection -1).

As far as horizontal divisors are concerned, the most interesting one to intersect with is the *canonical divisor* associated to the canonical sheaf, whose definition we recall below. If E is an effective vertical divisor in X, the adjunction formula gives a relation between the canonical sheaves of X/R and that of E/k. The main reason why the canonical divisor is interesting is that on a regular arithmetic surface, the canonical sheaf is a dualizing sheaf in the sense of the Grothendieck-Serre duality theory, therefore the adjunction formula translates, via the Riemann-Roch theorem, into an expression of the intersection of E with the canonical divisor of X in terms of the Euler-Poincaré characteristic χ of E. We will now explain this.

Let us first recall briefly the definition of the canonical sheaf of a regular arithmetic surface X, assuming that X is projective (it can be shown that this is always the case, see [Lic]). We choose a projective embedding $i : X \hookrightarrow P := \mathbb{P}^n_R$ and note that since X and P are regular, then i is a regular immersion. It follows that the conormal sheaf $\mathcal{C}_{X/P} = i^*(\mathcal{I}/\mathcal{I}^2)$ is locally free over X, where $\mathcal{I}$ denotes the ideal sheaf of X in P. Also since P is smooth over R, the sheaf of differential 1-forms $\Omega^1_{P/R}$ is locally free over R. Thus the maximal exterior powers of the sheaves $\mathcal{C}_{X/P}$ and $i^*\Omega^1_{P/R}$, also called their *determinant*, are invertible sheaves on X. The canonical sheaf is defined to be the invertible sheaf

$$\omega_{X/R} := \det(\mathcal{C}_{X/P})^\vee \otimes \det(i^*\Omega^1_{P/R})$$

where $(\cdot)^\vee = \mathcal{H}om(\cdot, \mathcal{O}_X)$ is the linear dual. It can be proved that $\omega_{X/R}$ is independent of the choice of a projective embedding for X, and that it is a dualizing sheaf. Any divisor K on X such that $\mathcal{O}_X(K) \simeq \omega_{X/R}$ is called a *canonical divisor*.

Theorem 2.3.4. *Let X be a regular arithmetic surface over R, E a vertical positive Cartier divisor with $0 < E \leq X_k$, and $K_{X/R}$ a canonical divisor. Then we have the adjunction formula*

$$-2\chi(E) = E \cdot (E + K_{X/R}) \ .$$

Proof. In fact, the definition of $\omega_{X/R}$ is valid as such for an arbitrary local complete intersection (lci) morphism. Moreover, for a composition of two lci morphisms $f : X \to Y$ and $g : Y \to Z$ we have the general adjunction formula $\omega_{X/Z} \simeq \omega_{X/Y} \otimes_{\mathcal{O}_X} f^*\omega_{Y/Z}$, see [Liu], Section 6.4.2. In particular we have $\omega_{E/R} \simeq \omega_{E/k} \otimes$

$f^*\omega_{k/R} \simeq \omega_{E/k}$ where $f : E \to \mathrm{Spec}(k)$ is the structure morphism. A useful particular case of computation of the canonical sheaf is $\omega_{D/X} = \mathcal{O}_X(D)|_D$ for an effective Cartier divisor D in a locally noetherian scheme X (this is left as an exercise). Using this particular case and the general adjunction formula for the composition $E \to X \to \mathrm{Spec}(R)$, we have

$$\omega_{E/k} \simeq \omega_{E/R} \simeq \omega_{E/X} \otimes \omega_{X/R}|_E \simeq (\mathcal{O}_X(E) \otimes \omega_{X/R})|_E \ .$$

By the Riemann-Roch theorem, we have $\deg(\omega_{E/k}) = -2\chi(E)$ and the asserted formula follows, by taking degrees. □

2.4. Blow-up, blow-down, contraction

We assume that the reader has some familiarity with blow-ups, and we recall only the features that will be useful to us. Let X be a noetherian scheme and $i : Z \hookrightarrow X$ a closed subscheme with sheaf of ideals $\mathcal{I}$. The *blow-up of X along Z* is the morphism $\pi : \widetilde{X} \to X$ with $\widetilde{X} = \mathrm{Proj}(\oplus_{d\geq 0}\mathcal{I}^d)$. The *exceptional divisor* is $E := V(\mathcal{I}\mathcal{O}_{\widetilde{X}})$; it is a Cartier divisor. If i is a regular immersion, then the conormal sheaf $\mathcal{C}_{Z/X} = i^*(\mathcal{I}/\mathcal{I}^2)$ is locally free and $E \simeq \mathbb{P}(i^*(\mathcal{I}/\mathcal{I}^2))$ as a projective fibre bundle over Z; it carries a sheaf $\mathcal{O}_E(1)$. In this case, one can see that the sheaf $\mathcal{O}_{\widetilde{X}}(E)|_E$ is naturally isomorphic to $\mathcal{O}_E(-1)$, because $\mathcal{O}_{\widetilde{X}}(E) \simeq (\mathcal{I}\mathcal{O}_{\widetilde{X}})^{-1}$.

Example 2.4.1. Let X be a regular arithmetic surface and $Z = \{x\}$ a regular closed point of the special fibre. Then $\widetilde{X}$ is again a regular arithmetic surface and the exceptional divisor is a projective line over k, with self-intersection -1.

Example 2.4.2. Let x be a nodal singularity in the special fibre of a normal arithmetic surface. The completed local ring is isomorphic to $\mathcal{O} = R[[a,b]]/(ab - \pi^n)$ for some $n \geq 1$. We call the integer n the *thickness* of the node. We blow up $\{x\}$ inside $X = \mathrm{Spec}(\mathcal{O})$. If $n = 1$, the point x is regular so we are in the situation of the preceding example. If $n \geq 2$, the point x is a singular normal point and it is an exercise to compute that the blow-up of X at this point is

$$\widetilde{X} = \mathrm{Proj}(\mathcal{O}[[u,v,w]]/(uv - \pi^{n-2}w^2, av - bu, bw - \pi v, aw - \pi u)) \ .$$

If $n = 2$, the exceptional divisor is a smooth conic over k with self-intersection -2. If $n \geq 3$, the exceptional divisor is composed of two projective lines intersecting in a nodal singularity of thickness $n-2$, each meeting the rest of the special fibre in one point.

Remark 2.4.3. We saw that among the nodal singularities $ab - \pi^n$, the regular one for $n = 1$ shows a different behaviour. Here is one more illustration of this fact. Let X be a regular arithmetic surface and assume that X_K has a rational point $\mathrm{Spec}(K) \to X$. By the valuative criterion of properness, this point extends to a section $\mathrm{Spec}(R) \to X$, and we denote by $x : \mathrm{Spec}(k) \to X$ the reduction. Let $\mathcal{O} = \mathcal{O}_{X,x}$, $i : R \to \mathcal{O}$ the structure morphism, m the maximal ideal of R, n the maximal ideal of $\mathcal{O}$. Thus we have a map $s : \mathcal{O} \to R$ such that $s \circ i = \mathrm{id}$, and one checks that this forces to have an injection of cotangent k-vector spaces $m/m^2 \subset n/n^2$. Therefore we can choose a basis of n/n^2 containing the image of

π, in other words we can choose a system of parameters for $\mathcal{O}$ containing π. This proves that $\mathcal{O}/\pi = \mathcal{O}_{X_k,x}$ is regular. To sum up, the reduction of a K-rational point on a regular surface X is a regular point of X_k. Of course, this is false as soon as $n \geq 2$, since the point with coordinates $a = \pi$, $b = \pi^{n-1}$ reduces to the node.

The process of blowing-up is a prominent tool in the birational study of regular surfaces. For obvious reasons, it is also very desirable to reverse this operation and examine the possibility to *blow down*, that is to say to characterize those divisors $E \subset X$ in regular surfaces that are exceptional divisors of some blow-up of a *regular* scheme. Note that if $f : X \to Y$ is the blow-up of a point y, then π is also the blow-down of $E := f^{-1}(y)$ and the terminology is just a way to put emphasis on (Y, y) or on (X, E).

As a first step, it is a general fact that one can *contract* the component E, and the actual difficult question is the nature of the singularity that one gets. We choose to present contractions in their natural setting, and then we will state without proof the classical results of Castelnuovo, Artin and Lipman on the control of the singularities.

Definition 2.4.4. Let X be a normal arithmetic surface. Let $\mathcal{E}$ be a set of irreducible components of the special fibre X_k. A *contraction* is a morphism $f : X \to Y$ such that Y is a normal arithmetic surface, $f(E)$ is a point for all $E \in \mathcal{E}$, and f induces an isomorphism

$$X \setminus \bigcup_{E \in \mathcal{E}} E \longrightarrow Y \setminus \bigcup_{E \in \mathcal{E}} f(E) \ .$$

Using the Stein factorization, it is relatively easy to see that f is unique if it exists, and that its fibres are connected. Under our assumption that R is complete with algebraically closed residue field, one can always construct an effective relative (i.e., R-flat) Cartier divisor D of X meeting exactly the components of X_k not belonging to $\mathcal{E}$. Indeed, for example if X_k is reduced, one can choose one smooth point in each component not in $\mathcal{E}$. Since R is henselian these points lift to sections of X over R, and we can take D to be the sum of these sections. If X_k is not reduced, a similar argument using Cohen-Macaulay points instead of smooth points does the job, cf. [BLR], Proposition 6.7/4. Thus, existence of contractions follows from the following result:

Theorem 2.4.5. *Let X be a normal arithmetic surface. Let $\mathcal{E}$ be a strict subset of the set of irreducible components of the special fibre X_k, and D an effective relative Cartier divisor of X over R meeting exactly the components of X_k not belonging to $\mathcal{E}$. Then the morphism*

$$f : X \to Y := \operatorname{Proj}\big(\bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)\big)$$

is a contraction of the components of $\mathcal{E}$.

Proof. We first explain what is f. Let us write $H^0(X, \mathcal{O}_X(nD))^\sim$ for the associated constant sheaf on X. Note that $\operatorname{Proj}(\oplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD))^\sim) \simeq Y \times_R X$,

and $\mathrm{Proj}(\oplus_{n\geq 0}\mathcal{O}_X(nD)) \simeq X$ canonically (see [Ha], Chap. II, Lemma 7.9). The restriction of sections gives a natural map of graded $\mathcal{O}_X$-algebras

$$\bigoplus_{n\geq 0} H^0(X,\mathcal{O}_X(nD))^{\sim} \to \bigoplus_{n\geq 0}\mathcal{O}_X(nD)\ .$$

We obtain f by taking Proj and composing with the projection $Y\times_R X\to Y$.

Since D_K has positive degree on X_K, it is ample and it follows that the restriction of f to the generic fibre is an isomorphism. Also, after some more work this implies that $\mathcal{O}_X(nD)$ is generated by its global sections if n is large enough; we will admit this point, and refer to [BLR], p. 168 for the details. Therefore the ring $A=\oplus_{n\geq 0}H^0(X,\mathcal{O}_X(nD))$ is of finite type over R by [EGA2], 3.3.1, and so Y is a projective R-scheme. Moreover X is covered by the open sets X_ℓ where ℓ does not vanish, for all global sections $\ell\in H^0(X,\mathcal{O}_X(nD))$, and f induces an isomorphism

$$A_{(\ell)} \xrightarrow{\sim} H^0(X_\ell,\mathcal{O}_X)\ .$$

If follows that $A_{(\ell)}$, and hence Y, is normal and flat over R. Moreover we see that $f_*\mathcal{O}_X\simeq\mathcal{O}_Y$, so by Zariski's connectedness principle (cf. [Liu], 5.3.15) it follows that the fibres of f are connected.

It remains to prove that f is a contraction of the components of $\mathcal{E}$. If $E\in\mathcal{E}$, then $\mathcal{O}_X(nD)|_E\simeq\mathcal{O}_E$ and hence any global section of $\mathcal{O}_X(nD)$ induces a constant function on E, since E is proper. It follows that the image $f(E)$ is a point. If $E\notin\mathcal{E}$, we may choose a point $x\in E\cap\mathrm{Supp}(D)$. Let ℓ be a global section that generates $\mathcal{O}_X(nD)$ on a neighbourhood U of x, for some n large enough. Then $1/\ell$ is a function on X_ℓ that, by definition, vanishes on $U\cap\mathrm{Supp}(D)$ (with order n) and is non-zero on $U-\mathrm{Supp}(D)$. Thus $f|_E$ is not constant, so it is quasi-finite. Since its fibres are connected, in fact $f|_E$ is birational, and since Y is normal we deduce that $f|_E$ is an isomorphism onto its image, by Zariski's main theorem (cf. [Liu], 4.4.6). □

The numerical information that we have collected about exceptional divisors in Subsection 2.3 is crucial to control the singularity at the image points of the components that are contracted, as in the following two results which we will use without proof. The first is Castelnuovo's criterion about blow-downs.

Theorem 2.4.6. *Let X be a regular arithmetic surface and E a vertical prime divisor. Then there exists a blow-down of E if and only if $E\simeq\mathbb{P}^1_k$ and $E^2=-1$.*

Proof. See [Lic], Theorem 3.9, or [Liu], Theorem 9.3.8. □

The second result which we want to mention is an improvement by Lipman [Lip1] of previous results of Artin [Ar] on contractions for algebraic surfaces. The statement uses the following fact, which we quote without proof (see [Liu], Lemma 9.4.12): for a regular arithmetic surface X and distinct vertical prime divisors $E_1,\dots,E_r$ such that the intersection matrix $(E_i\cdot E_j)$ is negative semi-definite, there exists a smallest effective divisor $C=\sum a_iE_i$ such that $C\geq\sum_i E_i$ and $C\cdot E_i\geq 0$ for all i. We call C the *fundamental divisor* for $\{E_i\}_i$.

Theorem 2.4.7. *Let X be a regular arithmetic surface and let $E_1, \ldots, E_r$ be distinct reduced vertical prime divisors with negative semi-definite intersection matrix. Assume that the Euler-Poincaré characteristic of the fundamental divisor C associated to the E_i is positive. Then the contraction of $E_1, \ldots, E_r$ is a normal arithmetic surface, and the resulting singularity is a regular point if and only if $-C^2 = H^0(C, \mathcal{O}_C)$.*

Proof. See [Lip1], Theorem 27.1, or [Liu], Theorem 9.4.15. Note that in the terminology of [Lip1], a rational double point, (i.e., a rational singularity with multiplicity 2) is none other than a node of the special fibre. □

2.5. Minimal regular models

We can now state the main results of the birational theory of arithmetic surfaces:

Theorem 2.5.1. *Let C be a smooth geometrically connected curve over K, of genus $g \geq 1$. Then C has a minimal regular model over R, unique up to a unique isomorphism.*

Proof. By Theorem 2.2.1, there exists a regular model for C. By successive blow-downs of exceptional divisors, we construct a regular model X that is relatively minimal. Let X' be another such model. Since any two regular models are dominated by a third ([Lic], Proposition 4.2) and any morphism between two models factors into a sequence of blow-ups ([Lic], Theorem 1.15), there exist sequences of blow-ups

$$Y = X_m \to X_{m-1} \to \cdots \to X_1 \to X_0 = X$$

and

$$Y = X'_n \to X'_{n-1} \to \cdots \to X'_1 \to X'_0 = X'$$

terminating at the same Y. We may choose Y such that $m+n$ is minimal. If $m > 0$, there is an exceptional curve E for the morphism $Y \to X_{m-1}$. Since X' has no exceptional curve, the image of E in X' is not an exceptional curve, hence there is an r such that the image of E in X'_r is the exceptional divisor of $X'_r \to X'_{r-1}$. Also, for all $i \in \{r, \ldots, n-1\}$ the image of E in the surface X'_i does not contain the center of the blow-up $X'_{i+1} \to X'_i$. Thus, we can rearrange the blow-ups so that E is the exceptional curve of $Y \to X'_{n-1}$. Therefore $X_{m-1} \simeq X'_{n-1}$ and this contradicts the minimality of $m+n$. It follows that $m = 0$, so there is a morphism $X \to X'$, and since X is relatively minimal we obtain $X \simeq X'$. □

Theorem 2.5.2. *Let C be a smooth geometrically connected curve over K, of genus $g \geq 1$. Then C has a minimal regular model with normal crossings over R. It is unique up to a unique isomorphism.*

Proof. In fact Theorem 2.2.1 asserts the existence of a regular model with normal crossings. Proceeding along the same lines as in the proof of the above theorem, one produces a minimal regular model with normal crossings. □

3. Stable reduction

In this section, C is a smooth geometrically connected curve over K, of genus $g \geq 2$.

3.1. Stable reduction is equivalent to semistable reduction

Proposition 3.1.1. *Let C be a smooth geometrically connected curve over K, of genus $g \geq 2$. Then the following conditions are equivalent:*

(1) *C has stable reduction,*
(2) *C has semistable reduction,*
(3) *the minimal regular model of C is semistable.*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3): let X be a semistable model of C over R. Replacing X by the repeated blow-down of all exceptional divisors in the regular locus of X, we may assume that it has no exceptional divisor. Then, by the deformation theory of the node (cf. [Liu], 10.3.22), the completed local ring of a singular point $x \in X_k$ is $\widehat{\mathcal{O}}_{X,x} \simeq R[[a,b]]/(ab - \pi^n)$ for some $n \geq 2$. By Example 2.4.2, blowing-up $[n/2]$ times the singularity leads to a regular scheme X' whose special fibre has $n-1$ new projective lines of self-intersection -2. This is the minimal regular model of C, which is therefore semistable.

(3) $\Rightarrow$ (1): let X be the minimal regular model of C. Consider the family of all components of the special fibre that are projective lines of self-intersection -2. A connected configuration of such lines is either topologically a circle, or a segment. Since $g \geq 2$, the first possibility can not occur. It follows that such a configuration has positive Euler-Poincaré characteristic, so by Theorem 2.4.7, the contraction of these lines is a normal surface with nodal singularities. $\square$

3.2. Proof of semistable reduction in characteristic 0

Theorem 3.2.1. *Assume that the residue field k has characteristic 0. Let X be the minimal regular model with normal crossings of C and let $n_1, \ldots, n_r$ be the multiplicities of the irreducible components of X_k. Let n be a common multiple of $n_1, \ldots, n_r$ and $R' = R[\rho]/(\rho^n - \pi)$. Then the normalization of $X \times_R R'$ is semistable.*

The key fact is that in residue characteristic 0, divisors with normal crossings have a particularly simple local shape. This is due to the possibility to extract nth roots.

Proof. Let $x \in X$ be a closed point of X_k and let A be the completion of its local ring in X. We will use two facts about A: firstly, since k is algebraically closed of characteristic 0 and A is complete, it follows from Hensel's lemma that one can extract nth roots in A for all integers $n \geq 1$. Note that by the same argument R contains all roots of unity. Secondly, since A is a regular noetherian local ring, it is a unique factorization domain, and each regular system of parameters (f, g) is composed of prime elements.

Since $(X_k)_{\mathrm{red}}$ is a normal crossings divisor, we have two possibilities. The first possibility is that $\sqrt{\pi}A = (f)$ for some regular system of parameters (f, g). In this case f is the only prime factor of π, so $\pi = uf^a$ for some unit $u \in A$. Since k is algebraically closed of characteristic 0 and A is complete, one sees that u is an ath power in A so that changing f if necessary we have $\pi = f^a$. Then one checks that the natural map $R[[u, v]]/(u^a - \pi) \to A$ taking u to f and v to g is an isomorphism. Here a is the multiplicity of the component of X_k containing x, so by assumption $n = am$ for some integer m. Then

$$A \otimes_R R' \simeq R'[[u, v]]/(u^a - \rho^{am}) \simeq R'[[u, v]]/(\Pi(u - \zeta\rho^m))$$

with the product ranging over the ath roots of unity ζ. The normalization of this ring is the product of the normal rings $R'[[u, v]]/(u - \zeta\rho^m) \simeq R'[[v]]$ so the normalization of $X \times_R R'$ is smooth at all points lying over x.

The second possibility is that $\sqrt{\pi}A = (fg)$ for some regular system of parameters (f, g). In this case f and g are the only prime factors of π, so $\pi = uf^a g^b$ for some unit $u \in A$ which as above may be chosen to be 1. Thus $\pi = f^a g^b$ and one checks that the natural map $R[[u, v]]/(u^a v^b - \pi) \to A$ taking u to f and v to g is an isomorphism. Again a and b are the multiplicities of the two components at x. Let $d = \gcd(a, b)$, $a = d\alpha$, $b = d\beta$, $n = d\alpha\beta m$. Then as above the normalization of $A \otimes_R R'$ is the product of the normalizations of the rings $R'[[u, v]]/(u^\alpha v^\beta - \zeta\rho^{\alpha\beta m})$ for all dth roots of unity ζ. If we introduce $\xi \in R$ such that $\xi^{\alpha\beta} = \zeta$ then the normalization is the morphism

$$A = R'[[u, v]]/(u^\alpha v^\beta - \zeta\rho^{\alpha\beta m}) \to B = R'[[x, y]]/(xy - \xi\rho^m)$$

given by $u \mapsto x^\beta$ and $v \mapsto y^\alpha$. Indeed, the ring B is normal and one may realize it in the fraction field of A by choosing i, j such that $i\alpha + j\beta = 1$ and setting

$$x = u^j(\xi^\alpha \rho^{\alpha m}/v)^i \quad \text{and} \quad y = v^i(\xi^\beta \rho^{\beta m}/u)^j \ . \qquad \square$$

3.3. Generalized jacobians

Let X be an arbitrary connected projective curve over an algebraically closed field k. It can be shown that the identity component $\underline{\mathrm{Pic}}^0(X)$ of the Picard functor is representable by a smooth connected algebraic group called the *generalized jacobian* of X and denoted $\mathrm{Pic}^0(X)$. In this subsection, which serves as a preparation for the next subsection, we will give a description of $\mathrm{Pic}^0(X)$. The first feature of $\mathrm{Pic}^0(X)$ which is readily accessible is its tangent space at the identity:

Lemma 3.3.1. *The tangent space of* $\mathrm{Pic}^0(X)$ *at the identity is canonically isomorphic to* $H^1(X, \mathcal{O}_X)$.

Proof. Let $k[\epsilon]$, with $\epsilon^2 = 0$, be the ring of dual numbers and let $X[\epsilon] := X \times_k k[\epsilon]$. Consider the exact sequence

$$0 \longrightarrow \mathcal{O}_X \overset{x \mapsto 1+\epsilon x}{\longrightarrow} \mathcal{O}_{X[\epsilon]}^\times \longrightarrow \mathcal{O}_X^\times \longrightarrow 0 \ .$$

In the associated long exact sequence, the map $H^0(\mathcal{O}_{X[\epsilon]}^\times) \to H^0(\mathcal{O}_X^\times)$ is surjective since the second group contains nothing else but the invertible constant functions.

It follows that the kernel of the morphism $H^1(\mathcal{O}^\times_{X[\epsilon]}) \to H^1(\mathcal{O}^\times_X)$ is isomorphic to $H^1(X, \mathcal{O}_X)$. Since $H^1(\mathcal{O}^\times_X) = \mathrm{Pic}(X)$ and $H^1(\mathcal{O}^\times_{X[\epsilon]}) = \mathrm{Pic}(X[\epsilon])$, the kernel is by definition the tangent space at the identity. □

In order to go further into the structure of $\mathrm{Pic}^0(X)$, we introduce an intermediary curve X' sandwiched between the reduced curve X_{red} and its normalization $\widetilde{X}$. This curve is obtained topologically as follows. Look at all points $x \in X_{\mathrm{red}}$ with $r \geq 2$ preimages $\widetilde{x}_1, \ldots, \widetilde{x}_r$ in $\widetilde{X}$, and glue these preimages transversally. The curve X' may be better described by its structure sheaf as a subsheaf of $\mathcal{O}_{\widetilde{X}}$: its functions are the functions on $\widetilde{X}$ taking the same value on $\widetilde{x}_1, \ldots, \widetilde{x}_r$ for all points x as above. Thus X' has only ordinary singularities, that is to say singularities that locally look like the union of the coordinate axes in some affine space $\mathbb{A}^r$. Note that the integer r, called the *multiplicity*, may be recovered as the dimension of the tangent space at the ordinary singularity. The curve X' is called the *curve with ordinary singularities associated to* X. It is also the largest curve between X_{red} and $\widetilde{X}$ which is universally homeomorphic to X_{red}. To sum up we have the picture:

$$\widetilde{X} \to X' \to X_{\mathrm{red}} \to X \ .$$

By pullback, we have morphisms $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_{\mathrm{red}}) \to \mathrm{Pic}^0(X') \to \mathrm{Pic}^0(\widetilde{X})$.

Lemma 3.3.2. *The morphism* $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_{\mathrm{red}})$ *is surjective with unipotent kernel of dimension* $\dim H^1(X, \mathcal{O}_X) - \dim H^1(X_{\mathrm{red}}, \mathcal{O}_{X_{\mathrm{red}}})$.

Proof. Let $\mathcal{I}$ be the ideal sheaf of X_{red} in X, i.e., the sheaf of nilpotent functions on X. Let $X_n \subset X$ be the closed subscheme defined by the sheaf of ideals $\mathcal{I}^{n+1}$. We use the filtration $\mathcal{I} \supset \mathcal{I}^2 \supset \cdots$. For each $n \geq 1$ we have an exact sequence

$$0 \to \mathcal{I}^n \to (\mathcal{O}_X/\mathcal{I}^{n+1})^\times \to (\mathcal{O}_X/\mathcal{I}^n)^\times \to 0$$

where the map $\mathcal{I}^n \to (\mathcal{O}_X/\mathcal{I}^{n+1})^\times$ takes x to $1 + x$. Since X is complete and connected the map $H^0(X, (\mathcal{O}_X/\mathcal{I}^{n+1})^\times) \to H^0(X, (\mathcal{O}_X/\mathcal{I}^n)^\times) = k^\times$ is surjective. Consequently the long exact sequence of cohomology gives a short exact sequence

$$0 \to H^1(X, \mathcal{I}^n) \to H^1(X_n, \mathcal{O}^\times_{X_n}) \to H^1(X_{n-1}, \mathcal{O}^\times_{X_{n-1}}) \to 0 \ .$$

Since the base is a field, all schemes are flat and hence this description is valid after any base change $S \to \mathrm{Spec}(k)$. So there is an induced exact sequence of algebraic groups

$$0 \to V_n \to \mathrm{Pic}^0(X_n) \to \mathrm{Pic}^0(X_{n-1}) \to 0$$

where V_n is the algebraic group which is the vector bundle over $\mathrm{Spec}(k)$ determined by the vector space $H^1(X, \mathcal{I}^n)$. Thus V_n is unipotent; note that the fact that V_n factors through the identity component of the Picard functor comes from the fact that it is connected. Finally $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_{\mathrm{red}})$ is surjective and the kernel is a successive extension of unipotent groups, so it is a unipotent group. The dimension count for the dimension of the kernel is immediate by inspection of the exact sequences. □

Remark 3.3.3. It is not true that $\mathrm{Pic}^0(X) \to \mathrm{Pic}^0(X_{\mathrm{red}})$ is an isomorphism if and only if $X_{\mathrm{red}} \hookrightarrow X$ is. For example if X is generically reduced, i.e., the sheaf of nilpotent functions has finite support, then $\mathrm{Pic}^0(X) \simeq \mathrm{Pic}^0(X_{\mathrm{red}})$.

Recall that the *arithmetic genus* of a projective curve over a field k is defined by the equality $p_a(X) = 1 - \chi(\mathcal{O}_X)$ where χ is the Euler-Poincaré characteristic.

Lemma 3.3.4. *The morphism* $\mathrm{Pic}^0(X_{\mathrm{red}}) \to \mathrm{Pic}^0(X')$ *is surjective with unipotent kernel of dimension* $p_a(X_{\mathrm{red}}) - p_a(X')$. *Moreover,* $p_a(X_{\mathrm{red}}) = p_a(X')$ *if and only if* $X' \to X_{\mathrm{red}}$ *is an isomorphism.*

Proof. Recall that the morphism $h : X' \to X_{\mathrm{red}}$ is a homeomorphism. We have an exact sequence

$$0 \to (\mathcal{O}_{X_{\mathrm{red}}})^\times \to (h_*\mathcal{O}_{X'})^\times \to \mathcal{F} \to 0$$

where the cokernel $\mathcal{F}$ has finite support, hence no higher cohomology. Since h is bijective and the curves X_{red}, X' are complete and connected we have

$$H^0(X_{\mathrm{red}}, (\mathcal{O}_{X_{\mathrm{red}}})^\times) = H^0(X', (\mathcal{O}_{X'})^\times) = k^\times$$

so the long exact sequence of cohomology gives

$$0 \to H^0(X_{\mathrm{red}}, \mathcal{F}) \to H^1(X_{\mathrm{red}}, (\mathcal{O}_{X_{\mathrm{red}}})^\times) \to H^1(X', (\mathcal{O}_{X'})^\times) \to 0\ .$$

Moreover $H^0(X_{\mathrm{red}}, \mathcal{F}) = \oplus \mathcal{O}_{X',x'}/\mathcal{O}_{X,x}$ where the direct sum runs over the non-ordinary singular points x of X_{red}, and x' is the unique point above x. Denoting by m_x the maximal ideal of the local ring of x, it is immediate to see that the inclusion $1 + m_{x'} \to \mathcal{O}_{X',x'}$ induces an isomorphism $\mathcal{O}_{X',x'}/\mathcal{O}_{X_{\mathrm{red}},x} \simeq (1 + m_{x'})/(1 + m_x)$. Using the fact that $\mathcal{O}_{X',x'}/m_x$ is an artinian ring, one may see that there is an integer $r \geq 1$ such that $(m_{x'})^r \subset m_r$. Then one introduces a filtration of $(1 + m_{x'})/(1+m_x)$ and proves as in the proof of Lemma 3.3.2 that the algebraic group U that represents $H^0(X_{\mathrm{red}}, \mathcal{F})$ is unipotent. We refer to [Liu], Lemmas 7.5.11 and 7.5.12 for the details of these assertions. Finally the exact sequence above induces an exact sequence of algebraic groups

$$0 \to U \to \mathrm{Pic}^0(X_{\mathrm{red}}) \to \mathrm{Pic}^0(X') \to 0$$

with U unipotent. The proof of the final statement about the dimension of the kernel can be found in [Liu], Lemma 7.5.18. □

Lemma 3.3.5. *The morphism* $\mathrm{Pic}^0(X') \to \mathrm{Pic}^0(\widetilde{X})$ *is surjective with toric kernel of dimension* $\mu - c + 1$, *where* μ *is the sum of the excess multiplicities* $m_x - 1$ *for all ordinary multiple points* $x \in X'$ *and* c *is the number of connected components of* $\widetilde{X}$.

Proof. Write $\pi : \widetilde{X} \to X'$ for the normalization map. We have an exact sequence

$$0 \to (\mathcal{O}_{X'})^\times \to (\pi_*\mathcal{O}_{\widetilde{X}})^\times \to \mathcal{F} \to 0$$

where the cokernel $\mathcal{F}$ has finite support, hence no higher cohomology. Let c be the number of connected components of $\widetilde{X}$. The long exact sequence of cohomology gives

$$0 \to k^\times \to (k^\times)^c \to H^0(X, \mathcal{F}) \to H^1(X', (\mathcal{O}_{X'})^\times) \to H^1(X', (\pi_* \mathcal{O}_{\widetilde{X}})^\times) \to 0 \ .$$

One has the following supplementary information: the map $k^\times \to (k^\times)^c$ is the diagonal inclusion, the sheaf $\mathcal{F}$ is supported at all ordinary multiple points and $H^0(X, \mathcal{F})$ is the sum $\oplus_{x \in X'} (k^\times)^{m_x - 1}$ over all these points, and

$$H^1(X', (\pi_* \mathcal{O}_{\widetilde{X}})^\times) = H^1(\widetilde{X}, (\mathcal{O}_{\widetilde{X}})^\times)$$

since π is affine. As above, these statements are valid after any base change $S \to \mathrm{Spec}(k)$, so we obtain an induced exact sequence of algebraic groups

$$0 \to \mathbb{G}_m \to (\mathbb{G}_m)^c \to \Pi\, (\mathbb{G}_m)^{m_x - 1} \to \mathrm{Pic}^0(X') \to \mathrm{Pic}^0(\widetilde{X}) \to 0$$

and this proves the lemma. □

3.4. Relation with semistable reduction of abelian varieties

Let C be a smooth geometrically connected curve over K, of genus $g \geq 2$. Let X be the minimal regular model of C. Its special fibre X_k may be singular, possibly nonreduced and we have seen the structure of its generalized jacobian in the previous subsection. This algebraic group turns out to be tightly linked to the reduction type of C. In fact, quite generally, classical results of Chevalley imply that any smooth connected commutative algebraic group over an algebraically closed field is an extension of an abelian variety by a product of a torus and a connected smooth unipotent group. In this section, following Deligne and Mumford, we will prove the following theorem:

Theorem 3.4.1. *Let C be a smooth geometrically connected curve over K, of genus $g \geq 2$, with a K-rational point. Let X be the minimal regular model of C. Then C has stable reduction over R if and only if* $\mathrm{Pic}^0(X_k)$ *has no unipotent subgroup.*

Proof. Assume that C has stable reduction. Then X_k is reduced and has only nodal singularities, by Proposition 3.1.1, so it is equal to its associated curve with ordinary singularities. Since the normalization of X_k is a smooth curve, its generalized jacobian is an abelian variety. Hence it follows from Lemma 3.3.5 that $\mathrm{Pic}^0(X_k)$ is an extension of an abelian variety by a torus, so it has no unipotent subgroup.

Conversely, assume that $\mathrm{Pic}^0(X_k)$ has no unipotent subgroup.

By Lemma 3.3.2 the morphism $\mathrm{Pic}^0(X_k) \to \mathrm{Pic}^0((X_k)_{\mathrm{red}})$ is an isomorphism. Thus by Lemma 3.3.1 we have $H^1(X_k, \mathcal{O}_{X_k}) = H^1((X_k)_{\mathrm{red}}, \mathcal{O}_{(X_k)_{\mathrm{red}}})$. But since X_k has at least one reduced component (the given K-rational point of C reduces by 2.4.3 to a regular point of X_k), we have also $H^0(X_k, \mathcal{O}_{X_k}) = H^0((X_k)_{\mathrm{red}}, \mathcal{O}_{(X_k)_{\mathrm{red}}}) = k$. In other words X_k and its reduced subscheme have

equal Euler-Poincaré characteristics. Let $E_1, \ldots, E_r$ be the irreducible components of X_k and $d_1, \ldots, d_r$ their multiplicities. By the adjunction formula of Theorem 2.3.4 we get

$$\Sigma\, d_i E_i \cdot (\Sigma\, d_i E_i + K) = \Sigma\, E_i \cdot (\Sigma\, E_i + K)$$

where K is a canonical divisor of X/R. Since $\sum d_i E_i = X_k$ is in the radical of the intersection form, we obtain

$$\Sigma\, (d_i - 1) E_i \cdot K = \Sigma\, E_i \cdot \Sigma\, E_i \ .$$

Now assume that $d_i > 1$ for some i. Then $\sum E_i \neq X_k$ and hence $\sum E_i \cdot \sum E_i < 0$, because the intersection form is negative semi-definite with isotropic cone generated by X_k. Therefore by the above equality, we must have $E_{i_0} \cdot K < 0$ for some i_0. Since also $E_{i_0} \cdot E_{i_0} < 0$, we have

$$-2 \geq E_{i_0} \cdot E_{i_0} + E_{i_0} \cdot K = E_{i_0} \cdot (E_{i_0} + K) = -2\chi(E_{i_0}) \geq -2 \ .$$

Finally $\chi(E_{i_0}) = -1$, so E_{i_0} is a projective line with self-intersection -1. This is impossible since X is the minimal regular model. It follows that $d_i = 1$ for all i, hence X_k is reduced. Again since $\mathrm{Pic}^0(X_k)$ has no unipotent subgroup, by Lemma 3.3.4 the curve X_k has ordinary multiple singularities. Since X_k lies on a regular surface, the dimension of the tangent space at all points is less than 2, hence the singular points are ordinary double points. This proves that C has stable reduction over R. □

We can now state the stable reduction theorem in full generality, and we will indicate how Deligne and Mumford deduce it from the above theorem (see [DM], Corollary 2.7).

Theorem 3.4.2. *Let C be a smooth geometrically connected curve over K, of genus $g \geq 2$. Then there exists a finite field extension L/K such that the curve C_L has a stable model. Furthermore, this stable model is unique.*

The unicity statement means that if C_L and C_M have stable models for some finite field extensions L, M then these models become isomorphic in the ring of integers of N, for all fields N containing L and M. This fact follows directly from the proof of the implication (3) ⇒ (1) of Proposition 3.1.1. Indeed, if C has stable reduction, the stable model is determined uniquely as the blow-down of all chains of projective lines with self-intersection -2 in the special fibre of the minimal regular model of C.

The proof of the existence part given in the article [DM] requires much more material from algebraic geometry, in particular it uses results on Néron models of abelian varieties. We give the sketch of the argument, for the readers acquainted with these notions. To prove the theorem, we may pass to a finite field extension and hence assume that C has a K-rational point. Moreover, a result of Grothendieck [SGA7] asserts that after a further finite field extension (again omitted from the notations), the Néron model $\mathcal{J}$ of the jacobian $J = \mathrm{Pic}^0(C/K)$ has a special fibre $\mathcal{J}_k$ without unipotent subgroup. Now, let X be the minimal

regular model of C over the ring of integers R of K. By properness there is a section $\mathrm{Spec}(R) \to X$ that extends the rational point of C, and the corresponding k-point is regular (Remark 2.4.3). In particular, this section hits the special fibre in a component of multiplicity 1. Under these assumptions, by a theorem of Raynaud [Ra1], the Picard functor $\mathrm{Pic}^0(X/R)$ is isomorphic to $\mathcal{J}$ (in particular, it is representable). It follows that the special fibre of $\mathrm{Pic}^0(X/R)$, in other words $\mathrm{Pic}^0(X_k)$, has no unipotent subgroup. By Theorem 3.4.1, C has stable reduction.

4. Application to moduli of curves and covers

4.1. Valuative criterion for the stack of stable curves

Let $g \geq 2$ be a fixed integer and let $\overline{\mathcal{M}}_g$ be the moduli stack of stable curves of genus g.

Once it is known that $\overline{\mathcal{M}}_g$ is separated (cf. the next subsection), the valuative criterion of properness for $\overline{\mathcal{M}}_g$ is the following statement: for all discrete valuation rings R with fraction field K, and all K-points $\mathrm{Spec}(K) \to \overline{\mathcal{M}}_g$, there exists a finite field extension K'/K such that $\mathrm{Spec}(K') \to \mathrm{Spec}(K) \to \overline{\mathcal{M}}_g$ extends to a point $\mathrm{Spec}(R') \to \overline{\mathcal{M}}_g$ where R' is the integral closure of R in K'.

Once it is known that $\overline{\mathcal{M}}_g$ is of finite type, it is enough to verify the valuative criterion for complete valuation rings R with algebraically closed residue field.

Finally, by the well-known Lemma 4.1.1 below, it is enough to test the criterion for points $\mathrm{Spec}(K) \to \overline{\mathcal{M}}_g$ that map into some open dense substack $U \subset \overline{\mathcal{M}}_g$. The deformation theory of stable curves proves that smooth curves are dense in $\overline{\mathcal{M}}_g$, hence we may take U to be the open substack of smooth curves. Then, the valuative criterion is just Theorem 3.4.2.

Lemma 4.1.1. *Let S be a noetherian scheme and let $\mathcal{X}$ be an algebraic stack of finite type and separated over S. Let $\mathcal{U}$ be a dense open substack. Then $\mathcal{X}$ is proper over S if and only if for all discrete valuation rings R with fraction field K and all S-morphisms $\mathrm{Spec}(K) \to \mathcal{U}$, there exists a finite extension K'/K and a morphism $\mathrm{Spec}(R') \to \mathcal{X}$, where R' is the integral closure of R in K', such that the following diagram is commutative:*

$$\begin{tikzcd}
\mathrm{Spec}(K') \arrow[r] \arrow[d] & \mathrm{Spec}(K) \arrow[r] & \mathcal{U} \arrow[r] & \mathcal{X} \arrow[d] \\
\mathrm{Spec}(R') \arrow[rrr] \arrow[urrr, dotted] & & & S.
\end{tikzcd}$$

Proof. For simplicity, we will prove the lemma in the case where $\mathcal{X}$ is a scheme X. The proof for an algebraic stack is exactly the same, but we want to avoid giving references to the literature on algebraic stacks for the necessary ingredients. It is enough to prove the *if* part. Since the notion of properness is local on the target, we may assume that S is affine. Then by [EGA2], 5.4.5, we may replace S by one of its reduced irreducible components Z and then X by one of the reduced

irreducible components of the preimage of Z in X. Thus we may assume that X and S are integral. By Chow's lemma [EGA2], 5.6.1, there exists a scheme X' quasi-projective over S and a projective, surjective, birational morphism $X' \to X$. It is easy to see that $X \to S$ is proper if and only if $X' \to S$ is proper, thus we may replace X by X' and assume X quasiprojective. Let $j : X \to P$ be an open dense immersion into a projective S-scheme. Then $X \to S$ is proper if and only if j is surjective. Let x be a point in P. Since U is dense in X hence also in P, there exists a point $y \in U$ and a morphism $\mathrm{Spec}(R) \to P$ where R is a discrete valuation ring with fraction field K, mapping the open point to y and the closed point to x (see [EGA2], 7.1.9). By the valuative criterion which is the assumption of the lemma, the map $\mathrm{Spec}(K) \to X$ extends (maybe after a finite extension) to $\mathrm{Spec}(R) \to X$. Since X is separated, such an extension is unique and this means that $x \in X$. So j is surjective and the lemma is proved. □

4.2. Automorphisms of stable curves

As a preparation for the next subsection, we need some preliminaries concerning automorphisms of stable curves. Not just the automorphism *groups*, but also the automorphism *functors*, are interesting. Even more generally, if X, Y are stable curves over a scheme S, then by Grothendieck's theory of the Hilbert scheme and related functors, the functor of isomorphisms between X and Y is representable by a quasi-projective S-scheme denoted $\mathrm{Isom}_S(X, Y)$. It is really this scheme that we want to describe.

Lemma 4.2.1. *Let X be a stable curve over a field k. Then, the group of automorphisms of X/k is finite and the group of global vector fields* $\mathrm{Ext}^0(\Omega_{X/k}, \mathcal{O}_X)$ *is zero.*

Proof. Let S be the set of singular points of X and let $\pi : \widetilde{X} \to X$ be the normalization morphism. Let A be the group of automorphisms of X and let A_0 be the subgroup of those automorphisms φ such that for all $x \in S$, we have $\varphi(x) = x$ and φ preserves the branches at x. Since S is finite, A_0 has finite index in A and hence it is enough to prove that A_0 is finite. Then elements of A_0 are the same as automorphisms of $\widetilde{X}$ acting trivially on $\pi^{-1}(S)$. Let us call the points of $\pi^{-1}(S)$ *marked points*. Since X is connected, the components of $\widetilde{X}$ are either smooth curves of genus $g \geq 2$ with maybe some marked points, or elliptic curves with at least one marked point, or rational curves with at least three marked points. Each of these has finitely many automorphisms, hence A_0 is finite.

A global vector field D on X is the same as a global vector field $\widetilde{D}$ on $\widetilde{X}$ which vanishes at all marked points. We proceed again by inspection of the three different types of components of $\widetilde{X}$. It is known that smooth curves of genus $g \geq 2$ have no vector field, elliptic curves have no vector field vanishing in one point, and smooth rational curves ones have no vector field vanishing in three points. Hence $\widetilde{D} = 0$ and $D = 0$. □

Lemma 4.2.2. *Let X, Y be a stable curves over a scheme S. Then, the isomorphism scheme* $\mathrm{Isom}_S(X, Y)$ *is finite and unramified over S.*

Proof. The scheme $\mathrm{Isom}_S(X, Y)$ is of finite type as an open subscheme of a Hilbert scheme. It is also proper, since the valuative criterion is exactly the unicity statement in Theorem 3.4.2. Hence in order to prove the lemma we may assume that S is the spectrum of an algebraically closed field k. Then, either $\mathrm{Isom}_S(X, Y)$ is empty or it is isomorphic to $\mathrm{Aut}_k(X)$. Hence, it is finite by Lemma 4.2.1. Let $k[\epsilon]$ with $\epsilon^2 = 0$ be the ring of dual numbers. In order to prove that $\mathrm{Aut}_k(X)$ is unramified, it is enough to prove that an automorphism φ of $X \times_k k[\epsilon]$ which is the identity modulo ϵ is the identity. Such a φ stabilizes each affine open subscheme $\mathrm{Spec}(A) \subset X$ and acts there via a ring homomorphism $\varphi^\sharp(a) = a + \lambda(a)\epsilon$. Since $\varphi^\sharp$ is multiplicative we get that λ is in fact a derivation. By gluing on all open affine, the various λ's define a global vector field, which is zero by Lemma 4.2.1 again. Hence φ is the identity. □

The stable reduction theorem for Galois covers which we will prove below is valid when the order of the Galois group is prime to all residue characteristics. In the proof, we will use the following lemma:

Lemma 4.2.3. *Let X be a reduced, irreducible curve over a field k and let x be a smooth closed point. Let φ be an automorphism of X of finite order n prime to the characteristic of k, belonging to the inertia group at x. Then the action of φ on the tangent space to X at x is via a primitive nth root of unity, i.e., it is faithful.*

Proof. We can assume that $n \geq 2$ and that x is a rational point, passing to a finite extension of k if necessary. Then the completed local ring of x is isomorphic to the ring of power series $k[[t]]$. The action of φ on the tangent space to C at x is done via multiplication by some mth root of unity ζ, with $m|n$. If $m \neq n$, then replacing φ by φ^m we reduce to the case where $\zeta = 1$. Since φ is not the trivial automorphism of C, there is an integer i and a nonzero scalar $a \in k$ such that $\varphi(t) = t + at^i$ modulo t^{i+1}. Then $\varphi^n(t) = t + nat^i$ modulo t^{i+1}. Since $\varphi^n(t) = t$ and n is not zero in k, this is impossible. Therefore, $m = n$. □

4.3. Reduction of Galois covers at good characteristics

We now give the applications to stable reduction of Galois covers of curves (by *cover* we mean a finite surjective morphism). To do this, we fix a finite group G of order n and we consider a cover of smooth, geometrically connected curves $f : C \to D$ which is Galois with group G. We assume as usual that the genus of C is $g \geq 2$. The case where the order n is divisible by the residue characteristic p of k brings some more complicated pathologies, and here we will rather have a look at the case where n is prime to p. We make the following definition.

Definition 4.3.1. Let k be a field of characteristic p, and G a finite group of order n prime to p. Let X be a stable curve over k endowed with an action of G, and for all nodes $x \in X$, let $H_x \subset G$ denote the subgroup of the inertia group of x composed of elements that preserve the branches at x. We say that the action is

stable, or that the Galois cover $X \to Y := X/G$ is *stable*, if the action of G on X is faithful and for all nodes $x \in X$, the action of H_x on the tangent space of X at x is faithful with characters on the two branches χ_1, χ_2 satisfying the relation $\chi_1\chi_2 = 1$.

Note that the stabilizer is cyclic when it preserves the branches at x, and dihedral when some elements of H permute the branches at x.

An extremely important consequence of the assumption $(n, p) = 1$ is that the formation of the quotient $X \to X/G$ commutes with base change. Consequently, the definition of a stable cover above makes sense in families, i.e., if $X \to S$ is a stable curve over a scheme S endowed with an action of G by S-automorphisms and $Y = X/G$, then we say that the cover $X \to Y$ is a stable Galois cover if and only if it is stable the fibre over each point $s \in S$. Then we arrive at the following stable reduction theorem for covers:

Theorem 4.3.2. *Let G be a finite group of order n prime to the characteristic of k, the residue field of R. Let $C \to D$ be a cover of smooth, geometrically connected curves which is Galois with group G, and assume that the genus of C is $g \geq 2$. Then after a finite extension of K, the cover $C \to D$ has a stable model $X \to Y$ over R. Furthermore, this model is unique.*

Proof. By the stable reduction theorem, there exists a finite field extension L/K such that C_L has a stable model X. Replacing K by L for notational simplicity, we reduce to the case $L = K$. Then by unicity of the stable model and by abstract nonsense, the group action extends to an action of G on X by R-automorphisms. By Lemma 4.2.2, the induced action of G on the special fibre X_k is faithful: indeed, if $\varphi \in G$ has trivial image in $\mathrm{Aut}_k(X)$, then by the property of unramification of the automorphism functor, it has trivial image in $\mathrm{Aut}_{R/m^n}(X \otimes_R R/m^n)$ for all $n \geq 1$, so since R is complete, it has trivial image in $\mathrm{Aut}_R(X)$. We define $Y = X/G$.

We now prove that the action is stable. Let $x \in X_k$ be a nodal point and let $H_x \subset G$ be the subgroup of the stabilizer of x composed of elements that preserve the branches at x. The completion of the local ring $\mathcal{O}_{X,x}$ is isomorphic to $R[[a, b]]/(ab - \pi^n)$ for some $n \geq 1$. Then the tangent action on the branches is obviously via multiplication by inverse roots of unity of order $|H_x|$. It remains to see that the kernel N of the action of H_x on the tangent space $T_{X_k,x}$ is trivial. In fact N acts trivially on the whole irreducible components containing x, as one sees by applying Lemma 4.2.3 to the normalization of X_k. Since X_k is connected, it follows at once that N acts trivially on X_k, hence $N = 1$. □

Moreover, one can prove, using deformation theory, that a stable Galois cover of curves over k can be deformed into a smooth curve over R with faithful G-action. For details about this point, we refer for example to [BR].

In the case where the order of G is divisible by the residue characteristic p, things are much more complicated. We will conclude by a simple example, which gives an idea of the local situation around a node of the special fibre. Assume that R contains a primitive pth root of unity ζ. We look at the affine R-curve

X with function ring $R[x,y]/(xy-a)$, for some fixed a in the maximal ideal of R. We consider the group $G = \mathbb{Z}/p\mathbb{Z}$, with generator σ, and the action on a neighbourhood of the node of X_k given by

$$\sigma(x) = \zeta x + a \qquad \text{and} \qquad \sigma(y) = \frac{y}{\zeta + y} \ .$$

Then the reduced action is given by $\sigma(x) = x$ and $\sigma(y) = y/(1+y)$, hence it is faithful on one branch but not on the other. Apparently some information on the group action is lost in reduction, but it is not clear what to do in order to recover it. At the moment, no "reasonable" stable reduction theorem for covers at "bad" characteristics is known.

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Galois Categories*

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Abstract. These notes describe the formalism of Galois categories and fundamental groups, as introduced by A. Grothendieck in [SGA1, Chap. V]. This formalism stems from Galois theory for topological covers and can be regarded as the natural categorical generalization of it. But, far beyond providing a uniform setting for the preexisting Galois theories as those of topological covers and field extensions, this formalism gave rise to the construction and theory of the étale fundamental group of schemes – one of the major achievements of modern algebraic geometry.

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1. Foreword

In Section 2, we give the axiomatic definition of a Galois category and state the main theorem which asserts that a Galois category is a category equivalent to the category of finite discrete Π-sets for some profinite group Π. In Section 3, we carry out in details the proof of the main theorem. In Section 4, we show that there is a natural equivalence of categories between the category of profinite groups and the category of Galois categories pointed with fibre functors. This gives a powerful dictionary to translate properties of a functor between two pointed Galois categories in terms of properties of the corresponding morphism of profinite groups (and conversely). In Section 5 we define the category of étale covers of a connected scheme and prove that it is a Galois category. In Section 6, we apply the formalism of Section 4 to describe the étale fundamental groups of specific classes of schemes such as abelian varieties or normal schemes. The short Section 7 is devoted to geometrically connected schemes of finite type over fields. These schemes have the property that their fundamental group decomposes into a geometric part and an arithmetic part. But the interplay between those two parts remains mysterious and is at the source of some of the most standard conjectures about fundamen-

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tal groups such as anabelian conjectures or the section conjecture. The four last sections are devoted to the study of the geometric part namely, the fundamental group of a connected scheme of finite type over an algebraically closed field. In Section 8, we state the main G.A.G.A. theorem, which describes what occurs over the complex numbers (and, basically, over any algebraically closed field of characteristic 0). In Section 9, we construct the specialization morphism from the étale fundamental group of the geometric generic fibre to the étale fundamental of the geometric special fibre of a scheme proper, smooth and geometrically connected over a trait and show that it is an epimorphism. We improve this result in Section 10, by showing that, in the smooth case, the specialization epimorphism induces an isomorphism on the prime-to-p completions (where p denotes the residue characteristic of the closed point). In the concluding Section 11, we apply the theory of specialization to show that the étale fundamental group of a connected proper scheme over an algebraically closed field is topologically finitely generated. In the appendix, we gather some results (without proof) from descent theory that are needed in the proofs of some of the elaborate theorems presented here.

The main source and guideline for these notes was [SGA1] but for several parts of the exposition, I am also indebted to [Mur67]. In particular, though the case of schemes is only considered there, I could extract a consequent part of Sections 3 and 4 from this source (complemented with Proposition 3.3, which is a categorical version of a scheme-theoretic result of J.-P. Serre). I also resorted to [Mur67] for Section 9. Another source is the first synthetic section of [Mi80], which I used for classical results on étale morphisms in Subsection 5.10 and normal schemes in Subsection 6.4. Also, at some points, I mention famous conjectures (some of which were proved recently) on étale fundamental groups, such as Abhyankar's conjecture, anabelian conjectures or the section conjecture. For this, I am indebted to the survey expositions in [Sz09] and [Sz10].

Among other introductions to étale fundamental groups (avoiding the language of Galois categories), I should mention the proceedings of the conference *Courbes semi-stables et groupe fondamental en géométrie algébrique* held in Luminy in 1998 [BLR00] and, in particular, the elementary self-contained introductory article of A. Mézard [Me00] as well as the nice book of T. Szamuely [Sz09], which emphasizes the parallel story of topological covers, field theory and schemes – especially curves.

The main contribution of these notes to the existing introductory literature on étale fundamental groups is that we privilege the categorical setting to the 'incarnated ones' (as exposed in [Me00] and [Sz09]). In particular, we provide detailed proofs of all the categorical statements in Sections 3 and 4. To our knowledge, such statements are only available in the original sources [SGA1] and [Mur67] and, there, their proofs are only sketched. Privileging the categorical setting is not only a matter of taste but stems from the conviction that elementary category theory, which is only involved in Galois categories, is much simpler than (even elementary) scheme theory.

Concerning scheme theory, there is nothing new in the material presented here but we tried to make the exposition both concise and exhaustive so that it becomes accessible to graduate students in algebraic geometry. In Section 5, 6, 7 and 10, we provide detailed proofs. Sections 8, 9 and 11 require more elaborate tools. In Section 8, we only provide the minimal material to understand the statement of the main G.A.G.A. theorem but in Sections 9 and 11 we state the main theorems involved and, relying on them, give detailed sketches of proof.

For Sections 2 to 4 only some familiarity with the language of categories and the notion of profinite groups are required. For Sections 5 to 7, the reader has to be familiar with the basics of commutative algebra as in [AM69] and the theory of schemes as in [Hart77, Chap. II]. As mentioned, Sections 8 to 11 rely on difficult theorems but to understand their statements, only a little more knowledge of the theory of schemes is needed – say as in [Hart77, Chap. III].

2. Galois categories

2.1. Definition and elementary properties

Given a category $\mathcal{C}$ and two objects $X,\ Y \in \mathcal{C}$, we will use the following notation:

$\mathrm{Hom}_{\mathcal{C}}(X,Y)$: Set of morphisms from X to Y in $\mathcal{C}$

$\mathrm{Isom}_{\mathcal{C}}(X,Y)$: Set of isomorphisms from X to Y in $\mathcal{C}$

$\mathrm{Aut}_{\mathcal{C}}(X) \quad := \mathrm{Isom}_{\mathcal{C}}(X,X)$

A morphism $u : X \to Y$ in a category $\mathcal{C}$ is a *strict epimorphism* if the fibre product $X \times_{u,Y,u} X$ exists in $\mathcal{C}$ and for any object Z in $\mathcal{C}$, the map $u\circ : \mathrm{Hom}_{\mathcal{C}}(Y,Z) \to \mathrm{Hom}_{\mathcal{C}}(X,Z)$ is injective and induces a bijection onto the set of all morphism $\psi : X \to Z$ in $\mathcal{C}$ such that $\psi \circ p_1 = \psi \circ p_2$, where $p_i : X \times_{u,Y,u} X \to X$ denotes the ith projection, $i = 1, 2$.

Let $FSets$ denote the category of finite sets.

Definition 2.1. A **Galois category** is a category $\mathcal{C}$ such that there exists a covariant functor $F : \mathcal{C} \to FSets$ satisfying the following axioms:

(1) $\mathcal{C}$ has a final object $e_{\mathcal{C}}$ and finite fibre products exist in $\mathcal{C}$.
(2) Finite coproducts exist in $\mathcal{C}$ and categorical quotients by finite groups of automorphisms exist in $\mathcal{C}$.
(3) Any morphism $u : Y \to X$ in $\mathcal{C}$ factors as $Y \xrightarrow{u'} X' \xrightarrow{u''} X$, where u' is a strict epimorphism and u'' is a monomorphism which is an isomorphism onto a direct summand of X.
(4) F sends final objects to final objects and commutes with fibre products.
(5) F commutes with finite coproducts and categorical quotients by finite groups of automorphisms and sends strict epimorphisms to strict epimorphisms.
(6) Let $u : Y \to X$ be a morphism in $\mathcal{C}$, then $F(u)$ is an isomorphism if and only if u is an isomorphism.

Remark 2.2. As the coproduct over the empty set $\emptyset$ is always an initial object, it follows from axiom (2) that $\mathcal{C}$ has an initial object $\emptyset_{\mathcal{C}}$.

2.1.1. Equivalent formulations of axioms (1), (2), (4), (5). (1) is equivalent to:

$$(1)' \text{ Finite projective limits exist in } \mathcal{C}.$$

(2) is implied by:

$$(2)' \text{ Finite inductive limits exist in } \mathcal{C}.$$

Let $\mathcal{C}_1$, $\mathcal{C}_2$ be two categories admitting finite projective limits (resp. finite inductive limits). A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is said to be *right exact* (resp. *left exact*) if it commutes with finite projective limits (resp. with finite inductive limits). Then, (4) is equivalent to:

$$(4)' \ F \text{ is right exact}$$

and (5) is implied by:

$$(5)' \ F \text{ is left exact.}$$

It will follow from Theorem 2.8 that (1)–(6) are equivalent to (1), (2)$'$, (3), (4), (5)$'$ and (6).

2.1.2. Unicity in axiom (3).

Lemma 2.3. *The decomposition* $Y \xrightarrow{u'} X' \xrightarrow{u''} X$ *in axiom* (3) *is unique in the sense that for any two such decompositions* $Y \xrightarrow{u'_i} X'_i \xrightarrow{u''_i} X = X'_i \sqcup X''_i$, $i = 1, 2$ *there exists a* (*necessarily*) *unique isomorphism* $\omega : X'_1 \tilde{\to} X'_2$ *such that* $\omega \circ u'_1 = u'_2$ *and* $u''_2 \circ \omega = u''_1$.

Proof. From the injectivity of $- \circ u' : \mathrm{Hom}_{\mathcal{C}}(X', X) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, X)$, any such decomposition $Y \xrightarrow{u'} X' \xrightarrow{u''} X$ is entirely determined by u, u'. Let $Y \xrightarrow{u'_i} X'_i \xrightarrow{u''_i} X = X'_i \sqcup X''_i$, $i = 1, 2$ be two such decompositions. Since $u = u''_1 \circ u'_1$ one gets:

$$u''_2 \circ (u'_2 \circ p_1) = u \circ p_1 = u \circ p_2 = u''_2 \circ (u'_2 \circ p_2),$$

where $p_i : Y \times_{u'_1, X'_1, u'_1} Y \to Y$ denotes the ith projection, $i = 1, 2$. As $u''_2 : X'_2 \hookrightarrow X$ is a monomorphism, this implies that $u'_2 \circ p_1 = u'_2 \circ p_2$ and, as $u'_1 : Y \to X'_1$ is a strict epimorphism, this in turn implies that $u'_2 : Y \to X'_2$ lies in the image of $u'_1 \circ - : \mathrm{Hom}_{\mathcal{C}}(X'_1, X'_2) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, X'_2)$ hence can be written in $\mathcal{C}$ as $u'_2 : Y \xrightarrow{u'_1} X'_1 \xrightarrow{\phi} X'_2$. From axiom (6), to prove that $\phi : X'_1 \tilde{\to} X'_2$ is an isomorphism in $\mathcal{C}$, it is enough to prove that $F(\phi) : F(X'_1) \twoheadrightarrow F(X'_2)$ is bijective. But $F(\phi) : F(X'_1) \twoheadrightarrow F(X'_2)$ is surjective since $F(u'_2)$ is, hence bijective since $|F(X'_1)| = |F(X'_2)| = |F(u)(F(Y))|$. $\square$

2.1.3. Artinian property. It follows from axiom (4) that a Galois category is always artinian. More precisely, one has the following elementary categorical lemma.

Lemma 2.4. *Let $\mathcal{C}$ be a category which admits finite fibre products and let $u : X \to Y$ be a morphism in $\mathcal{C}$. Then $u : X \to Y$ is a monomorphism if and only if the first projection $p_1 : X \times_Y X \to X$ is an isomorphism. In particular,*

(1) *A functor that commutes with fibre products sends monomorphisms to monomorphisms.*
(2) *If $u : X \to Y$ is both a monomorphism and a strict epimorphism then $u : X \to Y$ is an isomorphism.*

Proof. Let $\Delta_{X|Y} : X \to X \times_{u,Y,u} X$ denote the diagonal morphism. By definition, $p_1 \circ \Delta_{X|Y} = Id_X$ so, if $p_1 : X \times_Y X \to X$ is an isomorphism, its inverse is automatically $\Delta_{X|Y} : X \to X \times_Y X$. Assume first that $u : X \to Y$ is a monomorphism. Then, from $u \circ p_1 = u \circ p_2$, one deduces that $p_1 = p_2$. But, then, $p_1 \circ \Delta_{X|Y} \circ p_1 = Id_X \circ p_1 = p_1$ and:

$$p_2 \circ \Delta_{X|Y} \circ p_1 = Id_X \circ p_1 = p_1 = p_2.$$

So, from the uniqueness in the universal property of the fibre product, one gets $\Delta_{X|Y} \circ p_1 = Id_{X \times_Y X}$. Conversely, assume that $p_1 : X \times_Y X \tilde{\to} X$ is an isomorphism. Then, for any morphisms $f, g : W \to X$ in $\mathcal{C}$ such that $u \circ f = u \circ g$ there exists a unique morphism $(f, g) : W \to X \times_Y X$ such that $p_1 \circ (f, g) = f$ and $p_2 \circ (f, g) = g$. From the former equality, one obtains that $(f, g) = \Delta_{X|Y} \circ f$ and, from the latter, that $g = p_2 \circ (f, g) = p_2 \circ \Delta_{X|Y} \circ f = f$.

Assertion (1) follows straightforwardly from the fact that functors send isomorphisms to isomorphisms. It remains to prove assertion (2). Since $u : X \to Y$ is a strict epimorphism, the map $u\circ : \mathrm{Hom}_{\mathcal{C}}(Y, X) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, Y)$ induces a bijection onto the set of all morphisms $v : X \to X$ such that $v \circ p_1 = v \circ p_2$, where $p_i : X \times_Y X \to X$ is the ith projection, $i = 1, 2$. But since $u : X \to Y$ is also a monomorphism, the first projection $p_1 : X \times_Y X \tilde{\to} X$ is an isomorphism with inverse $\Delta_{X|Y} : X \to X \times_Y X$. So $\Delta_{X|Y} \circ p_1 = Id_{X \times_Y X}$, which yields:

$$\begin{aligned} p_2 \circ \Delta_{X|Y} \circ p_1 &= p_2 \\ &= Id_X \circ p_1 \quad = p_1. \end{aligned}$$

Thus $p_1 = p_2$, which implies that $u\circ : \mathrm{Hom}_{\mathcal{C}}(Y, X) \tilde{\to} \mathrm{Hom}_{\mathcal{C}}(Y, Y)$ is bijective. In particular, there exists $v : Y \to X$ such that $u \circ v = Id_Y$. But, then, $u \circ v \circ u = u = u \circ Id_X$ whence $v \circ u = Id_X$. □

Corollary 2.5. *A Galois category $\mathcal{C}$ is artinian.*

Proof. Let $F : \mathcal{C} \to FSets$ be a fibre functor for $\mathcal{C}$ and consider a decreasing sequence

$$\cdots \overset{t_{n+1}}{\hookrightarrow} T_n \overset{t_n}{\hookrightarrow} \cdots \overset{t_2}{\hookrightarrow} T_1 \overset{t_1}{\hookrightarrow} T_0$$

of monomorphisms in $\mathcal{C}$. We want to show that $t_{n+1} : T_{n+1} \hookrightarrow T_n$ is an isomorphism for $n \gg 0$. From axiom (6), it is enough to show that $F(t_{n+1}) : F(T_{n+1}) \to F(T_n)$ is an isomorphism for $n \gg 0$. But it follows from Lemma 2.4 (1) and axiom

(4) that $F(t_{n+1}) : F(T_{n+1}) \hookrightarrow F(T_n)$ is a monomorphism and, since $F(T_0)$ is finite, the monomorphism $F(t_{n+1}) : F(T_{n+1}) \hookrightarrow F(T_n)$ is actually an isomorphism for $n \gg 0$. $\square$

2.1.4. A reinforcement of axiom (6)**.** Combining axioms (3), (4) and (6), one also obtains that $F : \mathcal{C} \to FSets$ is "conservative" for strict epimorphisms, monomorphisms, final and initial objects:

Lemma 2.6.

(1) *If $u : Y \to X$ is a morphism in $\mathcal{C}$ then $F(u)$ is an epimorphism (resp. a monomorphism) if and only if u is a strict epimorphism (resp. a monomorphism).*
(2) *For any $X_0 \in \mathcal{C}$, one has:*
   - *$F(X_0) = \emptyset$ if and only if $X_0 = \emptyset_{\mathcal{C}}$;*
   - *$F(X_0) = *$ if and only if $X_0 = e_{\mathcal{C}}$, where $*$ denotes the final object in $FSets$.*

Proof. (1) The "if" implication for epimorphism follows from axiom (4) and the "if" implication for monomorphism from Lemma 2.4 (1) and axiom (4).

We now prove the "only if" implications. From axiom (3), any morphism $u : Y \to X$ in $\mathcal{C}$ factors as $Y \xrightarrow{u'} X' \xrightarrow{u''} X$, where u' is a strict epimorphism and u'' is a monomorphism which is an isomorphism onto a direct summand of X. So, if $F(u)$ is an epimorphism then $F(u'')$ is an epimorphism as well. But from the "if" implication, $F(u'')$ is also a monomorphism hence an isomorphism since we are in the category $FSets$. So u'' is an isomorphism by axiom (6). The proof for monomorphism is exactly the same.

(2) We first consider the case of initial objects. By definition of an initial object, for any $X \in \mathcal{C}$ there is exactly one morphism from $\emptyset_{\mathcal{C}}$ to X in $\mathcal{C}$; denote it by $u_X : \emptyset_{\mathcal{C}} \to X$.

Assume first that $F(X_0) = \emptyset$. Since, for any finite set E, there is a morphism from E to $\emptyset$ in $FSets$ if and only if $E = \emptyset$ and since $F(u_{X_0})$ is a morphism from $F(\emptyset_{\mathcal{C}})$ to $F(X_0) = \emptyset$ in $FSets$, one has $F(\emptyset_{\mathcal{C}}) = \emptyset$. But this forces $F(u_{X_0}) = Id_{\emptyset}$. In particular, $F(u_{X_0})$ is an isomorphism hence, by axiom (6) so is u_{X_0}.

Assume now that $X_0 = \emptyset_{\mathcal{C}}$. For any object $X \in \mathcal{C}$, one has a canonical isomorphism $(u_X, Id_X) : \emptyset_{\mathcal{C}} \sqcup X \tilde{\to} X$ (with inverse the canonical morphism $i_X : X \tilde{\to} \emptyset_{\mathcal{C}} \sqcup X$) thus $F((u_X, Id_X)) : F(\emptyset_{\mathcal{C}} \sqcup X) \tilde{\to} F(X)$ is again an isomorphism. But, it follows from axiom (5) that $F(\emptyset_{\mathcal{C}} \sqcup X) \simeq F(\emptyset_{\mathcal{C}}) \sqcup F(X)$, which forces $|F(\emptyset_{\mathcal{C}})| = 0$ hence $F(\emptyset_{\mathcal{C}}) = \emptyset$.

We consider now the case of final object. The fact that $F(e_{\mathcal{C}}) = *$ follows from axiom (4). Conversely, by definition of a final object, for any $X \in \mathcal{C}$ there is exactly one morphism from X to $e_{\mathcal{C}}$ in $\mathcal{C}$; denote it by $v_X : X \to e_{\mathcal{C}}$. So, $F(X) = *$ forces $F(v_X) : * \to *$ is the identity which, by axiom (6), implies that $v_X : X \tilde{\to} e_{\mathcal{C}}$ is an isomorphism. $\square$

2.2. Main theorem

Given a Galois category $\mathcal{C}$, a functor $F : \mathcal{C} \to FSets$ satisfying axioms (4), (5), (6) is called a *fibre functor for* $\mathcal{C}$. Given a fibre functor $F : \mathcal{C} \to FSets$ for $\mathcal{C}$, the *fundamental group of* $\mathcal{C}$ *with base point* F is the group – denoted by $\pi_1(\mathcal{C}; F)$ – of automorphisms of the functor $F : \mathcal{C} \to FSets$.

Also, given two fibre functors $F_i : \mathcal{C} \to FSets$ for $\mathcal{C}$, $i = 1, 2$ the *set of paths from* F_1 *to* F_2 *in* $\mathcal{C}$ is the set – denoted by $\pi_1(\mathcal{C}; F_1, F_2) := \mathrm{Isom}_{Fct}(F_1, F_2)$ – of isomorphisms of functors from $F_1 : \mathcal{C} \to FSets$ to $F_2 : \mathcal{C} \to FSets$.

Example 2.7.

1. For any connected, locally arcwise connected and locally simply connected topological space B, let $\mathcal{C}_B^{\mathrm{top}}$ denote the category of finite topological covers of B. Then $\mathcal{C}_B$ is Galois with fibre functors the usual "fibre at b" functors, $b \in B$:
$$\begin{array}{rlcl} F_b : & \mathcal{C}_B^{\mathrm{top}} & \to & FSets \\ & f : X \to B & \to & f^{-1}(b) \end{array}.$$
Let $\pi_1^{\mathrm{top}}(B; b)$ denote the topological fundamental group of B with base point b and group law defined as follows. For any $\gamma, \gamma' \in \pi_1^{\mathrm{top}}(B; b)$ with representatives $c_\gamma, c_{\gamma'} : [0, 1] \to B$ we define $\gamma' \cdot \gamma$ to be the homotopy class of:
$$\begin{array}{rlcl} c_{\gamma'} \circ c_\gamma : & [0,1] & \to & B \\ & 0 \le t \le \frac{1}{2} & \to & c_\gamma(2t) \\ & \frac{1}{2} \le t \le 1 & \to & c_{\gamma'}(2t-1) \end{array}$$
Then, with this convention, one has:
$$\pi_1(\mathcal{C}_B^{\mathrm{top}}; F_b) = \widehat{\pi_1^{\mathrm{top}}(B; b)}$$
(where $\widehat{(-)}$ denotes the profinite completion).
2. For any profinite group Π, let $\mathcal{C}(\Pi)$ denote the category of finite (discrete) sets with continuous left Π-action. Then $\mathcal{C}(\Pi)$ is Galois with fibre functor the forgetful functor $For : \mathcal{C}(\Pi) \to FSets$. And, in that case:
$$\pi_1(\mathcal{C}(\Pi); For) = \Pi.$$

Example 2.7 (2) is actually the typical example of Galois categories. Indeed, the fundamental group $\pi_1(\mathcal{C}; F)$ is equipped with a natural structure of profinite group. For this, set:
$$\Pi := \prod_{X \in Ob(\mathcal{C})} \mathrm{Aut}_{FSets}(F(X))$$
and endow Π with the product topology of the discrete topologies, which gives it a structure of profinite group. Considering the monomorphism of groups:
$$\begin{array}{lcl} \pi_1(\mathcal{C}; F) & \hookrightarrow & \Pi \\ \theta & \mapsto & (\theta(X))_{X \in Ob(\mathcal{C})} \end{array}$$

the group $\pi_1(\mathcal{C}; F)$ can be identified with the intersection of all:

$$\mathcal{C}_\phi := \{(\sigma_X)_{X \in Ob(\mathcal{C})} \in \Pi \mid \sigma_X \circ F(\phi) = F(\phi) \circ \sigma_Y\},$$

where $\phi : Y \to X$ describes the set of all morphisms in $\mathcal{C}$. By definition of the product topology, the $\mathcal{C}_\phi$ are closed. So $\pi_1(\mathcal{C}; F)$ is closed as well and, equipped with the topology induced from the product topology on Π, it becomes a profinite group.

By definition of this topology, a fibre functor $F : \mathcal{C} \to FSets$ for $\mathcal{C}$ factors as:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "F"'] & FSets \\ \mathcal{C}(\pi_1(\mathcal{C}; F)). \arrow[ur, "For"'] & \end{tikzcd}$$

Theorem 2.8 (Main theorem). *Let $\mathcal{C}$ be a Galois category. Then:*

(1) *Any fibre functor $F : \mathcal{C} \to FSets$ induces an equivalence of categories $F : \mathcal{C} \to \mathcal{C}(\pi_1(\mathcal{C}; F))$.*

(2) *For any two fibre functors $F_i : \mathcal{C} \to FSets$, $i = 1, 2$, the set of paths $\pi_1(\mathcal{C}; F_1, F_2)$ is non-empty. The profinite group $\pi_1(\mathcal{C}; F_1)$ is non-canonically isomorphic to $\pi_1(\mathcal{C}; F_2)$ with an isomorphism that is canonical up to inner automorphisms. In particular, the abelianization $\pi_1(\mathcal{C}; F)^{ab}$ of $\pi_1(\mathcal{C}; F)$ does not depend on F up to canonical isomorphism.*

3. Proof of the main theorem

Given a category $\mathcal{C}$ and X, $Y \in \mathcal{C}$, we will say that X *dominates* Y *in* $\mathcal{C}$ – and write $X \geq Y$ – if there is at least one morphism from X to Y in $\mathcal{C}$.

From now on, let $\mathcal{C}$ be a Galois category and let $F : \mathcal{C} \to FSets$ be a fibre functor for $\mathcal{C}$.

3.1. The pointed category associated with $\mathcal{C}$, F

We define the *pointed category associated with* $\mathcal{C}$ *and* F to be the category $\mathcal{C}^{pt}$ whose objects are pairs (X, ζ) with $X \in \mathcal{C}$ and $\zeta \in F(X)$ and whose morphisms from (X_1, ζ_1) to (X_2, ζ_2) are the morphisms $u : X_1 \to X_2$ in $\mathcal{C}$ such that $F(u)(\zeta_1) = \zeta_2$.

There is a natural forgetful functor:

$$For : \mathcal{C}^{pt} \to \mathcal{C}$$

and a 1-to-1 correspondence between sections of $For : Ob(\mathcal{C}^{pt}) \to Ob(\mathcal{C})$ and families:

$$\underline{\zeta} = (\zeta_X)_{X \in Ob(\mathcal{C})} \in \prod_{X \in Ob(\mathcal{C})} F(X).$$

The idea behind the notion of pointed categories is to replace the original category $\mathcal{C}$ by a category $\mathcal{C}^{pt}$ with more objects but *less* morphisms between objects.

Let $\mathcal{C}_o \subset \mathcal{C}$ denote the full subcategory of connected objects (see Subsection 3.2.1) and let $\mathcal{G} \subset \mathcal{C}_o$ denote the full subcategory of Galois objects (see Subsection 3.2.2). Then, it turns out that:

- For any two objects X, Y in $\mathcal{G}$ such that $X \geq Y$ and for any $\zeta_X \in F(X)$, $\zeta_Y \in F(Y)$ there is exactly one morphism from (X, ζ_X) to (Y, ζ_Y) in $\mathcal{G}^{pt}$;
- For any two objects $X, Y \in \mathcal{G}$ there exists an object $Z \in \mathcal{G}$ such that $Z \geq X$ and $Z \geq Y$.

As a result, any section $\underline{\zeta}$ of $For : Ob(\mathcal{C}^{pt}) \to Ob(\mathcal{C})$ endows $Ob(\mathcal{G})$ with a structure of projective system, that we denote by $\mathcal{G}^{\underline{\zeta}}$.

The two remarkable facts concerning $\mathcal{G}^{\underline{\zeta}}$ are:

(1) Any object in $\mathcal{C}_o^{pt}$ is dominated by an object in $\mathcal{G}^{\underline{\zeta}}$ (see Proposition 3.3);

(2) Given any object $X \in \mathcal{G}$, if we replace $\mathcal{C}$ by the full subcategory $\mathcal{C}^X \subset \mathcal{C}$ whose objects are the objects in $\mathcal{C}$ whose connected components are all dominated by X and $F : \mathcal{C} \to FSets$ by its restriction $F^X : \mathcal{C}^X \to FSets$ to $\mathcal{C}^X$ then (see Proposition 3.5),
   (a) the evaluation morphism: $ev_{\zeta_X} : \mathrm{Hom}_{\mathcal{C}}(X, -)|_{\mathcal{C}^X} \to F^X$ is an isomorphism;
   (b) $\mathcal{C}^X$ is a Galois category with fibre functor $F^X : \mathcal{C}^X \to FSets$ for which Theorem 2.8 holds.

(1) provides a well-defined morphism of functors:

$$ev_{\underline{\zeta}} : \varinjlim_{\mathcal{G}^{\underline{\zeta}}} \mathrm{Hom}_{\mathcal{C}}(X, -) \to F$$

and it will follow from (2) (a) that this is an isomorphism. But, then, the proof of Theorem 2.8 follows easily by combining (1) and (2) (b). Furthermore, this will give a natural description of $\pi_1(\mathcal{C}; F)$ as:

$$(\varprojlim_{\mathcal{G}^{\underline{\zeta}}} \mathrm{Aut}_{\mathcal{C}}(X))^{op}.$$

3.2. Connected and Galois objects

3.2.1. Connected objects. An object $X \in \mathcal{C}$ is *connected* if it cannot be written as a coproduct $X = X_1 \sqcup X_2$ with $X_i \neq \emptyset_{\mathcal{C}}$, $i = 1, 2$. We gather below elementary properties of connected objects.

Proposition 3.1 (Minimality and connected components). *An object $X_0 \in \mathcal{C}$ is connected if and only if for any $X \in \mathcal{C}$, $X \neq \emptyset_{\mathcal{C}}$ any monomorphism from X to X_0 in $\mathcal{C}$ is automatically an isomorphism.*

In particular, any object $X \in \mathcal{C}$, $X \neq \emptyset_{\mathcal{C}}$ can be written as:

$$X = \bigsqcup_{i=1}^{r} X_i,$$

with $X_i \in \mathcal{C}$ connected, $X_i \neq \emptyset_{\mathcal{C}}$, $i = 1, \ldots, r$ and this decomposition is unique (up to permutation). We say that the X_i, $i = 1, \ldots, r$ are the connected components of X.

Proof. We prove first the "only if" implication. Write $X_0 = X_0' \sqcup X_0''$ and assume, for instance, that $X_0' \neq \emptyset_{\mathcal{C}}$. From Lemma 2.6 (1), the canonical morphism $i_{X_0'} : X_0' \to X_0$ is a monomorphism hence automatically an isomorphism, which forces $F(X_0'') = \emptyset$ hence $X_0'' = \emptyset_{\mathcal{C}}$ by Lemma 2.6 (2).

We prove now the "if" implication. Assume that $X_0 \neq \emptyset_{\mathcal{C}}$ is connected and let $X \in \mathcal{C}$, $X \neq \emptyset_{\mathcal{C}}$. By axiom (3), any monomorphism $i : X \hookrightarrow X_0$ in $\mathcal{C}$ factors as $X \xrightarrow{i'} X_0' \xrightarrow{i''} X_0 = X_0' \sqcup X_0''$ with $i' : X \to X_0'$ a strict epimorphism and $i'' : X_0' \to X_0$ a monomorphism inducing an isomorphism onto X_0'. Since X_0 is connected either $X_0' = \emptyset_{\mathcal{C}}$ or $X_0'' = \emptyset_{\mathcal{C}}$. But if $X_0' = \emptyset_{\mathcal{C}}$ then $F(X) = \emptyset$, which, by Lemma 2.6 (2), forces $X = \emptyset_{\mathcal{C}}$ and contradicts our assumption. So $X_0'' = \emptyset_{\mathcal{C}}$ and $i'' : X_0' \hookrightarrow X_0$ is an isomorphism. But, then, $i : X \hookrightarrow X_0$ is both a monomorphism and a strict epimorphism hence an isomorphism by Lemma 2.4.

As for the last assertion, since $\mathcal{C}$ is artinian, for any $X \in \mathcal{C}$, $X \neq \emptyset$, there exists $X_1 \in \mathcal{C}$ connected, $X_1 \neq \emptyset_{\mathcal{C}}$ and a monomorphism $i_1 : X_1 \hookrightarrow X$. If i_1 is an isomorphism then X is connected. Else, from axiom (3), i_1 factors as $X_1 \xrightarrow{i_1'} X' \xrightarrow{i_1''} X = X' \sqcup X''$ with i_1' a strict epimorphism and i_1'' a monomorphism inducing an isomorphism onto X'. Since i_1 and i_1'' are monomorphism, i_1' is a monomorphism as well hence an isomorphism, by Lemma 2.4 (2). We then iterate the argument on X''. By axiom (5), this process terminates after at most $|F(X)|$ steps. So we obtain a decomposition:

$$X = \bigsqcup_{i=1}^{r} X_i$$

as a coproduct of finitely many non-initial connected objects, which proves the existence. For the unicity, assume that we have another such decomposition:

$$X = \bigsqcup_{i=1}^{s} Y_i.$$

For $1 \leq i \leq r$, let $1 \leq \sigma(i) \leq s$ such that $F(X_i) \cap F(Y_{\sigma(i)}) \neq \emptyset$. Then consider:

$$\begin{tikzcd}
X_i \arrow[r, hook, "i_{X_i}"] & X \\
X_i \times_X Y_{\sigma(i)} \arrow[u, "p"] \arrow[r, "q"'] \arrow[ur, phantom, "\square"] & Y_{\sigma(i)}. \arrow[u, hook, "i_{Y_{\sigma(i)}}"']
\end{tikzcd}$$

Since i_{X_i} is a monomorphism, q is a monomorphism as well. Also, by axiom (4) one has $F(X_i \times_X Y_{\sigma(i)}) = F(X_i) \cap F(Y_{\sigma(i)})$, which is nonempty by definition of $\sigma(i)$. So, from Lemma 2.6 (1), one has $X_i \times_X Y_{\sigma(i)} \neq \emptyset_{\mathcal{C}}$ and, since $Y_{\sigma(i)}$ is connected and q is a monomorphism, q is an isomorphism. Similarly, p is an isomorphism. $\square$

Proposition 3.2 (Morphisms from and to connected objects).

(1) (Rigidity) *For any $X_0 \in \mathcal{C}$ connected, $X_0 \neq \emptyset_{\mathcal{C}}$, for any $X \in \mathcal{C}$, $X \neq \emptyset_{\mathcal{C}}$ and for any $\zeta_0 \in F(X_0)$, $\zeta \in F(X)$, there is at most one morphism from (X_0, ζ_0) to (X, ζ) in $\mathcal{C}^{pt}$;*

(2) (Domination by connected objects) *For any $(X_i, \zeta_i) \in \mathcal{C}^{pt}$, $i = 1, \ldots, r$ there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with $X_0 \in \mathcal{C}$ connected such that $(X_0, \zeta_0) \geq (X_i, \zeta_i)$ in $\mathcal{C}^{pt}$, $i = 1, \ldots, r$.*
In particular, for any $X \in \mathcal{C}$, there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with $X_0 \in \mathcal{C}$ connected such that the evaluation map:

$$\begin{array}{rccc} ev_{\zeta_0}: & \mathrm{Hom}_{\mathcal{C}}(X_0, X) & \tilde{\to} & F(X) \\ & u : X_0 \to X & \mapsto & F(u)(\zeta_0) \end{array}$$

is bijective.

(3) (i) *If $X_0 \in \mathcal{C}$ is connected then any morphism $u : X \to X_0$ in $\mathcal{C}$ is a strict epimorphism;*
(ii) *If $u : X_0 \to X$ is a strict epimorphism in $\mathcal{C}$ and if X_0 is connected then X is also connected;*
(iii) *If $X_0 \in \mathcal{C}$ is connected then any endomorphism $u : X_0 \to X_0$ in $\mathcal{C}$ is automatically an automorphism.*

Proof. (1) It follows from axiom (1) that the equalizer $\ker(u_1, u_2) \xrightarrow{i} X$ of any two morphisms $u_i : X \to Y$, $i = 1, 2$ in $\mathcal{C}$ exists in $\mathcal{C}$. So, let $u_i : (X_0, \zeta_0) \to (X, \zeta)$ be two morphisms in $\mathcal{C}^{pt}$, $i = 1, 2$ and consider their equalizer $\ker(u_1, u_2) \xrightarrow{i} X_0$ in $\mathcal{C}$. From axiom (4), $F(\ker(u_1, u_2)) \xrightarrow{F(i)} F(X_0)$ is the equalizer of $F(u_i) : F(X_0) \to F(X)$, $i = 1, 2$ in $FSets$. But by assumption, $\zeta_0 \in \ker(F(u_1), F(u_2)) = F(\ker(u_1, u_2))$ so, in particular, $F(\ker(u_1, u_2)) \neq \emptyset$ and it follows from Lemma 2.6 (2) that $\ker(u_1, u_2) \neq \emptyset_{\mathcal{C}}$. Since an equalizer is always a monomorphism, it follows then from Proposition 3.1 that $i : \ker(u_1, u_2) \tilde{\to} X_0$ is an isomorphism that is, $u_1 = u_2$.

(2) Take $X_0 := X_1 \times \cdots \times X_r$, $\zeta_0 := (\zeta_1, \ldots, \zeta_r) \in F(X_1) \times \cdots \times F(X_r) = F(X_1 \times \cdots \times X_r)$ (by axiom (4)). The ith projection $pr_i : X_0 \to X_i$ then induces a morphism from (X_0, ζ_0) to (X_i, ζ_i) in $\mathcal{C}^{pt}$, $i = 1, \ldots, r$. So, it is enough to prove that for any $(X, \zeta) \in \mathcal{C}^{pt}$ there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with X_0 connected such that $(X_0, \zeta_0) \geq (X, \zeta)$ in $\mathcal{C}^{pt}$. If $X \in \mathcal{C}$ is connected then $Id : (X, \zeta) \to (X, \zeta)$ works. Else, write:

$$X = \bigsqcup_{i=1}^{r} X_i$$

as the coproduct of its connected components and let $i_{X_i} : X_i \hookrightarrow X$ denote the canonical monomorphism, $i = 1, \ldots, r$. Then, from axiom (2) one gets:

$$F(X) = \bigsqcup_{i=1}^{r} F(X_i)$$

hence, there exists a unique $1 \leq i \leq r$ such that $\zeta \in F(X_i)$ and $i_{X_i} : (X_i, \zeta) \hookrightarrow (X, \zeta)$ works.

(3)(i) It follows from axiom (3) that $u : X \to X_0$ factors as $X \overset{u'}{\to} X_0' \overset{u''}{\to} X_0' \sqcup X_0'' = X_0$, where u' is a strict epimorphism and u'' is a monomorphism inducing an isomorphism onto X_0'. Furthermore, $X \neq \emptyset_{\mathcal{C}}$ forces $X_0' \neq \emptyset_{\mathcal{C}}$ thus, since X_0 is connected, $X_0'' = \emptyset_{\mathcal{C}}$ hence $u'' : X_0' \tilde{\to} X_0$ is an isomorphism.

(ii) From axiom (6), it is enough to prove that $F(u) : F(X_0) \tilde{\to} F(X_0)$ is an isomorphism. But as $F(X_0)$ is finite, it is actually enough to prove that $F(u) : F(X_0) \twoheadrightarrow F(X_0)$ is an epimorphism. By axiom (3) write $u : X_0 \to X_0$ as $X_0 \overset{u'}{\to} X_0' \overset{u''}{\to} X_0 = X_0' \sqcup X_0''$ with $u' : X_0 \to X_0'$ a strict epimorphism and $u'' : X_0' \to X_0$ a monomorphism inducing an isomorphism onto X_0'. Since X_0 is connected either $X_0' = \emptyset_{\mathcal{C}}$ or $X_0'' = \emptyset_{\mathcal{C}}$. The former implies $X_0 = \emptyset_{\mathcal{C}}$ and then the claim is straightforward. The latter implies $X_0 = X_0'$ thus $u'' : X_0' \to X_0$ is an isomorphism and $u : X_0 \to X_0$ is a strict epimorphism so the conclusion follows from axiom (4).

(iii) If $X_0 = \emptyset_{\mathcal{C}}$, the claim is straightforward. Else, write $X = X' \sqcup X''$ in $\mathcal{C}$ with $X' \neq \emptyset_{\mathcal{C}}$ and let $i_{X'} : X' \hookrightarrow X$ denote the canonical monomorphism. Fix $\zeta' \in F(X')$ and $\zeta_0 \in F(X_0)$ such that $F(u)(\zeta_0) = \zeta'$. From (2), there exist $(X_0', \zeta_0') \in \mathcal{C}^{pt}$ with X_0' connected and morphisms $p : (X_0', \zeta_0') \to (X_0, \zeta_0)$ and $q : (X_0', \zeta_0') \to (X', \zeta')$ in $\mathcal{C}^{pt}$. From (3) (i) the morphism $p : X_0' \to X_0$ is automatically a strict epimorphism, so $u \circ p : X_0' \to X$ is also a strict epimorphism. From (1), one has: $u \circ p = i_{X'} \circ q$. So $i_{X'} \circ q$ is a strict epimorphism and, in particular, $F(X) = F(X')$, which forces $F(X'') = \emptyset$ hence, $X'' = \emptyset_{\mathcal{C}}$ by Lemma 2.6 (2). $\square$

3.2.2. Galois objects. It follows from Proposition 3.2 (1) and (3) (iii) that for any connected object $X_0 \in \mathcal{C}$, $X_0 \neq \emptyset_{\mathcal{C}}$ and for any $\zeta_0 \in F(X_0)$, the evaluation map:

$$\begin{array}{rccc} ev_{\zeta_0} : & \mathrm{Aut}_{\mathcal{C}}(X_0) & \hookrightarrow & F(X_0) \\ & u : X_0 \tilde{\to} X_0 & \mapsto & F(u)(\zeta_0) \end{array}$$

is injective. A connected object X_0 in $\mathcal{C}$ is *Galois in* $\mathcal{C}$ if for any $\zeta_0 \in F(X_0)$ the evaluation map $ev_{\zeta_0} : \mathrm{Aut}_{\mathcal{C}}(X_0) \hookrightarrow F(X_0)$ is bijective. This is equivalent to one of the following:

(1) $\mathrm{Aut}_{\mathcal{C}}(X_0)$ acts transitively on $F(X_0)$;
(2) $\mathrm{Aut}_{\mathcal{C}}(X_0)$ acts simply transitively on $F(X_0)$;
(3) $|\mathrm{Aut}_{\mathcal{C}}(X_0)| = |F(X_0)|$;
(4) $X_0/\mathrm{Aut}_{\mathcal{C}}(X_0)$ is final in $\mathcal{C}$.

The equivalence of (1), (2) and (3) follows from the fact hat $\mathrm{Aut}_{\mathcal{C}}(X_0)$ acts simply on $F(X_0)$. It follows from Lemma 2.6 (2) that (4) is equivalent to $F(X_0/\mathrm{Aut}_{\mathcal{C}}(X_0)) = *$.

But, from axiom (5), this is also equivalent to $F(X_0)/\mathrm{Aut}_{\mathcal{C}}(X_0) = *$, which is (1). Note that (4) shows that the notion of Galois object does not depend on the fibre functor $F : \mathcal{C} \to FSets$.

Proposition 3.3 (Galois closure). *For any $X \in \mathcal{C}$ connected, there exists $\hat{X} \in \mathcal{C}$ Galois dominating X in $\mathcal{C}$ and minimal among the Galois objects dominating X in $\mathcal{C}$.*

Proof. From Lemma 3.2 (2) there exists $(X_0, \zeta_0) \in \mathcal{C}^{pt}$ with $X_0 \in \mathcal{C}$ connected such that the evaluation map $ev_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \tilde{\to} F(X)$ is bijective. Write:

$$\mathrm{Hom}_{\mathcal{C}}(X_0, X) = \{u_1, \ldots, u_n\}.$$

Set $\zeta_i := F(u_i)(\zeta_0)$, $i = 1, \ldots, n$ and let $pr_i : X^n \to X$ denote the ith projection, $i = 1, \ldots, n$. By the universal property of product, there exists a unique morphism $\pi := (u_1, \ldots, u_n) : X_0 \to X^n$ such that $pr_i \circ \pi = u_i$, $i = 1, \ldots, n$.

By axiom (3), one can decompose $\pi : X_0 \to X^n$ as $X_0 \xrightarrow{\pi'} \hat{X} \xrightarrow{\pi''} X^n = \hat{X} \sqcup \hat{X}'$ with π' a strict epimorphism and π'' a monomorphism inducing an isomorphism onto $\hat{X}$. We claim that $\hat{X}$ is Galois and is minimal for morphisms from Galois objects to X.

It follows from Lemma 3.2 (3) (ii) that $\hat{X}$ is connected in $\mathcal{C}$. Set $\hat{\zeta}_0 := F(\pi')(\zeta_0) = (\zeta_1, \ldots, \zeta_n) \in F(\hat{X})$; we are to prove that the evaluation map $ev_{\hat{\zeta}_0} : \mathrm{Aut}_{\mathcal{C}}(\hat{X}) \to F(\hat{X})$ is surjective that is, for any $\zeta \in F(\hat{X})$ there exists $\omega \in \mathrm{Aut}_{\mathcal{C}}(\hat{X})$ such that $F(\omega)(\hat{\zeta}_0) = \zeta$. From Proposition 3.2 (2) there exists $(\tilde{X}_0, \tilde{\zeta}_0) \in \mathcal{C}^{pt}$ with $\tilde{X}_0 \in \mathcal{C}$ connected such that $(\tilde{X}_0, \tilde{\zeta}_0) \geq (X_0, \zeta_0)$ and $(\tilde{X}_0, \tilde{\zeta}_0) \geq (\hat{X}, \zeta)$, $\zeta \in F(\hat{X})$ in $\mathcal{C}^{pt}$. So, up to replacing (X_0, ζ_0) with $(\tilde{X}_0, \tilde{\zeta}_0)$, we may assume that there are morphisms $\rho_\zeta : (X_0, \zeta_0) \to (\hat{X}, \zeta)$ in $\mathcal{C}^{pt}$, $\zeta \in F(\hat{X})$. So, on the one hand, one can write $F(\omega)(\hat{\zeta}_0) = F(\omega \circ \pi')(\zeta_0)$ and, on the other hand, $\zeta = F(\rho_\zeta)(\zeta_0)$. But then, by Lemma 3.2 (1), there exists $\omega \in \mathrm{Aut}_{\mathcal{C}}(\hat{X})$ such that $F(\omega)(\hat{\zeta}_0) = \zeta$ if and only if there exists $\omega \in \mathrm{Aut}_{\mathcal{C}}(\hat{X})$ such that $\omega \circ \pi' = \rho_\zeta$. To prove the existence of such an ω observe that:

$$(*) \quad \{pr_1 \circ \pi'' \circ \rho_\zeta, \ldots, pr_n \circ \pi'' \circ \rho_\zeta\} = \{u_1, \ldots, u_n\}.$$

Indeed, the inclusion $\subset$ is straightforward and to prove the converse inclusion $\supset$, it is enough to prove that the $pr_i \circ \pi'' \circ \rho_\zeta$, $1 \leq i \leq n$ are all distinct. But since $pr_i \circ \pi'' \circ \pi' = u_i \neq u_j = pr_j \circ \pi'' \circ \pi'$, $1 \leq i \neq j \leq n$ and $\pi' : X_0 \to \hat{X}$ is a strict epimorphism, $pr_i \circ \pi'' \neq pr_j \circ \pi''$, $1 \leq i \neq j \leq n$ as well. And, as X_0 is connected, $\rho_\zeta : X_0 \to \hat{X}$ is automatically a strict epimorphism hence $pr_i \circ \pi'' \circ \rho_\zeta \neq pr_j \circ \pi'' \circ \rho_\zeta$, $1 \leq i \neq j \leq n$. From $(*)$, there exists a permutation $\sigma \in \mathcal{S}_n$ such that $pr_{\sigma(i)} \circ \pi'' \circ \rho_\zeta = pr_i \circ \pi'' \circ \pi'$, $i = 1, \ldots, n$ and from the universal property of product there exist a unique isomorphism $\sigma : X^n \tilde{\to} X^n$ such that $pr_i \circ \sigma = pr_{\sigma(i)}$, $i = 1, \ldots, n$. Hence $pr_i \circ \pi'' \circ \pi' = pr_i \circ \sigma \circ \pi'' \circ \rho_\zeta$, $i = 1, \ldots, n$, which forces $\pi'' \circ \pi' = \sigma \circ \pi'' \circ \rho_\zeta$. But, then, from the unicity of the decomposition in axiom (3), there exists an automorphism $\omega : \hat{X} \tilde{\to} \hat{X}$ satisfying $\sigma \circ \pi'' = \pi'' \circ \omega$ and $\omega \circ \pi' = \rho_\zeta$.

It remains to prove the minimality of $\hat{X}$. Let $Y \in \mathcal{C}$ Galois and $q : Y \to X$ a morphism in $\mathcal{C}$. Fix $\eta_i \in F(Y)$ such that $F(q)(\eta_i) = \zeta_i$, $i = 1, \ldots, n$. Since $Y \in \mathcal{C}$ is Galois, there exists $\omega_i \in \mathrm{Aut}_{\mathcal{C}}(Y)$ such that $F(\omega_i)(\eta_1) = \eta_i$, $i = 1, \ldots, n$.

This defines a unique morphism $\kappa := (q \circ \omega_1, \ldots, q \circ \omega_n) : Y \to X^n$ such that $pr_i \circ \kappa = q \circ \omega_i$, $i = 1, \ldots, n$. By axiom (3), $\kappa : Y \to X^n$ factors as $Y \xrightarrow{\kappa'} Z' \xrightarrow{\pi''} X^n = Z' \sqcup Z''$ with π' a strict epimorphism in $\mathcal{C}$ and π'' a monomorphism inducing an isomorphism onto Z'. It follows from Lemma 3.2 (3) (ii) that Z' is connected and $F(\kappa)(\eta_1) = (\zeta_1, \ldots, \zeta_n) = \hat{\zeta}_0$ hence Z' is the connected component of $\hat{\zeta}_0$ in X^n that is $\hat{X}$. □

In particular, $\hat{X}$ is unique up to isomorphism; it is called the *Galois closure of* X.

The following lemma will allow us to restrict to connected objects.

Let $X_0,\ X_1, \ldots, X_r \in \mathcal{C}$ connected, set:

$$X := \bigsqcup_{i=1}^{r} X_i$$

and let $i_{X_i} : X_i \hookrightarrow X$ denote the canonical monomorphism, $i = 1, \ldots, r$. One has a well-defined injective map:

$$\sqcup_{i=1}^{r} i_{X_i} \circ : \bigsqcup_{i=1}^{r} \mathrm{Hom}_{\mathcal{C}}(X_0, X_i) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X_0, X).$$

And, actually:

Lemma 3.4. *The map:*

$$\sqcup_{i=1}^{r} i_{X_i} \circ : \bigsqcup_{i=1}^{r} \mathrm{Hom}_{\mathcal{C}}(X_0, X_i) \tilde{\to} \mathrm{Hom}_{\mathcal{C}}(X_0, X)$$

is bijective

Proof. From axiom (3), any $u : X_0 \to X$ factors as $X_0 \xrightarrow{u'} X' \xrightarrow{u''} X = X' \sqcup X''$ with u' a strict epimorphism and u'' a monomorphism inducing an isomorphism onto X'. As X_0 is connected, it follows from Lemma 3.2 (3) (ii) that X' is also connected, so X' is one of the connected component X_i, $i = 1, \ldots, r$ of X. This shows that the above injective map is surjective hence bijective as claimed. □

For any $X_0 \in \mathcal{C}$ Galois let $\mathcal{C}^{X_0} \subset \mathcal{C}$ denote the full subcategory whose objects are the $X \in \mathcal{C}$ such that X_0 dominates any connected component of X in $\mathcal{C}$. Write $F^{X_0} := F|_{\mathcal{C}^{X_0}} : \mathcal{C}^{X_0} \to FSets$ for the restriction of $F : \mathcal{C} \to FSets$ to $\mathcal{C}^{X_0}$. The next proposition is the "finite level" version of Theorem 2.8 and can be regarded as the core of its proof.

Proposition 3.5 (Galois correspondence).

(1) *Any $\zeta_0 \in F(X_0)$ induces a functor isomorphism:*

$$ev_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, -)|_{\mathcal{C}^{X_0}} \tilde{\to} F^{X_0}.$$

In particular, this induces an isomorphism of groups:

$$u^{\zeta_0} : \mathrm{Aut}_{Fct}(F^{X_0}) \tilde{\to} \mathrm{Aut}_{Fct}(\mathrm{Hom}_{\mathcal{C}}(X_0, -)|_{\mathcal{C}^{X_0}}) = \mathrm{Aut}_{\mathcal{C}}(X_0)^{op}$$

(*where the second equality is just the Yoneda lemma*) *and which can be explicitly described:*

$$u^{\zeta_0}(\theta) = ev_{\zeta_0}^{-1}(\theta(X_0)(\zeta_0)).$$

(2) *The functor* $F^{X_0} : \mathcal{C}^{X_0} \to FSets$ *factors through an equivalence of categories:*

$$\begin{tikzcd} \mathcal{C}^{X_0} \arrow[r, "F^{X_0}"] \arrow[d, "F^{X_0}"'] & FSets \\ \mathcal{C}(\mathrm{Aut}_{\mathcal{C}}(X_0)^{op}) \arrow[ur, "For"'] & \end{tikzcd}$$

Proof. (1) For any morphism $u : Y \to X$ in $\mathcal{C}^{X_0}$, it follows from the fact that $F : \mathcal{C} \to FSets$ is a functor that the following diagram commutes:

$$\begin{tikzcd} F(Y) \arrow[r, "F(u)"] & F(X) \\ \mathrm{Hom}_{\mathcal{C}}(X_0, Y) \arrow[u, "ev_{\zeta_0(Y)}"] \arrow[r, "u\circ"'] & \mathrm{Hom}_{\mathcal{C}}(X_0, X), \arrow[u, "ev_{\zeta_0(X)}"'] \end{tikzcd}$$

that is, $ev_{\zeta_0} : \mathrm{Hom}_{\mathcal{C}}(X_0, -)|_{\mathcal{C}^{X_0}} \tilde{\to} F^{X_0}$ is a functor morphism.

Also, since X_0 is connected, $ev_{\zeta_0}(X) : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \hookrightarrow F(X)$ is injective, $X \in \mathcal{C}^{X_0}$.

- If X is connected it follows from Lemma 3.2 (3) (i) that any morphism $u : X_0 \to X$ in $\mathcal{C}$ is automatically a strict epimorphism. Write $F(X) = \{\zeta_1, \ldots, \zeta_n\}$ and let $\zeta_{0i} \in F(X_0)$ such that $F(u)(\zeta_{0i}) = \zeta_i$, $i = 1, \ldots, n$. Since $X_0 \in \mathcal{C}$ is Galois, there exists $\omega_i \in \mathrm{Aut}_{\mathcal{C}}(Y)$ such that $F(\omega_i)(\zeta_0) = \zeta_{0i}$, $i = 1, \ldots, n$, which proves that $ev_{\zeta_0}(X) : \mathrm{Hom}_{\mathcal{C}}(X_0, X) \twoheadrightarrow F(X)$ is surjective hence bijective.
- If X is not connected, the conclusion follows from Proposition 3.1, Lemma 3.4 and axiom (5).

(2) For simplicity set $G := \mathrm{Aut}_{\mathcal{C}}(X_0)$. From (1), we can identify $F^{X_0} : \mathcal{C}^{X_0} \to FSets$ with:

$$\mathrm{Hom}_{\mathcal{C}}(X_0, -)|_{\mathcal{C}^{X_0}} : \mathcal{C}^{X_0} \to FSets,$$

over which G^{op} acts naturally *via* composition on the right, whence a factorization:

$$\begin{tikzcd} \mathcal{C}^{X_0} \arrow[r, "F^{X_0}"] \arrow[d, "F^{X_0}"'] & FSets \\ \mathcal{C}(G^{op}). \arrow[ur, "For"'] & \end{tikzcd}$$

We will write "$\circ$" for the composition law in G and "$\vee$" for the composition law in G^{op}. It remains to prove that $F^{X_0} : \mathcal{C}^{X_0} \to \mathcal{C}(G^{op})$ is an equivalence of categories.

– F^{X_0} is essentially surjective: Let $E \in \mathcal{C}(G^{op})$. By the same argument as in (1), one may assume that E is connected in $\mathcal{C}(G^{op})$ that is a transitive left G^{op}-set. Thus we get an epimorphism in G^{op}-Sets:

$$\begin{array}{rccc} p_e^0 : & G^{op} & \twoheadrightarrow & E \\ & \omega & \mapsto & \omega \cdot e. \end{array}$$

Set $f_e := p_e^0 \circ ev_{\zeta_0}^{-1} : F(X_0) \twoheadrightarrow E$. Then, for any $s \in S_e := \mathrm{Stab}_{G^{op}}(e)$, and $\omega \in G$, one has:

$$\begin{aligned} f_e \circ F(s)(ev_{\zeta_0}(\omega)) &= p_e^0 \circ ev_{\zeta_0}^{-1} \circ ev_{\zeta_0}(s \circ \omega) \\ &= (s \circ \omega) \cdot e \\ &= (\omega \vee s) \cdot e \\ &= \omega \cdot (s \cdot e) \\ &= \omega \cdot e \\ &= f_e(ev_{\zeta_0}(\omega)). \end{aligned}$$

So, by the universal property of quotient, $f_e : F(X_0) \twoheadrightarrow E$ factors through:

$$\begin{tikzcd} G^{op} \arrow[r, two heads, "ev_{\zeta_0}"] \arrow[dr, two heads, "p_e^0"'] & F(X_0) \arrow[r] \arrow[d, two heads, "f_e"] & F(X_0)/S_e \arrow[dl, two heads, "\overline{f}_e."] \\ & E & \end{tikzcd}$$

But if $p_e : X_0 \to X_0/S_e$ denotes the categorical quotient of X_0 by $S_e \subset G$ assumed to exist by axiom (2), it follows from axiom (5) that $F(X_0) \twoheadrightarrow F(X_0)/S_e$ is $F(p_e) : F(X_0) \twoheadrightarrow F(X_0/S_e)$. Since X_0 is connected, G acts simply on $F(X_0)$ hence:

$$|F(X_0)/S_e| = |F(X_0)|/|S_e| = [G : S_e] = |E|.$$

So $\overline{f}_e : F(X_0)/S_e = F(X_0/S_e) \twoheadrightarrow E$ is actually an isomorphism in G^{op}-Sets.

– F^{X_0} is fully faithful: Let $X,\ Y \in \mathcal{C}^{X_0}$. Again, by the same argument as in (1), one may assume that $X,\ Y$ are connected in $\mathcal{C}$. The faithfulness of F^{X_0} directly follows from Proposition 3.2 (1). As for the fullness, for any morphism $u : F(X) \to F(Y)$ in $\mathcal{C}(G^{op})$, fix $e \in F(X)$. Since $u : F(X) \to F(Y)$ in a morphism in $\mathcal{C}(G^{op})$ one has $S_e \subset S_{u(e)}$ hence $p_{u(e)} : X_0 \to X_0/S_{u(e)}$ factors through:

$$\begin{tikzcd} X_0 \arrow[r, "p_e"] \arrow[d, "p_{u(e)}"'] & X_0/S_e \arrow[dl, "p_{e,u(e)}"] \\ X_0/S_{u(e)} & \end{tikzcd}$$

whence, from the proof of essential surjectivity, one gets the commutative diagram:

$$\begin{tikzcd}
 & F(X_0) \arrow[dl, "F(p_e)"'] \arrow[dr, "F(p_{u(e)})"] & \\
F(X_0/S_e) \arrow[rr, "F(p_{e,u(e)})"] \arrow[d, "\overline{f}_e"', "\simeq"] & & F(X_0/S_{u(e)}) \arrow[d, "\simeq"', "\overline{f}_{u(e)}"] \\
F(X) \arrow[rr, "u"'] & & F(Y).
\end{tikzcd}$$

□

Exercise 3.6. Let $X_0 \in \mathcal{C}$ Galois and $X \in \mathcal{C}^{X_0}$ which, from Proposition 3.5 can be identified with the quotient of X_0 by a subgroup $S_X \subset \mathrm{Aut}_{\mathcal{C}}(X_0)$. Show that X is Galois in $\mathcal{C}$ if and only if S_X is normal in $\mathrm{Aut}_{\mathcal{C}}(X_0)$ and that then, one has a short exact sequence of finite groups:

$$1 \to S_X \to \mathrm{Aut}_{\mathcal{C}}(X_0) \to \mathrm{Aut}_{\mathcal{C}}(X) \to 1.$$

3.3. Strict pro-representability of $F : \mathcal{C} \to FSets$

The category $Pro(\mathcal{C})$ associated with $\mathcal{C}$ is the category whose objects are projective systems $\underline{X} = (\phi_{i,j} : X_i \to X_j)_{i,j\in I,\ i\geq j}$ in $\mathcal{C}$ indexed by partially ordered filtrant sets $(I, \leq)$ and whose morphisms from $\underline{X} = (\phi_{i,j} : X_i \to X_j)_{i,j\in I,\ i\geq j}$ to $\underline{X}' = (\phi'_{i,j} : X'_i \to X'_j)_{i,j\in I',\ i\geq j}$ are:

$$\mathrm{Hom}_{Pro(\mathcal{C})}(\underline{X}, \underline{X}') := \varprojlim_{i'\in I'} \varinjlim_{i\in I} \mathrm{Hom}_{\mathcal{C}}(X_i, X'_{i'}).$$

Note that $\mathcal{C}$ can be regarded canonically as a full subcategory of $Pro(\mathcal{C})$ and that $F : \mathcal{C} \to FSets$ extends canonically to a functor $Pro(F) : Pro(\mathcal{C}) \to Pro(FSets)$.

The functor $F : \mathcal{C} \to FSets$ is said to be *pro-representable in* $\mathcal{C}$ if there exists $\underline{X} = (\phi_{i,j} : X_i \to X_j)_{i,j\in I,\ i\geq j} \in Pro(\mathcal{C})$ and a functor isomorphism:

$$\mathrm{Hom}_{Pro(\mathcal{C})}(\underline{X}, -)|_{\mathcal{C}} \tilde{\to} F$$

and it is said to be *strictly pro-representable in* $\mathcal{C}$ if it is pro-representable in $\mathcal{C}$ by an object $\underline{X} = (\phi_{i,j} : X_i \to X_j)_{i,j\in I,\ i\geq j} \in Pro(\mathcal{C})$ whose transition morphisms $\phi_{i,j} : X_i \twoheadrightarrow X_j$ are epimorphisms, $i, j \in I,\ i \geq j$.

3.3.1. Projective structures on Galois objects. Let $\mathcal{G}$ denote the set of all Galois objects (or more precisely, a system of representatives of the isomorphism classes of Galois objects) in $\mathcal{C}$. From Proposition 3.2 (2) and Proposition 3.3, $(\mathcal{G}, \leq)$ is directed. Fix $\underline{\zeta} = (\zeta_X)_{X\in\mathcal{G}} \in \prod_{X\in\mathcal{G}} F(X)$. Then, from Proposition 3.2 (1), for any $X,\ Y \in \mathcal{G}$ with $X \leq Y$, there exists a unique $\phi^{\zeta}_{X,Y} : Y \to X$ in $\mathcal{C}$ such that $\phi^{\zeta}_{X,Y}(\zeta_Y) = \zeta_X$. The unicity of $\phi^{\zeta}_{X,Y} : Y \to X$ implies that for any $X,\ Y,\ Z \in \mathcal{G}$ with $X \leq Y \leq Z$ one has:

$$\phi^{\zeta}_{X,Y} \circ \phi^{\zeta}_{Y,Z} = \phi^{\zeta}_{X,Z}.$$

This endows $\mathcal{G}$ with a structure of projective system

$$\mathcal{G}^{\underline{\zeta}} := (\phi^{\underline{\zeta}}_{X,Y} : Y \twoheadrightarrow X)_{X,\, Y \in \mathcal{G},\, X \leq Y}$$

and one has:

Proposition 3.7. *The fibre functor $F : \mathcal{C} \to FSets$ is strictly pro-representable in $\mathcal{C}$ by $\mathcal{G}^{\underline{\zeta}}$. More precisely, the evaluation morphisms $ev_{\zeta_X} : \mathrm{Hom}_{\mathcal{C}}(X, -)|_{\mathcal{C}^X} \to F|_{\mathcal{C}^X}$, $X \in \mathcal{G}$ induce a functor isomorphism:*

$$ev_{\underline{\zeta}} : \varinjlim \mathrm{Hom}_{\mathcal{C}}(X, -)|_{\mathcal{C}} \tilde{\to} F.$$

Proof. From Proposition 3.2 (3) (i), the transition morphisms are automatically strict epimorphisms. The remaining part of the assertion follows directly from the construction and Proposition 3.5. □

The projective structure $\mathcal{G}^{\underline{\zeta}}$ also induces a projective structure on the $\mathrm{Aut}_{\mathcal{C}}(X)$, $X \in \mathcal{G}$. More precisely, we have:

Lemma 3.8. *For any X, $Y \in \mathcal{G}$ with $X \leq Y$, for any morphisms ϕ, $\psi : Y \to X$ in $\mathcal{C}$ and for any $\omega_Y \in \mathrm{Aut}_{\mathcal{C}}(Y)$ there is a unique automorphisms $\omega_X := r_{\phi,\psi}(\omega_Y) : X \tilde{\to} X$ in $\mathcal{C}$ such that the following diagram commutes:*

$$\begin{tikzcd} Y \arrow[r, "\omega_Y"] \arrow[d, "\psi"'] & Y \arrow[d, "\phi"] \\ X \arrow[r, "\omega_X"'] & X. \end{tikzcd}$$

Proof. Since X is connected, $\psi : Y \to X$ is automatically a strict epimorphism and, in particular, the map:

$$\circ\psi : \mathrm{Aut}_{\mathcal{C}}(X) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

is injective. But it follows from Proposition 3.5 that $|\mathrm{Hom}_{\mathcal{C}}(Y, X)| = |F(X)|$ and from the fact that X is Galois that $|F(X)| = |\mathrm{Aut}_{\mathcal{C}}(X)|$. As a result the map:

$$\circ\psi : \mathrm{Aut}_{\mathcal{C}}(X) \tilde{\to} \mathrm{Hom}_{\mathcal{C}}(Y, X)$$

is actually bijective and, in particular, there exists a unique automorphism $\omega_X : X \tilde{\to} X$ in $\mathcal{C}$ such that $\phi \circ \omega_Y = \omega_X \circ \psi$. □

So one gets a well-defined surjective map:

$$r_{\phi,\psi} : \mathrm{Aut}_{\mathcal{C}}(Y) \twoheadrightarrow \mathrm{Aut}_{\mathcal{C}}(X),$$

which is automatically a group epimorphism when $\phi = \psi$. In particular, one gets a projective system of finite groups:

$$(r^{\underline{\zeta}}_{X,Y} := r_{\phi^{\underline{\zeta}}_{X,Y}, \phi^{\underline{\zeta}}_{X,Y}} : \mathrm{Aut}_{\mathcal{C}}(Y) \twoheadrightarrow \mathrm{Aut}_{\mathcal{C}}(X))_{X,Y \in \mathcal{G},\, X \leq Y}.$$

Set:

$$\Pi := \varprojlim \mathrm{Aut}_{\mathcal{C}}(X).$$

Then Π^{op} acts naturally on:

$$\varinjlim \mathrm{Hom}_{\mathcal{C}}(X,-)|_{\mathcal{C}}$$

by composition on the right, which induces a group monomorphism:

$$\Pi^{op} \hookrightarrow \mathrm{Aut}_{Fct}(\varinjlim \mathrm{Hom}_{\mathcal{C}}(X,-)|_{\mathcal{C}})$$

and the functor isomorphism

$$ev_{\underline{\zeta}} : \varinjlim \mathrm{Hom}_{\mathcal{C}}(X,-)|_{\mathcal{C}} \xrightarrow{\sim} F$$

thus induces a group monomorphism:

$$\begin{array}{rccc} u^{\underline{\zeta}}: & \pi_1(\mathcal{C};F) & \hookrightarrow & \Pi^{op} \\ & \theta & \mapsto & (ev_{\zeta_X}^{-1}(\theta(X)(\zeta_X)))_{X\in\mathcal{G}} \end{array}$$

and, actually:

Proposition 3.9. *The group monomorphism $u^{\underline{\zeta}} : \pi_1(\mathcal{C};F) \to \Pi^{op}$ is an isomorphism of profinite groups.*

Proof. We first show that $u_{\underline{\zeta}} : \pi_1(\mathcal{C};F) \to \Pi^{op}$ is a group isomorphism by constructing an inverse. Let $\underline{\omega} := (\omega_X)_{X\in\mathcal{G}} \in \Pi$. For any $Z \in \mathcal{C}$ connected, let $\hat{Z}$ denote the Galois closure of Z in $\mathcal{C}$ and consider the bijective map:

$$\theta_{\underline{\omega}}(Z) : F(Z) \overset{ev_{\zeta_{\hat{Z}}}^{-1}}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathcal{C}}(\hat{Z},Z) \overset{\circ\omega_{\hat{Z}}}{\xrightarrow{\sim}} \mathrm{Hom}_{\mathcal{C}}(\hat{Z},Z) \overset{ev_{\zeta_{\hat{Z}}}}{\xrightarrow{\sim}} F(Z).$$

One checks that this defines a functor automorphism and that $u^{\underline{\zeta}}(\theta_{\underline{\omega}}) = \underline{\omega}$.

Next, we show that $u_{\underline{\zeta}} : \pi_1(\mathcal{C};F) \to \Pi^{op}$ is continuous. For this, it is enough to check that the:

$$\pi_1(\mathcal{C};F) \overset{u^{\underline{\zeta}}}{\to} \Pi^{op} \to \mathrm{Aut}_{\mathcal{C}}(X)^{op},\ X \in \mathcal{G}$$

are, which is straightforward by the definition of the topology on $\pi_1(\mathcal{C};F)$. Finally, since $\pi_1(\mathcal{C};F)$ is compact, $u_{\underline{\zeta}}^{-1}$ is continuous as well. □

3.3.2. Conclusion. We can now carry out the proof of Theorem 2.8

1. From Proposition 3.7 and Proposition 3.9, this amount to showing that:

$$F^{\underline{\zeta}} : \mathrm{Hom}_{Pro(\mathcal{C})}(\underline{G^{\underline{\zeta}}},-)|_{\mathcal{C}} : \mathcal{C} \to FSets$$

factors through an equivalence of category $F^{\underline{\zeta}} : \mathcal{C} \to \mathcal{C}(\Pi^{op})$. But this follows almost straightforwardly from Proposition 3.5. Indeed,
– $F^{\underline{\zeta}}$ is essentially surjective: For any $E \in \mathcal{C}(\Pi^{op})$ since E is equipped with the discrete topology, the action of Π^{op} on E factors through a finite quotient $\mathrm{Aut}_{\mathcal{C}}(X)$ with $X \in \mathcal{G}$ and we can apply Proposition 3.5 in $\mathcal{C}^X$.
– $F^{\underline{\zeta}}$ is fully faithful: For any $Z,\ Z' \in \mathcal{C}$, there exists $X \in \mathcal{G}$ such that $X \geq Z$, $X \geq Z'$ and, again, this allows us to apply Proposition 3.5 in $\mathcal{C}^X$.

2. This immediately follows from Proposition 3.7. Indeed, let $F_i : \mathcal{C} \to FSets$, $i = 1, 2$ be fibre functors. Then any $\underline{\zeta}^i \in \prod_{X \in \mathcal{G}} F^i(X)$ induces a functor isomorphism:

$$ev^{F_i}_{\underline{\zeta}^i} : \mathrm{Hom}_{Pro(\mathcal{C})}(\underline{G}^{\underline{\zeta}^i}, -)|_{\mathcal{C}} \tilde{\to} F_i.$$

So it is enough to prove that $\underline{\mathcal{G}}^{\underline{\zeta}^1}$ and $\underline{\mathcal{G}}^{\underline{\zeta}^2}$ are isomorphic in $Pro(\mathcal{C})$. But one has:

$$\varprojlim_X \varinjlim_Y \mathrm{Hom}_{\mathcal{C}}(Y, X) = \varprojlim_X \varinjlim_Y \mathrm{Aut}_{\mathcal{C}}(X) = \varprojlim_X \mathrm{Aut}_{\mathcal{C}}(X) \ ,$$

where the first equality follows from Proposition 3.5 (1). So it is actually enough to prove that

$$\varprojlim \mathrm{Aut}_{\mathcal{C}}(X) \neq \emptyset,$$

where the structure of projective system on the $\mathrm{Aut}_{\mathcal{C}}(X)$, $X \in \mathcal{G}$ is given by the surjective maps defined in Lemma 3.8:

$$r_{\phi^1_{X,Y}, \phi^2_{X,Y}} : \mathrm{Aut}_{\mathcal{C}}(Y) \twoheadrightarrow \mathrm{Aut}_{\mathcal{C}}(X), \ X, \ Y \in \mathcal{G}, \ X \leq Y.$$

And this follows from the fact that a projective system of non-empty finite sets is non-empty. □

4. Fundamental functors and functoriality

4.1. Fundamental functors

Let $\mathcal{C}$, $\mathcal{C}'$ be two Galois categories. Then a covariant functor $H : \mathcal{C} \to \mathcal{C}'$ is a *fundamental* (or *exact*, according to the terminology of [SGA1]) functor from $\mathcal{C}$ to $\mathcal{C}'$ if there exists a fibre functor $F' : \mathcal{C}' \to FSets$ for $\mathcal{C}'$ such that $F' \circ H : \mathcal{C} \to FSets$ is again a fibre functor for $\mathcal{C}$ or, equivalently (since, from Theorem 2.8 (2), two fibre functors are always isomorphic), if for any fibre functor $F' : \mathcal{C}' \to FSets$ for $\mathcal{C}'$ the functor $F' \circ H : \mathcal{C} \to FSets$ is again a fibre functor for $\mathcal{C}$.

Let $u : \Pi' \to \Pi$ be a morphism of profinite groups. Then any $E \in \mathcal{C}(\Pi)$ can be endowed with a continuous action of Π' *via* $u : \Pi' \to \Pi$, which defines a canonical fundamental functor:

$$H_u : \mathcal{C}(\Pi) \to \mathcal{C}(\Pi').$$

Conversely, let $H : \mathcal{C} \to \mathcal{C}'$ be a fundamental functor. Let $F' : \mathcal{C}' \to FSets$ be a fibre functor for $\mathcal{C}'$ and set $F := F' \circ H : \mathcal{C} \to FSets$, $\Pi := \pi_1(\mathcal{C}; F)$, $\Pi' := \pi_1(\mathcal{C}'; F')$. Then for any $\Theta' \in \Pi'$, one has $\Theta' \circ H \in \Pi$, which defines a canonical morphism of profinite groups:

$$u_H : \Pi' \to \Pi.$$

One checks that $u_{H_u} = u$ and that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C}(\Pi) \arrow[r, "H_{u_H}"] & \mathcal{C}(\Pi') \\ \mathcal{C} \arrow[u, "F"] \arrow[r, "H"'] & \mathcal{C}'. \arrow[u, "F'"'] \end{tikzcd}$$

Furthermore, given a fibre functor $F' : \mathcal{C}' \to FSets$ for $\mathcal{C}'$ and two fundamental functors $H_1, H_2 : \mathcal{C} \to \mathcal{C}'$ such that $F' \circ H_1 = F' \circ H_2 =: F$, any morphism of functors $\alpha : H_1 \to H_2$ induces an endomorphism of functor $u_\alpha : F \to F$ such that:

$$u_\alpha \circ u_{H_1}(\theta') = u_{H_2}(\theta') \circ u_\alpha,\ \theta' \in \Pi'.$$

Thus, one the one hand, let Gal denote the 2-category whose objects are Galois categories pointed with fibre functors and where 1-morphisms from $(\mathcal{C}; F)$ to $(\mathcal{C}'; F')$ are fundamental functors $H : \mathcal{C} \to \mathcal{C}'$ such that $F' \circ H = F$ and 2-morphisms are isomorphisms between fundamental functors. And, on the other hand, let Pro denote the 2-category whose objects are profinite groups and where 1-morphisms are morphisms of profinite groups and 2-morphisms from $u_1 : \Pi' \to \Pi$ to $u_2 : \Pi' \to \Pi$ are elements $\theta \in \Pi$ such that $\theta \circ u_1(-) \circ \theta^{-1} = u_2$. Then, the functor $(\mathcal{C}, F) \to \pi_1(\mathcal{C}; F)$ from Gal to Pro is an equivalence of 2-categories with pseudo-inverse $\Pi \to (\mathcal{C}(\Pi), For)$. In the next subsection, we compare the properties of the fundamental functor $H : \mathcal{C} \to \mathcal{C}'$ and of the corresponding morphism of profinite groups $u : \Pi' \to \Pi$.

Example 4.1. Any continuous map $\phi : B' \to B$ of connected, locally arcwise connected and locally simply connected topological spaces defines a canonical functor:

$$\begin{array}{rlcl} H: & \mathcal{C}_B^{\text{top}} & \to & \mathcal{C}_{B'}^{\text{top}} \\ & f : X \to B & \mapsto & p_2 : X \times_{f,B,\phi} B' \to B'. \end{array}$$

and for any $b' \in B'$, one has:

$$\begin{aligned} F_{b'} \circ H(f) &= p_2^{-1}(b') \\ &= \{(x, b') \mid x \in X \text{ such that } f(x) = \phi(b')\} \\ &= f^{-1}(\phi(b')). \end{aligned}$$

Hence $H : \mathcal{C}_B \to \mathcal{C}_{B'}$ is a fundamental functor. In that case, the corresponding morphism of profinite groups is just the canonical morphism:

$$\hat{\phi} : \widehat{\pi_1^{\text{top}}(B'; b')} \to \widehat{\pi_1^{\text{top}}(B; \phi(b'))}$$

induced from $\phi : \pi_1^{\text{top}}(B'; b') \to \pi_1^{\text{top}}(B; \phi(b'))$.

4.2. Functoriality

From Subsection 4.1, one may assume that $\mathcal{C} = \mathcal{C}(\Pi)$, $\mathcal{C}' = \mathcal{C}(\Pi')$ and $H = H_u$ for some morphism of profinite groups $u : \Pi' \to \Pi$.

Given $(X, \zeta) \in \mathcal{C}^{pt}$, we will write $(X, \zeta)_0 := (X_0, \zeta)$, where X_0 denotes the connected component of ζ in X.

We will say that an object $X \in \mathcal{C}$ *has a section in* $\mathcal{C}$ if $e_{\mathcal{C}} \geq X$ and that an object $X \in \mathcal{C}$ is *totally split in* $\mathcal{C}$ if it is isomorphic to a finite coproduct of final objects.

Lemma 4.2. *With the above notation:*

(1) *For any open subgroup* $U \subset \Pi$,
- $\mathrm{im}(u) \subset U$ *if and only if* $(e_{\mathcal{C}'}, *) \geq (H(\Pi/U), 1))$ *in* $\mathcal{C}'^{pt}$*;*
- *Let:*
$$\mathrm{K}_{\Pi}(\mathrm{im}(u)) \lhd \Pi$$
denote the smallest normal subgroup in Π *containing* $\mathrm{im}(u)$*. Then* $\mathrm{K}_{\Pi}(\mathrm{im}(u)) \subset U$ *if and only if* $H(\Pi/U)$ *is totally split in* $\mathcal{C}'$*.*

In particular, $u : \Pi' \to \Pi$ *is trivial if and only if for any object* X *in* $\mathcal{C}$*,* $H(X)$ *is totally split in* $\mathcal{C}'$*.*

(2) *For any open subgroup* $U' \subset \Pi'$,
- $\ker(u) \subset U'$ *if and only if there exists an open subgroup* $U \subset \Pi$ *such that:* $(H(\Pi/U), 1)_0 \geq (\Pi'/U', 1)$ *in* $\mathcal{C}'^{pt}$*.*
- *if, furthermore,* $u : \Pi' \twoheadrightarrow \Pi$ *is an epimorphism, then* $\mathrm{Ker}(u) \subset U'$ *if and only if there exists an open subgroup* $U \subset \Pi$ *and an isomorphism* $(H(\Pi/U), 1)_0 \xrightarrow{\sim} (\Pi'/U', 1)$ *in* $\mathcal{C}'^{pt}$*.*

In particular,
- $u : \Pi' \hookrightarrow \Pi$ *is a monomorphism if and only if for any connected object* $X' \in \mathcal{C}'$ *there exists a connected object* $X \in \mathcal{C}$ *and a connected component* $H(X)_0$ *of* $H(X)$ *in* $\mathcal{C}$ *such that* $H(X)_0 \geq X'$ *in* $\mathcal{C}'$*.*
- *if, furthermore,* $u : \Pi' \twoheadrightarrow \Pi$ *is an epimorphism, then* $u : \Pi' \twoheadrightarrow \Pi$ *is an isomorphism if and only if* $H : \mathcal{C} \to \mathcal{C}'$ *is essentially surjective.*

Proof. Recall that, given a profinite group Π, a closed subgroup $S \subset \Pi$ is the intersection of all the open subgroups $U \subset \Pi$ containing S thus, in particular, $\{1\}$ is the intersection of all open subgroups of Π. This yields the characterization of trivial morphisms and monomorphisms from the preceding assertions in (1) and (2).

(1) For the first assertion of (1), note that $e_{\mathcal{C}'} = *$ and that $(e_{\mathcal{C}'}, *), \geq (H(\Pi/U), 1))$ in $\mathcal{C}'^{pt}$ if and only if the unique map $\phi : * \to H(\Pi/U)$ sending $*$ to U is a morphism in $\mathcal{C}'$ that is, if and only if for any $\theta' \in \Pi'$,
$$U = \phi(*) = \phi(\theta' \cdot *) = \theta' \cdot \phi(*) = u(\theta')U.$$
For the second assertion of (1), note that $\mathrm{K}_{\Pi}(\mathrm{Im}(u)) \subset U$ if and only if for any $g \in \Pi/U$, the map $\phi_g : * \to H(\Pi/U)$ sending $*$ to gU is a morphism in $\mathcal{C}'$. This yields a surjective morphism $\sqcup_{g \in \Pi/U} \phi_g : \sqcup_{g \in \Pi/U} * \to H(\Pi/U)$ in $\mathcal{C}'$, which is automatically injective by cardinality. Conversely, for any isomorphism $\sqcup_{i \in I} \phi_i : \sqcup_{i \in I} * \to H(\Pi/U)$ in $\mathcal{C}'$, set $i_i : * \to H(\Pi/U)$ for the morphism $* \hookrightarrow \sqcup_{i \in I} * \to H(\Pi/U)$ in $\mathcal{C}'$; by construction $i_i = \phi_{i_i(*)}$.

(2) Since U' is closed of finite index in Π' and both Π and Π' are compact, $u(U')$ is closed of finite index in $\mathrm{im}(u)$ hence open. So there exists an open subgroup

$U \subset \Pi$ such that $U \cap \mathrm{im}(u) \subset u(U')$. By definition, the connected component of 1 in $H(\Pi/U)$ in $\mathcal{C}'$ is:

$$\mathrm{im}(u)U/U \simeq \mathrm{im}(u)/(U \cap \mathrm{im}(u)) \simeq \Pi'/u^{-1}(U).$$

But $u^{-1}(U) = u^{-1}(U \cap \mathrm{Im}(u)) \subset U'$, whence a canonical epimorphism $(\mathrm{Im}(u)U/U, 1) \to (\Pi'/U', 1)$ in $\mathcal{C}'^{pt}$. If, furthermore, $\mathrm{im}(u) = \Pi$, then one can take $U = u(U')$ and ϕ is nothing but the inverse of the canonical isomorphism $\Pi'/U' \tilde{\to} \Pi/U$. Conversely, assume that there exists an open subgroup $U \subset \Pi$ and a morphism $\phi : (\mathrm{Im}(u)U/U, 1) \to (\Pi'/U', 1)$ in $\mathcal{C}'^{pt}$. Then, for any $g' \in \Pi'$, one has: $\phi(u(g')U) = g' \cdot \phi(1) = g'U'$. In particular, if $u(g') \in U$ then $g'U = \phi(u(g')U) = \phi(U) = U'$ whence $\ker(u) \subset u^{-1}(U) \subset U'$. Eventually, note that since $\ker(u)$ is normal in Π', the condition $\ker(u) \subset U'$ does not depend on the choice of $\zeta \in F(X)$ defining the isomorphism $X' \tilde{\to} \Pi'/U'$. □

Proposition 4.3.

(1) *The following three assertions are equivalent:*
   (i) $u : \Pi' \twoheadrightarrow \Pi$ *is an epimorphism;*
   (ii) $H : \mathcal{C} \to \mathcal{C}'$ *sends connected objects to connected objects;*
   (iii) $H : \mathcal{C} \to \mathcal{C}'$ *is fully faithful.*

(2) $u : \Pi' \hookrightarrow \Pi$ *is a monomorphism if and only if for any object* X' *in* $\mathcal{C}'$ *there exists an object* X *in* $\mathcal{C}$ *and a connected component* X'_0 *of* $H(X)$ *which dominates* X' *in* $\mathcal{C}'$.

(3) $u : \Pi' \tilde{\to} \Pi$ *is an isomorphism if and only if* $H : \mathcal{C} \to \mathcal{C}'$ *is an equivalence of categories.*

(4) *If* $\mathcal{C} \overset{H}{\to} \mathcal{C}' \overset{H'}{\to} \mathcal{C}''$ *is a sequence of fundamental functors of Galois categories with corresponding sequence of profinite groups* $\Pi \overset{u}{\leftarrow} \Pi' \overset{u'}{\leftarrow} \Pi''$. *Then,*
   – $\ker(u) \supset \mathrm{im}(u')$ *if and only if* $H'(H(X))$ *is totally split in* $\mathcal{C}''$, $X \in \mathcal{C}$;
   – $\ker(u) \subset \mathrm{im}(u')$ *if and only if for any connected object* $X' \in \mathcal{C}'$ *such that* $H'(X')$ *has a section in* $\mathcal{C}''$, *there exists* $X \in \mathcal{C}$ *and a connected component* X'_0 *of* $H(X)$ *which dominates* X' *in* $\mathcal{C}'$.

Proof. Assertion (2) and (4) follow from Lemma 4.2 (2). Assertions (3) follows from Lemma 4.2 and (1). So we are only to prove assertion (1). We will show that (i) ⇒ (ii) ⇒ (iii) ⇒ (i). For (i) ⇒ (ii), assume that $u : \Pi' \twoheadrightarrow \Pi$ is an epimorphism. Then, for any connected object X in $\mathcal{C}(\Pi)$, the group Π acts transitively on X. But $H(X)$ is just X equipped with the Π'-action $g' \cdot x = u(g') \cdot x$, $g' \in \Pi'$. Hence Π' acts transitively on $H(X)$ as well or, equivalently, $H(X)$ is connected. For (ii) ⇒ (i), assume that if $X \in \mathcal{C}$ is connected then $H(X)$ is also connected in $\mathcal{C}'$. This holds, in particular, for any finite quotient Π/N of Π with N a normal open subgroup of Π that is, the canonical morphism $u_N : \Pi' \overset{u}{\to} \Pi \overset{pr_N}{\twoheadrightarrow} \Pi/N$ is a continuous epimorphism. Hence so is $u = \varprojlim u_N$. The implication ⇒ (iii) is straightforward. Finally, for (iii) ⇒ (i), observe that given an open subgroup $U \subset \Pi$, $U \neq \Pi$ there is no morphism from $*$ to Π/U in $\mathcal{C}$. Hence, if $H : \mathcal{C} \to \mathcal{C}'$ is fully (faithful), there

is no morphism as well from $*$ to $H(\Pi/U)$ in $\mathcal{C}'$. But, from Lemma 4.2, this is equivalent to $\mathrm{im}(u) \not\subset U$. $\square$

Exercise 4.4. Given a Galois category $\mathcal{C}$ with fibre functor $F : \mathcal{C} \to FSets$ and $X_0 \in \mathcal{C}$ connected, let $\mathcal{C}_{X_0}$ denote the *category of X_0-objects* that is the category whose objects are morphism $\phi : X \to X_0$ in $\mathcal{C}$ and whose morphisms from $\phi' : X' \to X_0$ to $\phi : X \to X_0$ are the morphisms $\psi : X' \to X$ in $\mathcal{C}$ such that $\phi \circ \psi = \phi'$. For any $\zeta \in F(X_0)$, set

$$\begin{array}{rccl} F_{(X_0,\zeta)} : & \mathcal{C}_{X_0} & \to & FSets \\ & \phi : X \to X_0 & \mapsto & F(\phi)^{-1}(\zeta). \end{array}$$

Then,

1. show that $\mathcal{C}_{X_0}$ is Galois with fibre functors $F_{(X_0,\zeta)} : \mathcal{C}_{X_0} \to FSets$, $\zeta \in F(X_0)$ and that, furthermore, the canonical functor
$$\begin{array}{rccl} H : & \mathcal{C} & \to & \mathcal{C}_{X_0} \\ & X & \mapsto & p_2 : X \times X_0 \to X_0 \end{array}$$
has the property that $F_{(X_0,\zeta)} \circ H = F$, $\zeta \in F(X_0)$ and induces a profinite group monomorphism: $\pi_1(\mathcal{C}_{X_0}; F_{(X_0,\zeta)}) \hookrightarrow \pi_1(\mathcal{C}; F)$ with image $\mathrm{Stab}_{\pi_1(\mathcal{C};F)}(\zeta)$;
2. show that $H(\hat{X}_0)$ is totally split in $\mathcal{C}_{X_0}$ and that if X_0 is the Galois closure $\hat{X}$ of some connected object $X \in \mathcal{C}$ then $H(X)$ is totally split in $\mathcal{C}_{\hat{X}}$.

5. Etale covers

The aim of this section is to prove that the category of finite étale covers of a connected scheme is Galois (see Theorem 5.10). The proof of this result is carried out in Subsection 5.3. In Subsections 5.1 and 5.2, we introduce the notion of étale covers and give some of their elementary properties.

Convention: All the schemes are *locally noetherian*. We make this hypothesis for simplicity and will not repeat it later. For instance, it will sometimes be used explicitly in the proofs but not mentioned in the corresponding statement. Be aware that some results stated in the following sections remain valid without the noetherianity assumptions but *some do not*.

5.1. Etale algebras

Given a ring R, let Alg/R denote the category of R-algebras. Also, given a ring R, we write $R^\times$ for the group of invertible elements in R.

Lemma 5.1. *Let A be a finite-dimensional algebra over a field k. Then the following properties are equivalent:*

(1) *A is isomorphic (as k-algebra) to a finite product of finite separable field extensions of k;*
(2) *$A \otimes_k \overline{k}$ is isomorphic (as $\overline{k}$-algebra) to a finite product of copies of $\overline{k}$;*
(3) *$A \otimes_k \overline{k}$ is reduced;*
(4) $\Omega_{A|k} = 0$.

Proof. We first prove that a finite-dimensional algebra A over a field k is reduced if and only if it is isomorphic (as k-algebra) to a finite product of finite field extensions of k. The 'if' part is straightforward. As for the 'only if' part, write $A = \prod_{i=1}^{r} A_i$ as the finite product of its connected components. Since it is enough to prove that A_i is (as k-algebra) a finite field extension of k, *i.e.*, that $A_i \backslash \{0\} = A_i^{\times}$, $i = 1, \ldots, r$, we may assume that A is a finite-dimensional connected algebra over k. Let $a \in A \backslash \{0\}$. Since A is finite dimensional over k, it is artinian hence $Aa^n = Aa^{n+1}$ for $n \gg 0$. In particular, there exists $b \in A$ such that $a^n = ba^{n+1} = ba^n a = b^2 a^{n+2} = \cdots = b^n a^{2n}$ hence $a^n b^n = (a^n b^n)^2$, which forces $a^n b^n = 0$ or 1 since A has no non-trivial idempotent. But $a^n b^n = 0$ would imply $a^n = (a^n b^n) a^n = 0$, which is impossible since $a \neq 0$ and A is reduced. Hence $a(a^{n-1} b^n) = a^n b^n = 1$ so $a \in A^{\times}$. This proves that A is a field and, as it is also finite dimensional over k, it is a finite field extension of k. This already proves (2) $\Leftrightarrow$ (3). We are going to prove (2) $\Rightarrow$ (1) $\Rightarrow$ (4) $\Rightarrow$ (1).

(2) $\Rightarrow$ (1): Set $\overline{A} := A/\sqrt{0}$. Then $\overline{A}$ is reduced hence, from the above, is isomorphic (as k-algebra) to $\prod_{i=1}^{r} K_i$ with K_i a finite field extension of k, $i = 1, \ldots, r$. Now, any morphism $A \to \overline{k}$ of k-algebras factors through one of the K_i hence

$$N := |\mathrm{Hom}_{Alg/k}(A, \overline{k})| = \sum_{i=1}^{r} |\mathrm{Hom}_{Alg/k}(K_i, \overline{k})|.$$

Since:

$$|\mathrm{Hom}_{Alg/k}(K_i, \overline{k})| \leq [K_i : k]$$

with equality if and only if K_i is a finite separable field extension of k and

$$\dim_k(\overline{A}) = \sum_{i=1}^{r} [K_i : k] \leq \dim_k(A),$$

one has $N \leq \dim_k(A)$ and $N = \dim_k(A)$ if and only if $A = \overline{A}$ and:

$$|\mathrm{Hom}_{Alg/k}(K_i, \overline{k})| = [K_i : k], \ i = 1, \ldots r$$

that is, if and only if $A = \overline{A}$ and K_i is a finite separable field extension of k, $i = 1, \ldots, r$. But the universal property of tensor product implies that:

$$\mathrm{Hom}_{Alg/k}(A, \overline{k}) = \mathrm{Hom}_{Alg/\overline{k}}(A \otimes_k \overline{k}, \overline{k})$$

hence:

$$N = |\mathrm{Hom}_{Alg/\overline{k}}(A \otimes_k \overline{k}, \overline{k})| = \dim_{\overline{k}}(A \otimes_k \overline{k}) = \dim_k(A).$$

(1) $\Rightarrow$ (4): Write:

$$A = \prod_{i=1}^{r} K_i$$

as a finite product of finite separable field extensions of k. Then the maximal ideals of A are the kernel of the projection maps $\mathfrak{m}_i := \ker(A \twoheadrightarrow K_i)$, $i = 1, \ldots, r$ and $\Omega^1_{A|k} = 0$ if and only if $(\Omega^1_{A|k})_{\mathfrak{m}_i} = \Omega_{K_i|k} = 0$, $i = 1, \ldots, r$. Hence, one can assume that $A = K$ is a finite separable field extension of k. But, then, by the primitive

element theorem, $K = k[X]/P$ for some irreducible separable polynomial $P \in k[X]$ hence $\Omega^1_{K|k} = KdT/P'(t)dT$ (where t denotes the image of X in k) with $P'(t) \neq 0$ since P is separable.

(4) $\Rightarrow$ (3): $\Omega_{A|k} = 0$ implies that $\Omega_{A\otimes_k \overline{k}|\overline{k}} = \Omega_{A|k} \otimes_k \overline{k} = 0$. So, one may assume that $k = \overline{k}$ is algebraically closed. Since A is artinian any prime ideal is maximal and $|\mathrm{spec}(A)| < +\infty$. Write $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ for the finitely many prime (=maximal) ideals of A. Then, by the Chinese remainder theorem, one has the short exact sequence of A-modules:

$$0 \to \sqrt{0} \to A \xrightarrow{\phi} \prod_{i=1}^{r} A/\mathfrak{m}_i \to 1.$$

As $[A/\mathfrak{m}_i : k] < +\infty$ and k is algebraically closed, one actually has $A/\mathfrak{m}_i = k$, $i = 1, \ldots, r$. Let $e_i \in A$, $i = 1, \ldots r$ such that (i) $\phi(e_i) = (\delta_{i,j})_{1\leq j\leq r}$, $i = 1, \ldots, r$, (ii) $e_i e_j \in (\sqrt{0})^2$, $1 \leq i \neq j \leq r$ and (iii) $e_i - e_i^2 \in (\sqrt{0})^2$, $i = 1, \ldots, r$. Such a r-tuple can always be constructed. Indeed, start from $e_i \in A$, $i = 1, \ldots, r$ satisfying (i); then the e_i^2, $i = 1, \ldots, r$ satisfy (i) and (ii). Also, as A is artinian and thus, for all $i = 1, \ldots, r$ the chain of ideals:

$$\langle e_i \rangle \supset \langle e_i^2 \rangle \supset \cdots$$

stabilizes, we can find $n \geq 1$ and $a_i \in A$ such that for all $i = 1, \ldots, r$ one has:

$$a_i e_i^{2n} = e_i^n.$$

We set $\epsilon_i := (a_i e_i^n)^2 (= a_i e_i^n)$. Then $\phi(\epsilon_i) = \delta_{ij}$, $\epsilon_i \epsilon_j \in (\sqrt{0})^2$ for $1 \leq i \neq j \leq r$ and:

$$\epsilon_i^2 = (a_i e_i^n)^2 = a_i (a_i e_i^{2n}) = a_i e_i^n = \epsilon_i.$$

Hence the ϵ_i, $i = 1, \ldots, r$ satisfy (i), (ii), (iii). Let $\lambda_i : A \to A/\mathfrak{m}_i$ denote the ith component of ϕ and, for every $a \in A$, define $\lambda(a) := \sum_{i=1}^r \lambda_i(a) e_i$. Then, by definition, $a - \underline{\lambda}(a) \in \sqrt{0}$, $a \in A$ and one can check that the following map:

$$\begin{array}{rccl} d: & A & \to & \sqrt{0}/(\sqrt{0})^2 \\ & a & \to & (a - \underline{\lambda}(a)) \bmod (\sqrt{0})^2 \end{array}$$

defines a k-derivation hence is 0 by assumption, which forces $\sqrt{0} = (\sqrt{0})^2$. But, as A is an artinian, $\sqrt{0}$ is nilpotent hence $\sqrt{0} = (\sqrt{0})^2$ implies $\sqrt{0} = 0$ that is $A = \overline{A}$. □

A finite-dimensional algebra A over a field k satisfying the equivalent properties of Lemma 5.1 is said to be *étale over* k. We will write $FEAlg/k \subset Alg/k$ for the full subcategory of finite étale algebras over k.

5.2. Etale covers

Let Sch denote the category of schemes and, given a scheme S, let Sch/S denote the category of S-schemes.

Given a scheme S, we will write $\mathcal{O}_S$ for its structural sheaf and, given a point $s \in S$, we will write $\mathcal{O}_{S,s}$, $\mathfrak{m}_s$ and $k(s)$ for the local ring, maximal ideal

and residue field at s respectively. Also, we will write $\overline{s}$ for any geometric point associated with s, that is any morphism $\overline{s} : \operatorname{spec}(\Omega) \to S$ with image s and such that Ω is an algebraically closed field.

A morphism $\phi : X \to S$ that is locally of finite type is *unramified at* $x \in X$ if $\mathfrak{m}_{\phi(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$ and $k(x)$ is a finite separable extension of $k(\phi(x))$ (or, equivalently, if $\mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,\phi(x)}} k(\phi(x))$ is a finite separable field extension of $k(s)$) and it is *unramified* if it is unramified at all $x \in X$. A morphism $\phi : X \to S$ that is locally of finite type is *étale at* $x \in X$ if $\phi : X \to S$ is both flat and unramified at $x \in X$ and it is *étale* if it is étale at all $x \in X$. A morphism $\phi : X \to S$ is an *étale cover of* S if it is finite, surjective and étale.

We will often use the following characterization of finite flat morphisms and finite unramified morphisms respectively. Recall that, given a finite morphism $\phi : X \to S$, the $\mathcal{O}_S$-module $\phi_*\mathcal{O}_X$ is coherent.

Lemma 5.2. *Let $\phi : X \to S$ be a finite morphism. Then,*

(1) *$\phi : X \to S$ is flat if and only if $\phi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module;*
(2) *The following properties are equivalent:*
   (a) *$\phi : X \to S$ is unramified;*
   (b) *$\Omega^1_{X|S} = 0$;*
   (c) *$\Delta_{X|S} : X \to X \times_S X$ is an open immersion (hence induces an isomorphism onto an open and closed subscheme of $X \times_S X$).*
   (d) *$(\phi_*\mathcal{O}_X)_s \otimes_{\mathcal{O}_{S,s}} \kappa(s) = \mathcal{O}_{X_s}(X_s)$ is a finite étale algebra over $\kappa(s)$, $s \in S$;*

Proof. (1) As the question is local on X we may assume that $\phi : X \to S$ is induced by a finite, flat A-algebra $\phi^\# : A \to B$ with A noetherian. Then B is a flat A-module if and only if $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module, $\mathfrak{p} \in S$. But as $A_\mathfrak{p}$ is a local noetherian ring and $B_\mathfrak{p}$ is a finitely generated $A_\mathfrak{p}$-module, $B_\mathfrak{p}$ is a flat $A_\mathfrak{p}$-module if and only if $B_\mathfrak{p}$ is a free $A_\mathfrak{p}$-module. To conclude, for each $\mathfrak{p} \in S$, write:

$$B_\mathfrak{p} = \bigoplus_{i=1}^{r} A_\mathfrak{p} \frac{b_i}{s},$$

where $s \in A \setminus \mathfrak{p}$. This defines an exact sequence of A_s-modules:

$$0 \to K \to A_s^r \xrightarrow{(\frac{b_1}{s}, \ldots, \frac{b_r}{s})} B_s \to Q \to 0.$$

As A_s is noetherian, K is a finitely generated A_s-module hence its support $\operatorname{supp}(K)$ is the closed subset $V(\operatorname{Ann}(K)) \subset \operatorname{spec}(A_s)$. Similarly, as B_s is a finitely generated A_s-module, Q is a finitely generated A_s-module as well hence with closed support $\operatorname{supp}(Q) = V(\operatorname{Ann}(Q)) \subset \operatorname{spec}(A_s)$. But, by definition of the support, $U_\mathfrak{p} := \operatorname{supp}(K) \cap \operatorname{supp}(Q)$ is an open neighborhood of $\mathfrak{p}$ in S such that:

$$\phi_*\mathcal{O}_X|_{U_\mathfrak{p}} \simeq \mathcal{O}_{U_\mathfrak{p}}.$$

This shows that if $\phi : X \to S$ is flat then $\phi_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module. The converse implication is straightforward.

(2) We prove (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b): Since $\Omega^1_{X|S} = 0$ if and only if $\Omega_{X|S,x} = 0$, $x \in X$, one may again assume that $\phi : X \to S$ is induced by a finite A-algebra $\phi^\# : A \to B$ with A noetherian. Also, as $\Omega^1_{B|A}$ is a finitely generated B-module, by the Nakayama lemma, it is enough to show that:

$$\Omega^1_{B|A} \otimes_B k(\mathfrak{q}) = 0,\ \mathfrak{q} \in X.$$

But it follows from the fact that $f : X \to S$ is unramified that for any $\mathfrak{q} \in X$ above $\mathfrak{p} \in S$ one has:

$$B_\mathfrak{q} \otimes_{A_\mathfrak{p}} k(\mathfrak{p}) = k(\mathfrak{q}).$$

Whence:

$$\begin{aligned} \Omega^1_{B|A} \otimes_B k(\mathfrak{q}) &= \Omega^1_{B|A} \otimes_A k(\mathfrak{p}) \\ &= \Omega_{B \otimes_A k(\mathfrak{p})|k(\mathfrak{p})} \\ &= \Omega_{k(\mathfrak{q})|k(\mathfrak{p})} \\ &= 0, \end{aligned}$$

where the last equality follows from the fact that $k(\mathfrak{p}) \hookrightarrow k(\mathfrak{q})$ is a finite separable field extension.

(b) $\Rightarrow$ (c): As $\phi : X \to S$ is separated, the diagonal morphism $\Delta_{X|S} : X \to X \times_S X$ is a closed immersion and, in particular:

$$\Delta_{X|S}(X) = \mathrm{supp}(\Delta_{X|S*}\mathcal{O}_X).$$

Let:

$$\mathcal{I} := \mathrm{Ker}(\Delta^\#_{X|S} : \mathcal{O}_{X \times_S X} \to (\Delta_{X|S})_*\mathcal{O}_X) \subset \mathcal{O}_{X \times_S X}$$

denote the corresponding sheaf of ideals. By assumption $\Omega^1_{X|S} = 0 = \Delta^*_{X|S}(\mathcal{I}/\mathcal{I}^2)$. In particular,

$$\mathcal{I}_{\Delta_{X|S}(x)}/\mathcal{I}^2_{\Delta_{X|S}(x)} = (\Delta^*_{X|S}(\mathcal{I}/\mathcal{I}^2))_x = 0,\ x \in X$$

or, equivalently, $\mathcal{I}_{\Delta_{X|S}(x)} = \mathcal{I}^2_{\Delta_{X|S}(x)}$, $x \in X$. But, as S is noetherian and $\phi : X \to S$ is finite, X is noetherian hence $\mathcal{I}$ is coherent. So, by Nakayama,

$$\mathcal{I}_{\Delta_{X|S}(x)} = \mathcal{I}^2_{\Delta_{X|S}(x)},\ x \in X$$

forces

$$\mathcal{I}_{\Delta_{X|S}(x)} = 0,\ x \in X.$$

Thus $\Delta_{X|S}(X)$ is contained in the open subset $U := X \times_S X \setminus \mathrm{supp}(\mathcal{I})$. On the other hand, for all $u \in U$, the morphism induced on stalks:

$$\Delta^\#_{X|S,u} : \mathcal{O}_{X \times_S X,u} \tilde{\to} (\Delta_{X|S*}\mathcal{O}_X)_u$$

is an isomorphism. So U is contained in $\mathrm{supp}(\Delta_{X|S*}\mathcal{O}_X) = \Delta_{X|S}(X)$ hence $\Delta_{X|S}(X) = U$ and $\Delta_{X|S} : X \hookrightarrow X \times_S X$ is an open immersion.

(c) $\Rightarrow$ (d): For any geometric points $\overline{s} : \mathrm{spec}(\Omega) \to S$ and $\overline{x} : \mathrm{spec}(\Omega) \to X_{\overline{s}}$, consider the cartesian diagram:

$$\begin{tikzcd}
X \arrow[d, "\Delta_{X|S}"'] \arrow[dr, phantom, "\square"] & X_{\overline{s}} \arrow[l] \arrow[d, "\Delta_{X_{\overline{s}}|\Omega}"] \arrow[dr, phantom, "\square"] & \mathrm{spec}(\Omega) \arrow[l, "\overline{x}"'] \arrow[d, "(Id\times\overline{x})"] \\
X \times_S X & X_{\overline{s}} \times_\Omega X_{\overline{s}} \arrow[l] & \mathrm{spec}(\Omega) \times_\Omega X_{\overline{s}}. \arrow[l, "(\overline{x}\times Id)"]
\end{tikzcd}$$

Since open immersions are stable under base changes, $\overline{x} : \mathrm{spec}(\Omega) \to X_{\overline{s}}$ is again an open immersion hence induces an isomorphism onto a closed and open subscheme of $X_{\overline{s}}$ that is, since $\mathrm{spec}(\Omega)$ is connected and $X_{\overline{s}}$ is finite, a connected component of $X_{\overline{s}}$. As a result,

$$X_{\overline{s}} = \coprod_{\overline{x}:\mathrm{spec}(\Omega)\to X_{\overline{s}}} \mathrm{spec}(\Omega)$$

is a coproduct of $|X_{\overline{s}}|$ copies of $\mathrm{spec}(\Omega)$.

(d) $\Rightarrow$ (a): As the question is local on X, we may assume, one more time, that $\phi : X \to S$ is induced by a finite A-algebra $\phi^\# : A \to B$ with A noetherian. By assumption,

$$B \otimes_A k(\mathfrak{p}) = \prod_{1\le i\le n} k_i$$

is, as a $k(\mathfrak{p})$-algebra, the product of finitely many finite separable field extensions of $k(\mathfrak{p})$. In particular, any ideal in $\mathrm{spec}(B \otimes_A k(\mathfrak{p}))$ is maximal and equal to one of the:

$$\mathfrak{m}_j := \ker(\prod_{1\le i\le n} k_i \twoheadrightarrow k_j),\ j = 1, \ldots, n.$$

But, then, for any $\mathfrak{q} \in X$ above $\mathfrak{p} \in S$ whose image in $\mathrm{spec}(B \otimes_A k(\mathfrak{p}))$ is $\mathfrak{m}_j$ for some $1 \le j \le n$, one has:

$$B_{\mathfrak{q}} \otimes_{A_{\mathfrak{p}}} k(\mathfrak{p}) = (B \otimes_A k(\mathfrak{p}))_{\mathfrak{m}_j} = k_j,$$

which, by assumption, is a finite separable field extension of $k(\mathfrak{p})$. $\square$

Remark 5.3. The equivalences (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) also hold for morphisms which are locally of finite type.

Example 5.4. Assume that $S = \mathrm{spec}(A)$ is affine and let $P \in A[T]$ be a monic polynomial such that $P' \neq 0$. Set $B := A[T]/PA[T]$ and $C := B_b$ where $b \in B$ is such that $P'(t)$ becomes invertible in B_b (here t denotes the image of T in B). Then $\mathrm{spec}(C) \to S$ is an étale morphism. Such morphisms are called *standard étale morphisms.*

Actually, any étale morphism is locally of this type.

Theorem 5.5. (Local structure of étale morphisms) *Let A be a noetherian local ring and set $S = \mathrm{spec}(A)$. Let $\phi : X \to S$ an unramified* (*resp. étale*) *morphism.*

Then, for any $x \in X$, there exists an open neighborhood U of x such that one has a factorization:

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "\phi"'] & \mathrm{spec}(C), \arrow[dl] \\ S & \end{tikzcd}$$

where $\mathrm{spec}(C) \to S$ *is a standard étale morphism and* $U \hookrightarrow \mathrm{spec}(C)$ *is an immersion* (*resp. an open immersion*).

Proof. See [Mi80, Thm. 3.14 and Rem. 3.15]. □

For any étale cover $\phi : X \to S$, the rank function:

$$\begin{array}{rcl} r_-(\phi) : S & \to & \mathbb{Z}_{\geq 0} \\ s & \mapsto & r_s(\phi) := \mathrm{rank}_{\mathcal{O}_{S,s}}((\phi_* \mathcal{O}_X)_s) = \mathrm{rank}_{k(s)}(\mathcal{O}_{X_s}(X_s)) \\ & & \qquad = \dim_{\overline{k(s)}}(\mathcal{O}_{X_s}(X_s) \otimes_{k(s)} \overline{k(s)}) = |X_{\overline{s}}| \end{array}$$

is locally constant hence constant, since S is connected; we say that $r(\phi)$ is the *rank of* $\phi : X \to S$.

Eventually, let us recall the following two standard lemmas.

Lemma 5.6. (Stability) *If P is a property of morphisms of schemes which is* (i) *stable under composition and* (ii) *stable under arbitrary base-change then* (iv) *P is stable by fibre products. If furthermore* (iii) *closed immersions have P then,* (v) *for any $X \xrightarrow{f} Y \xrightarrow{g} Z$, if g is separated and $g \circ f$ has P then f has P.*

The properties $P =$ surjective, flat, unramified, étale satisfy (i) and (ii) hence (iv). The properties $P =$ separated, proper, finite satisfy (i), (ii), (iii) hence (iv) and (v).

Lemma 5.7. (Topological properties of finite morphisms)

(1) *A finite morphism is closed;*
(2) *A finite flat morphism is open.*

Remark 5.8.

(1) Since being finite is stable under base-change, Lemma 5.7 (1) shows that a finite morphism is universally closed. Since finite morphisms are affine hence separated, this shows that finite morphisms are proper.
(2) Lemma 5.7 (2) also hold for flat morphisms which are locally of finite type.

Corollary 5.9. *Let S be a connected scheme. Then any finite étale morphism $\phi : X \to S$ is automatically an étale cover. Furthermore, $\phi : X \tilde{\to} S$ is an isomorphism if and only if $r(\phi) = 1$.*

Proof. From Lemma 5.7, the set $\phi(X)$ is both open and closed in S, which is connected. Hence $\phi(X) = S$. As for the second part of the assertion, the "if"

implication is straightforward so we are only to prove the "only if" part. The condition $r(\phi) = 1$ already implies that $\phi : X \to S$ is bijective. But as $\phi : X \to S$ is continuous and, by Lemma 5.7 (2), open, it is automatically an homeomorphism. So $\phi : X \to S$ is an isomorphism if and only if $\phi_s^{\#} : \mathcal{O}_{S,s} \tilde{\to} (\phi_* \mathcal{O}_X)_s$ is an isomorphism, $s \in S$. This amounts to showing that any finite, faithfully flat A-algebra $A \hookrightarrow B$ such that $B = Ab$ as A-module is surjective that is $b \in A$. By assumption, there exists $a \in A$ such that $ab = 1$ and, as B is finite over A, there exists a monic polynomial $P_b = T^d + \sum_{i=0}^{d-1} r_i T^i \in A[T]$ such that $P_b(b) = 0$ hence, multiplying this equality by a^{d-1}, one gets $b = -\sum_{i=0}^{d-1} r_i a^{d-1-i} \in A$. $\square$

5.3. The category of étale covers of a connected scheme

5.3.1. Statement of the main theorem. Let S be a connected scheme and denote by $\mathcal{C}_S \subset Sch/S$ the full subcategory whose objects are étale covers of S.

Given a geometric point $\overline{s} : \mathrm{spec}(\Omega) \to S$, the underlying set associated to the scheme $X_{\overline{s}} := X \times_{\phi,S,\overline{s}} \mathrm{spec}(\Omega)$ will be denoted by $X_{\overline{s}}^{set}$. One thus obtains a functor:

$$\begin{array}{rlcl} F_{\overline{s}} : & \mathcal{C}_S & \to & FSets \\ & \phi : X \to S & \to & X_{\overline{s}}^{set}. \end{array}$$

Theorem 5.10. *The category of étale covers of S is Galois. And for any geometric point $\overline{s} : \mathrm{spec}(\Omega) \to S$, the functor $F_{\overline{s}} : \mathcal{C}_S \to FSets$ is a fibre functor for $\mathcal{C}_S$.*

Remark 5.11. For any geometric point $s : \mathrm{spec}(\Omega) \to S$, the functor $F_{\overline{s}} : \mathcal{C}_S \to FSets$ is a fibre functor for $\mathcal{C}_S$ but all fibre functors are not necessarily of this form. For instance, given an algebraically closed field Ω and a morphism $f : \mathbb{P}^1_\Omega \to S$ then the functor:

$$\begin{array}{rlcl} F_f : & \mathcal{C}_S & \to & FSets \\ & \phi : X \to S & \to & \pi_0(X \times_{\phi,S,f} \mathbb{P}^1_\Omega) \end{array}$$

is also a fibre functor for $\mathcal{C}_S$.

By analogy with topology, for any geometric point $\overline{s} : \mathrm{spec}(\Omega) \to S$, the profinite group:

$$\pi_1(S; \overline{s}) := \pi_1(\mathcal{C}_S; F_{\overline{s}})$$

is called the *étale fundamental group of S with base point $\overline{s}$*. Similarly, for any two geometric points $\overline{s}_i : \mathrm{spec}(\Omega_i) \to S$, $i = 1, 2$, the set:

$$\pi_1(S; \overline{s}_1, \overline{s}_2) := \pi_1(\mathcal{C}_S; F_{\overline{s}_1}, F_{\overline{s}_2})$$

is called the set of *étale paths from $\overline{s}_1$ to $\overline{s}_2$*. (Note that Ω_1 and Ω_2 may have different characteristics.)

From Theorem 2.8, the set of étale paths $\pi_1(S; \overline{s}_1, \overline{s}_2)$ from $\overline{s}_1$ to $\overline{s}_2$ is non-empty and the profinite group $\pi_1(S; \overline{s}_1)$ is noncanonically isomorphic to $\pi_1(S; \overline{s}_2)$ with an isomorphism that is canonical up to inner automorphisms.

Eventually, given a morphism $f : S' \to S$ of connected schemes and a geometric point $\overline{s}' : \mathrm{spec}(\Omega) \to S'$, the universal property of fibre product implies

that the base change functor $f^* : \mathcal{C}_S \to \mathcal{C}_{S'}$ satisfies $F_{\overline{s}'} \circ f^* = F_{f(\overline{s}')}$. Hence $f^* : \mathcal{C}_S \to \mathcal{C}_{S'}$ is a fundamental functor and one gets, correspondingly, a morphism of profinite groups:

$$\pi_1(f) : \pi_1(S'; \overline{s}') \to \pi_1(S; \overline{s}),$$

whose properties can be read out of those of $f : S' \to S$ using the results of Subsection 4.2.

5.3.2. Proof. We check axioms (1) to (6) of the definition of a Galois category.
Axiom (1): The category of étale covers of S has a final object: $Id_S : S \to S$ and, from Lemma 5.6, the fibre product (in the category of S-schemes) of any two étale covers of S over a third one is again an étale cover of S.
Axiom (2): The category of étale covers of S has an initial object: $\emptyset$ and the coproduct (in the category of S-schemes) of two étale covers of S is again an étale cover of S. A more delicate point is:

Lemma 5.12. *Categorical quotients by finite groups of automorphisms exist in $\mathcal{C}_S$.*

Proof of the lemma. Let $\phi : X \to S$ be an étale cover and let $G \subset \mathrm{Aut}_{Sch/S}(\phi)$ be a finite subgroup.
Step 1: Assume first that $S = \mathrm{spec}(A)$ is an affine scheme. Since étale cover are, in particular, finite hence affine morphisms, $\phi : X \to S$ is induced by a finite A-algebra $\phi^\# : A \to B$. But, then, it follows from the equivalence of category between the category of affine S-schemes and $(Alg/A)^{op}$ that the factorization

$$\begin{tikzcd} X \arrow[r, "p_G"] \arrow[d, "\phi"'] & \mathrm{spec}(B^{G^{op}}) =: G \setminus X \arrow[dl, "\phi_G"] \\ S & \end{tikzcd}$$

is the categorical quotient of $\phi : X \to S$ by G in the category of affine S-schemes. So, as $\mathcal{C}_S$ is a full subcategory of the category of affine S-schemes, to prove that $\phi_G : G \setminus X \to S$ is the categorical quotient of $\phi : X \to S$ by G in $\mathcal{C}_S$ it only remains to prove that $\phi_G : G \setminus X \to S$ is in $\mathcal{C}_S$.
Step 1-1 (trivialization): *An affine, surjective morphism $\phi : X \to S$ is an étale cover of S if and only if there exists a finite faithfully flat morphism $f : S' \to S$ such that the first projection $\phi' : X' := S' \times_{f,S,\phi} X \to S'$ is a totally split étale cover of S'.*

In other words, an affine surjective morphism $\phi : X \to S$ is an étale cover if and only if it is locally trivial for the Grothendieck topology whose covering families are finite, faithfully flat morphisms.

Proof. We first prove the "only if" implication. As $f : S' \to S$ is finite and faithfully flat, it follows from Lemma 5.2 (1) that for any $s \in S$ there exists an open affine neighborhood $U = \mathrm{spec}(A)$ of s such that $f|^U_{f^{-1}(U)} : f^{-1}(U) \to U$ is induced by a finite A-algebra $f^\# : A \hookrightarrow A'$ with $A' = A^r$. Also, as $\phi : X \to S$ is affine and surjective, $\phi|^U_{\phi^{-1}(U)} : \phi^{-1}(U) \to U$ corresponds to a A-algebra $\phi^\# : A \hookrightarrow B$. By

assumption $B \otimes_A A' = A'^s$ as A'-algebras hence $B \otimes_A A' = A^{rs}$ as A-modules. But, on the other hand, $B \otimes_A A' = B \otimes_A A^r = B^r$ as B-modules hence as A-modules. In particular, B is a direct factor of A^{rs} as A-module hence is flat over A. This shows that $\phi : X \to S$ is flat. Also, as B is a submodule of the finitely generated A-module A^{rs} and A is noetherian, B is also a finitely generated A-module. This shows that $\phi : X \to S$ is finite. With the notation:

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "\phi'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\phi"] \\ S' \arrow[r, "f"'] & S, \end{tikzcd}$$

it follows from Lemma 5.2 (2) (c) that $f'^*\Omega_{X|S} = \Omega_{X'|S'} = 0$ that is, $(f'^*\Omega_{X|S})_{x'} = \Omega_{X|S,f'(x')} = 0$, $x' \in X'$. But $f' : X' \to X$ is the base change of the surjective morphism $f : S' \to S$ hence it is surjective as well, which implies $\Omega_{X|S} = 0$. This shows that $\phi : X \to S$ is finite étale.

We now prove the "if" implication by induction on $r(\phi) \geq 1$. If $r(\phi) = 1$ it follows from Corollary 5.9 that $\phi : X \xrightarrow{\sim} S$ is an isomorphism and the statement is straightforward with $f = Id_S$. If $r(\phi) > 1$, from Lemma 5.2 (2) (d), the diagonal morphism $\Delta_{X|S} : X \hookrightarrow X \times_S X$ is both a closed and open immersion hence $X \times_S X$ can be written as a coproduct $X \sqcup X'$, where $\Delta_{X|S}(X)$ is identified with X and $X' := X \times_S X \setminus \Delta_{X|S}(X)$. In particular, $i_{X'} : X' \hookrightarrow X \times_S X$ is both a closed and open immersion as well hence a finite étale morphism. Also, as $\phi : X \to S$ is finite étale, its base change $p_1 : X \times_{\phi,S,\phi} X \to X$ is finite étale as well so the composite $\phi' : X' \xrightarrow{i_{X'}} X \times_S X \xrightarrow{p_1} X$ is finite étale. But as $\Delta_{X|S} : X \hookrightarrow X \times_S X$ is a section of $p_1 : X \times_S X \to X$, one has: $r(\phi') = r(p_1) - 1 = r(\phi) - 1$. So, by induction hypothesis, there exists a finite faithfully flat morphism $f : S' \to X$ such that $S' \times_{f,X,\phi'} X' \to S'$ is a totally split étale cover of S'. But, then, the composite $\phi \circ f : S' \to S$ is also finite and faithfully flat. Hence the conclusion follows from the formal computation based on elementary properties of fibre product of schemes:

$$\begin{aligned} S' \times_{\phi \circ f,S,\phi} X &= S' \times_{f,X,p_1} (X \times_S X) = S' \times_{f,X,p_1} (X \sqcup X') \\ &= (S' \times_{f,X,p_1} X) \sqcup (S' \times_{f,X,p_1} X'). \end{aligned} \qquad \square$$

<u>Step 1-2:</u> We want to apply step 1-1 to the quotient morphism $\phi_G : G \setminus X \to S$. For this, apply first step 1-1 to the étale cover $\phi : X \to S$ to obtain a faithfully flat A-algebra $A \to A'$ such that $B \otimes_A A' = A'^n$ as A'-algebras. Tensoring the exact sequence of A-algebras:

$$0 \to B^{G^{op}} \to B \xrightarrow{\sum_{g \in G^{op}} (Id_B - g\cdot)} \bigoplus_{g \in G^{op}} B$$

by the flat A-algebra A', one gets the exact sequence of B'-algebras:

$$0 \to B^{G^{op}} \otimes_A A' \to B \otimes_A A' \xrightarrow{\sum_{g \in G^{op}} (Id_B - g\cdot) \otimes_A Id_{A'}} \bigoplus_{g \in G} B \otimes_A A',$$

whence:

$$(*) \quad B^{G^{op}} \otimes_A A' = (B \otimes_A A')^{G^{op}} = (A'^n)^{G^{op}}.$$

But G^{op} is a subgroup of $\mathrm{Aut}_{Alg/A'}(A'^n)$, which is nothing but the symmetric group $\mathcal{S}_n$ acting on the canonical coordinates $E := \{1, \ldots, n\}$ in A'^n. Hence:

$$(A'^E)^{G^{op}} = \bigoplus_{G \backslash E} A'.$$

In terms of schemes, if $f : S' \to S$ denotes the faithfully flat morphism corresponding to $A \hookrightarrow A'$ then $S' \times_{f,S,\phi} X$ is just the coproduct of n copies of S' over which G acts by permutation and $(*)$ becomes:

$$S' \times_{f,S,\phi_G} (G \backslash X) = G \backslash \left(\bigsqcup_E S' \right) = \bigsqcup_{G \backslash E} S'.$$

Step 2: Reduce to step 1 by covering S with affine open subschemes (local existence) and using the unicity of categorical quotient up to canonical isomorphism (gluing). □

Remark 5.13. One can actually show that, in the affine case, $G \backslash X = \mathrm{spec}(B^{G^{op}})$ is actually the categorical quotient of $\phi : X \to S$ by G is the category of all S-schemes (*cf.* [MumF82, Prop. 0.1]).

Exercise 5.14. Show that categorical quotients of étale covers by finite groups of automorphisms commute with arbitrary base-changes.

Axiom (3): Before dealing with axiom (3), let us recall that, in the category of S-schemes, open immersions are monomorphisms and that:

Theorem 5.15. (Grothendieck – see [Mi80, Thm. 2.17]) *In the category of S-schemes, faithfully flat morphisms of finite type are strict epimorphisms.*

Lemma 5.16. *Given a commutative diagram of schemes:*

$$\begin{tikzcd} Y \arrow[r, "u"] \arrow[d, "\psi"'] & X \arrow[dl, "\phi"] \\ S, & \end{tikzcd}$$

if $\phi : X \to S$, $\psi : Y \to S$ are finite étale morphisms then $u : Y \to X$ is a finite étale morphism as well.

Proof of the lemma. Write $u = p_2 \circ \Gamma_u$, where $\Gamma_u : Y \to Y \times_S X$ is the graph of u, identified with the base-change:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "\Gamma_u"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "\Delta_{X|S}"] \\ Y \times_S X \arrow[r, "u \times_S Id_X"'] & X \times_S X \end{tikzcd}$$

and $p_2 : Y \times_S X \to X$ is the base-change defined by:

$$\begin{tikzcd} Y \times_S X \arrow[r] \arrow[d, "p_2"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "\psi"] \\ X \arrow[r, "\phi"'] & S. \end{tikzcd}$$

From Lemma 5.2 (2) (d), the diagonal morphism $\Delta_{X|S} : X \to X \times_S X$ is finite étale hence it follows from the first part of Lemma 5.6 that $\Gamma_u : Y \to Y \times_S X$ is finite, étale as well. Similarly, as $\psi : Y \to S$ is finite étale, $p_2 : Y \times_S X \to X$ is finite étale as well. Hence, the conclusion follows from the second part of Lemma 5.6. $\square$

For any two étale covers $\phi : X \to S$, $\psi : Y \to S$ and for any morphism $u : X \to Y$ over S, it follows from Lemma 5.16 that $u : Y \to X$ is a finite, étale morphism hence is both open (flatness) and closed (finite). In particular, one can write X as a coproduct $X = X' \sqcup X''$, where $X' := u(Y)$, $X'' := X \setminus X'$ are both open and closed in X and u factors as $u : Y \overset{u|^{X'}=u'}{\to} X' \overset{i'_{X'}=u''}{\to} X = X' \sqcup X''$ with u' a faithfully flat morphism hence a strict epimorphism in $R_S^{\text{ét}}$ and u'' an open immersion hence a monomorphism in $\mathcal{C}_S$. $\square$

<u>Axiom (4):</u> For any étale cover $\phi : X \to S$ one has $F_{\overline{s}}(\phi) = *$ if and only if $r(\phi) = 1$, which, in turn, is equivalent to $\phi : X \tilde{\to} S$. Also, it follows straightforwardly from the universal property of fibre product and the definition of $F_{\overline{s}}$ that $F_{\overline{s}}$ commutes with fibre products.

<u>Axiom (5):</u> The fact that $F_{\overline{s}}$ commutes with finite coproducts and transforms strict epimorphisms into strict epimorphisms is straightforward. So it only remains to prove that $F_{\overline{s}}$ commutes with categorical quotients by finite groups of automorphisms. Let $\phi : X \to S$ be an étale cover and $G \subset \mathrm{Aut}_{Sch/S}(\phi)$ a finite subgroup. Since the assertion is local on S, it follows from step 1-1 in axiom (2) that we may assume that $\phi : X \to S$ is totally split and that G acts on X by permuting the copies of S. But, then, the assertion is immediate since $G \setminus X = \bigsqcup_{G \setminus F_{\overline{s}}(\phi)} S$.

<u>Axiom (6):</u> For any two étale covers $\phi : X \to S$, $\psi : Y \to S$ let $u : X \to Y$ be a morphism over S such that $F_{\overline{s}}(u) : F_{\overline{s}}(\psi) \tilde{\to} F_{\overline{s}}(\phi)$ is bijective. It follows from Lemma 5.16 that $u : Y \to X$ is finite étale but, by assumption, it is also surjective hence $u : Y \to X$ is an étale cover. Moreover, still by assumption, it has rank 1 hence it is an isomorphism by Corollary 5.9. $\square$

6. Examples

Given a scheme X over an affine scheme $\mathrm{spec}(A)$, we will write $X \to A$ instead of $X \to \mathrm{spec}(A)$ for the structural morphism and given a A-algebra $A \to B$, we will write X_B for $X \times_A \mathrm{spec}(B)$. Similarly, given a morphism $f : X \to Y$ of schemes over $\mathrm{spec}(A)$, we will write $f_B : X_B \to Y_B$ for its base-change by $\mathrm{spec}(B) \to \mathrm{spec}(A)$. Also, given a morphism $f : Y \to X$ and a morphism $X \to X'$ we will often say that $f' : Y' \to X'$ is a model of $f : Y \to X$ over X' if there is a cartesian square:

$$\begin{tikzcd} Y \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "f'"] \\ X \arrow[r] & X'. \end{tikzcd}$$

6.1. Spectrum of a field

Let k be a field, $k \hookrightarrow \overline{k}$ a fixed algebraic closure of k and $k^s \subset \overline{k}$ the separable closure of k in $\overline{k}$; write $\Gamma_k := \mathrm{Aut}_{Alg/k}(k^s)$ for the absolute Galois group of k. Set $S := \mathrm{spec}(k)$. Then the datum of $k \hookrightarrow \overline{k}$ defines a geometric point $\overline{s} : \mathrm{spec}(\overline{k}) \to S$ and:

Proposition 6.1. *There is a canonical isomorphism of profinite groups:*

$$c_{\overline{s}} : \pi_1(S; \overline{s}) \tilde{\to} \Gamma_k.$$

Proof. The Galois objects in $\mathcal{C}_S$ are the $\mathrm{spec}(K) \to S$ induced by finite Galois field extensions $k \hookrightarrow K$; write $\mathcal{G}_S \subset \mathcal{C}_S$ for the full subcategory of Galois objects. The datum of $k \hookrightarrow \overline{k}$ allows us to identify k with a subfield of $\overline{k}$ and define a canonical section of the forgetful functor: $For : \mathcal{G}_S^{pt} \to \mathcal{G}_S$ by associating to each Galois object $\mathrm{spec}(K) \to S$ its isomorphic copy $\mathrm{spec}(K_\Omega) \to S$, where K_Ω is the unique subfield of $\overline{k}$ containing k and isomorphic to K as k-algebra. Then, on the one hand, the restriction morphisms $|_{K_\Omega} : \Gamma_k \to \mathrm{Aut}_{Alg/k}(K_\Omega)$ induce an isomorphism of profinite groups:

$$\Gamma_k \tilde{\to} \varprojlim_{K_\Omega} \mathrm{Aut}_{Alg/k}(K_\Omega).$$

And, on the other hand, by the equivalence of categories:

$$\begin{array}{ccc} \mathcal{C}_S & \to & (FEAlg/k)^{op} \\ \phi : X \to S & \mapsto & \phi^{\#}(X) : k \hookrightarrow \mathcal{O}_X(X) \end{array}$$

one can identify:

$$\mathrm{Aut}_{Alg/k}(K_\Omega) = \mathrm{Aut}_{\mathcal{C}_S}(\mathrm{spec}(K_\Omega))^{op}.$$

But then, from Proposition 3.9, one also has the canonical evaluation isomorphism of profinite groups:

$$\pi_1(S; \overline{s}) \tilde{\to} \varprojlim_{K_\Omega} \mathrm{Aut}_{\mathcal{C}_S}(\mathrm{spec}(K_\Omega))^{op},$$

which concludes the proof. $\square$

6.2. The first homotopy sequence and applications

6.2.1. Stein factorization. A scheme X over a field k is *separable over* k if, for any field extension K of k the scheme $X \times_k K$ is reduced. This is equivalent to requiring that X be reduced and that, for any generic point η of X, the extension $k \hookrightarrow k(\eta)$ be separable (recall that an arbitrary field extension $k \hookrightarrow K$ is separable if any finitely generated subextension admits a separating transcendence basis and that any field extension of a perfect field is separable). In particular, if k is perfect, this is equivalent to requiring that X be reduced. More generally, a scheme X over a scheme S is *separable over* S if it is flat over S and for any $s \in S$ the scheme X_s is separable over $k(s)$. Separable morphisms satisfy the following elementary properties:

– Any base change of a separable morphism is separable.
– If $X \to S$ is separable and $X' \to X$ is étale then $X' \to S$ is separable.

Theorem 6.2. (Stein factorization of a proper morphism) *Let $f : X \to S$ be a morphism such that $f_*\mathcal{O}_X$ is a quasicoherent $\mathcal{O}_S$-algebra. Then $f_*\mathcal{O}_X$ defines an S-scheme:*

$$p : S' = \operatorname{spec}(f_*\mathcal{O}_X) \to S$$

and $f : X \to S$ factors canonically as:

$$\begin{tikzcd} S & S' \arrow[l, "p"'] \\ X. \arrow[u, "f"] \arrow[ur, "f'"'] & \end{tikzcd}$$

Furthermore,

(1) *If $f : X \to S$ is proper then*

(a) *$p : S' \to S$ is finite and $f' : X \to S'$ is proper and with geometrically connected fibres;*

(b) *– The set of connected components of X_s is one-to-one with S'^{set}_s, $s \in S$;*
– The set of connected components of $X_{\overline{s}}$ is one-to-one with $S'^{set}_{\overline{s}}$, $s \in S$.
In particular, if $f_*\mathcal{O}_X = \mathcal{O}_S$ then $f : X \to S$ has geometrically connected fibres.

(2) *If $f : X \to S$ is proper and separable then $p : S' \to S$ is an étale cover. In particular, $f_*\mathcal{O}_X = \mathcal{O}_S$ if and only if $f : X \to S$ has geometrically connected fibres.*

Corollary 6.3. *Let $f : X \to S$ be a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_S$. Then, if S is connected, X is connected as well.*

Proof. It follows from (1) (b) of Theorem 6.2 that, if $f_*\mathcal{O}_X = \mathcal{O}_S$ then $f : X \to S$ is geometrically connected and, in particular, has connected fibres. But, as $f : X \to S$ is proper, it is closed and $f_*\mathcal{O}_X$ is coherent hence:

$$f(X) = \operatorname{supp}(f_*\mathcal{O}_X).$$

So $f_*\mathcal{O}_X = \mathcal{O}_S$ also implies that $f : X \to S$ is surjective. As a result, if $f_*\mathcal{O}_X = \mathcal{O}_S$ the morphism $f : X \to S$ is closed, surjective, with connected fibres so, if S is connected, this forces X to be connected as well. $\square$

6.2.2. The first homotopy sequence. Let S be a connected scheme, $f : X \to S$ a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_S$ and $s \in S$. Fix a geometric point $x_\Omega : \mathrm{spec}(\Omega) \to X_{\overline{s}}$ with image again denoted by x_Ω in X and s_Ω in S.

Theorem 6.4. (First homotopy sequence) *Consider the canonical sequence of profinite groups induced by* $(X_{\overline{s}}, x_\Omega) \to (X, x_\Omega) \to (S, s_\Omega)$*:*

$$\pi_1(X_{\overline{s}}; x_\Omega) \xrightarrow{i} \pi_1(X; x_\Omega) \xrightarrow{p} \pi_1(S; s_\Omega).$$

Then $p : \pi_1(X; x_\Omega) \twoheadrightarrow \pi_1(S; s_\Omega)$ *is an epimorphism and* $\mathrm{im}(i) \subset \ker(p)$*. If, furthermore,* $f : X \to S$ *is separable then* $\mathrm{im}(i) = \ker(p)$*.*

A first consequence of Theorem 6.4 is that the étale fundamental group of a connected, proper scheme over k is invariant by algebraically closed field extension. More precisely, let k be an algebraically closed field, X a scheme connected and proper over k and $k \hookrightarrow \Omega$ an algebraically closed field extension of k. Fix a geometric point $x_\Omega : \mathrm{spec}(\Omega) \to X_\Omega$ with image again denoted by x_Ω in X.

Corollary 6.5. *The canonical morphism of profinite groups:*

$$\pi_1(X_\Omega; x_\Omega) \xrightarrow{\sim} \pi_1(X; x_\Omega)$$

induced by $(X_\Omega; x_\Omega) \to (X; x_\Omega)$ *is an isomorphism.*

Proof. We first prove:

Lemma 6.6 (Product). *Let* k *be an algebraically closed field,* X *a connected, proper scheme over* k *and* Y *a connected scheme over* k*. For any* $x : \mathrm{spec}(k) \to X$ *and* $y : \mathrm{spec}(k) \to Y$*, the canonical morphism of profinite groups:*

$$\pi_1(X \times_k Y; (x, y)) \to \pi_1(X; x) \times \pi_1(Y; y)$$

induced by the projections $p_X : X \times_k Y \to X$ *and* $p_Y : X \times_k Y \to Y$ *is an isomorphism.*

Proof of the lemma. From Theorem A.2, one may assume that X is reduced hence, as k is algebraically closed, that X is separable over k. As X is proper, separable, geometrically connected and surjective over k, so is its base change $p_Y : X \times_k Y \to Y$. So, it follows from Theorem 6.2 (2) that $p_{Y*}\mathcal{O}_{X \times_k Y} = \mathcal{O}_Y$. Thus, one can apply Theorem 6.4 to $p_Y : X \times_k Y \to Y$ to get an exact sequence:

$$\pi_1((X \times_k Y)_y; x) \to \pi_1(X \times_k Y; (x, y)) \to \pi_1(Y; y) \to 1.$$

Furthermore, $X = (X \times_k Y)_y \to X \times_k Y \xrightarrow{p_X} X$ is the identity so $p_X : X \times_k Y \to X$ yields a section of $\pi_1(X; x) \to \pi_1(X \times_k Y; (x, y))$. $\square$

Note that if $y : \mathrm{spec}(\Omega) \to Y$ is any geometric point then the above only shows that $\pi_1(X \times_k Y; (x, y)) \xrightarrow{\sim} \pi_1(X_\Omega; x) \times \pi_1(Y; y)$.

Proof of Corollary 6.5. We apply the criterion of Proposition 4.3.

Surjectivity: Let $\phi : Y \to X$ be a connected étale cover. We are to prove that Y_Ω is again connected. But, as k is algebraically closed, if Y is connected then it is automatically geometrically connected over k and, in particular, Y_Ω is connected.

Injectivity: One has to prove that for any connected étale cover $\phi : Y \to X_\Omega$, there exists an étale cover $\tilde{\phi} : \tilde{Y} \to X$ which is a model of ϕ over X. We begin with a general lemma.

Lemma 6.7. *Let X be a connected scheme of finite type over a field k and let $k \hookrightarrow \Omega$ be a field extension of k. Then, for any étale cover $\phi : Y \to X_\Omega$, there exists a finitely generated k-algebra R contained in Ω and an affine morphism of finite type $\tilde{\phi} : \tilde{Y} \to X_R$ which is a model of $\phi : Y \to X_\Omega$ over X_R. Furthermore, if η denotes the generic point of* $\mathrm{spec}(R)$*, then $\tilde{\phi}_{k(\eta)} : \tilde{Y}_{k(\eta)} \to X_{k(\eta)}$ is an étale cover.*

Proof of the lemma. Since X is quasi-compact, there exists a finite covering of X by Zariski-open subschemes $X_i := \mathrm{spec}(A_i) \hookrightarrow X$, $i = 1, \ldots, n$, where the A_i are finitely generated k-algebra. As $\phi : Y \to X_\Omega$ is affine, we can write $U_i := \phi^{-1}(X_{i\Omega}) = \mathrm{spec}(B_i)$, where B_i is of the form:

$$B_i = A_i \otimes_k \Omega[\underline{T}]/\langle P_{i,1}, \ldots, P_{i,r_i}\rangle.$$

For each $1 \leq j \leq r_i$, the $\underline{\alpha}$th coefficient of $P_{i,j}$ is of the form:

$$\sum_k r_{i,j,\underline{\alpha},k} \otimes_k \lambda_{i,j,\underline{\alpha},k}$$

with $r_{i,j,\underline{\alpha},k} \in A_i$, $\lambda_{i,j,\underline{\alpha},k} \in \Omega$. So, let R_i denote the sub k-algebra of Ω generated by the $\lambda_{i,j,\underline{\alpha},k}$ then B_i can also be written as:

$$B_i = A_i \otimes_k R_i[\underline{T}]/\langle P_{i,1}, \ldots, P_{i,r_i}\rangle \otimes_{R_i} \Omega.$$

Let R denote the sub-k-algebra of Ω generated by the R_i, $i = 1, \ldots, n$. Then $k \hookrightarrow R$ is a finitely generated k-algebra and up to enlarging R, one may assume that the gluing data on the $U_i \cap U_j$ descend to R then one can construct $\tilde{\phi}$ by gluing the $\mathrm{spec}(A_i \otimes_k R[\underline{T}]/\langle P_{i,1}, \ldots, P_{i,r_i}\rangle)$ along these descended gluing data. By construction $\tilde{\phi}$ is affine.

To conclude, since $k(\eta) \hookrightarrow \Omega$ is faithfully flat and $\phi : Y \to X_\Omega$ is finite and faithfully flat, the same is automatically true for $\tilde{\phi}_{k(\eta)} : \tilde{Y}_{k(\eta)} \to X_{k(\eta)}$, which is then étale since $\phi : Y \to X_\Omega$ is. □

So, applying Lemma 6.7 to $\phi : Y \to X_\Omega$ and up to replacing R by R_r for some $r \in R \setminus \{0\}$, one may assume that $\phi : Y \to X_\Omega$ is the base-change of some étale cover $\phi^0 : Y^0 \to X_R$. Note that, since $Y^0_\Omega = Y$ is connected, both Y^0_η and Y^0 are connected as well. Fix $s : \mathrm{spec}(k) \to S$. Since the fundamental group does not depend on the fibre functor, one can assume that $k(x) = k$. Then, from Lemma 6.6, one gets the canonical isomorphism of profinite groups:

$$\pi_1(X \times_k S; (x, s)) \tilde{\to} \pi_1(X; x) \times \pi_1(S; s).$$

Let $U \subset \pi_1(X \times_k S; (x,s))$ be the open subgroup corresponding to the étale cover $\phi^0 : Y^0 \to X \times_k S$ and let $U_X \subset \pi_1(X;x)$ and $U_S \subset \pi_1(S;s)$ be open subgroups such that $U_X \times U_S \subset U$. Then U_X and U_S correspond to connected étale covers $\psi_X : \tilde{X} \to X$ and $\psi_S : \tilde{S} \to S$ such that $\phi^0 : Y^0 \to X \times_k S$ is a quotient of $\psi_X \times_k \psi_S : \tilde{X} \times_k \tilde{S} \to X \times_k S$. Consider the following cartesian diagram:

$$\begin{tikzcd}
 & & \tilde{X} \times_k \tilde{S} \arrow[dl] \arrow[dl, dotted] \arrow[ddl] \\
Y^0 \arrow[d] \arrow[r, phantom, "\square"] & \tilde{Y}^0 \arrow[l] \arrow[d] & \\
X \times_k S & X \times_k \tilde{S}. \arrow[l] &
\end{tikzcd}$$

Since $k(\eta) \subset \Omega$ and Ω is algebraically closed, one may assume that any point $\tilde{s} \in \tilde{S}$ above $s \in S$ has residue field contained in Ω and, in particular, one can consider the associated Ω-point $\tilde{s}_\Omega : \mathrm{spec}(\Omega) \to \tilde{S}$. Then, one has the cartesian diagram:

$$\begin{tikzcd}
\tilde{Y}^0_S \arrow[d] \arrow[r, phantom, "\square"] & Y_\Omega \arrow[l] \arrow[d] \\
X \times_k S & X_\Omega. \arrow[l, "Id_X \times_k \tilde{s}_\Omega"']
\end{tikzcd}$$

Again, since Y_Ω is connected, $\tilde{Y}^0$ is connected as well, from which it follows that $\tilde{Y}^0 \to X \times_k \tilde{S}$ corresponds to an open subgroup $V \subset \pi_1(X \times_k \tilde{S}) = \pi_1(X) \times U_S$ containing $\pi_1(\tilde{X} \times_k \tilde{S}) = U_X \times U_S$. Hence $V = U \times U_S$ for some open subgroup $U_X \subset U \subset \pi_1(X)$ hence $\tilde{Y}^0 \to X \times_k \tilde{S}$ is of the form $\tilde{Y} \times_k \tilde{S} \to X \times_k \tilde{S}$ for some étale cover $\tilde{\phi} : \tilde{Y} \to X$. $\square$

Remark 6.8. An argument due to F. Pop [Sz09, pp. 190–191] shows that Corollary 6.5 remains true for connected schemes of finite type over k as soon as $\pi_1(X; x_\Omega)$ (or $\pi_1(X_\Omega; x_\Omega)$) is finitely generated. However, in general, Corollary 6.5 is no longer true for non-proper schemes. Indeed, let k be an algebraically closed field of characteristic $p > 0$. From the long cohomology exact sequence associated with Artin-Schreier short exact sequence:

$$0 \to (\mathbb{Z}/p)_{\mathbb{A}^1_k} \to \mathbb{G}_{a,\mathbb{A}^1_k} \xrightarrow{\wp} \mathbb{G}_{a,\mathbb{A}^1_k} \to 0$$

(and taking into account that, as $\mathbb{A}^1_k$ is affine, $\mathrm{H}^1(\mathbb{A}^1_k, \mathbb{G}_a) = 0$) one gets:

$$k[T]/\wp k[T] = \mathrm{H}^0(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k})/\wp \mathrm{H}^0(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}) \tilde{\to} \mathrm{H}^1_{et}(\mathbb{A}^1_k, \mathbb{Z}/p) = \mathrm{Hom}(\pi_1(\mathbb{A}^1_k, 0), \mathbb{Z}/p).$$

An additive section of the canonical epimorphism $k[T] \twoheadrightarrow k[T]/\wp k[T]$ is given by the representatives:

$$\sum_{n>0,(n,p)=1} a_n T^n, \ a_n \in k,$$

which shows that $\pi_1(\mathbb{A}^1_k, 0)$ is not of finite type and depends on the base field k. More generally, one can show [Bo00], [G00] that if S is a smooth connected curve over an algebraically closed field of characteristic $p > 0$ then the pro-p completion $\pi_1(S)^{(p)}$ of $\pi_1(S)$ is a free pro-p group of rank r, where:

- if S is proper over k then r is the p-rank of the jacobian variety $\mathrm{J}_{S|k}$;
- if S is affine over k then r is the cardinality of k.

This determines completely the pro-p completion $\pi_1(S)^{(p)}$ of $\pi_1(S)$. In Sections 8, 9 and 10, we will see that the prime-to-p completion $\pi_1(S)^{(p)'}$ of $\pi_1(S)$ is also completely determined. However, except when $\pi_1(S)$ is abelian, this does not determine $\pi_1(S)$ entirely (see Remark 11.5).

6.2.3. Proof of Theorem 6.4. We apply, again, the criterion of Proposition 4.3. We begin with an elementary lemma, stating that the inclusion $\mathrm{im}(i) \subset \ker(p)$ is true under less restrictive hypotheses.

Lemma 6.9. *Let X, S be connected schemes, $f : X \to S$ a geometrically connected morphism and $s \in S$. Fix a geometric point $x_\Omega : \mathrm{spec}(\Omega) \to X_{\overline{s}}$ with image again denoted by x_Ω in X and s_Ω in S and consider the canonical sequence of profinite groups induced by $(X_{\overline{s}}, x_\Omega) \to (X, x_\Omega) \to (S, s_\Omega)$:*

$$\pi_1(X_{\overline{s}}; x_\Omega) \xrightarrow{i} \pi_1(X; x_\Omega) \xrightarrow{p} \pi_1(S; s_\Omega).$$

Then, one always has $\mathrm{im}(i) \subset \ker(p)$.

Proof. Let $\phi : S' \to S$ be an étale cover and consider the following notation:

$$\begin{tikzcd}
S'_{\overline{s}} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\square"] & & S' \arrow[d, "\phi"] \\
X_{\overline{s}} \arrow[r] \arrow[dr] \arrow[drr, phantom, "\square"] & X \arrow[r, "f"'] & S \\
& k(\overline{s}). \arrow[ur, "\overline{s}"'] &
\end{tikzcd}$$

We are to prove that $\overline{S}' \to X_{\overline{s}}$ is totally split. But, this is just formal computation based on elementary properties of fibre product of schemes:

$$\begin{aligned}
S'_{\overline{s}} = X_{\overline{s}} \times_{S,\phi} S' &= (X \times_{f,S,\overline{s}} \mathrm{spec}(k(\overline{s}))) \times_{S,\phi} S' \\
&= X \times_{f,S} (\mathrm{spec}(k(\overline{s})) \times_{\overline{s},S,\phi} S') \\
&= X \times_{f,S} \sqcup_{S'_{\overline{s}}} \mathrm{spec}(k(\overline{s})) \\
&= \sqcup_{S'_{\overline{s}}} X_{\overline{s}}.
\end{aligned}$$

□

We return to the proof of Theorem 6.4. For simplicity, write $\overline{X} := X_{\overline{s}}$.

Exactness on the right: We are to prove that for any connected étale cover $\phi : S' \to S$ and with the notation for base change:

$$\begin{tikzcd} X' \arrow[r, "\phi'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, "\phi"'] & S, \end{tikzcd}$$

the scheme X' is again connected. But, one has:

$$f'_*(\mathcal{O}_{X'}) = f'_*(\phi'^*\mathcal{O}_X) = \phi^* f_* \mathcal{O}_X \overset{(*)}{=} \phi^* \mathcal{O}_S = \mathcal{O}_{S'},$$

where (*) follows from the assumption that $f_*\mathcal{O}_X = \mathcal{O}_S$. Hence, as $f' : X' \to S'$ is proper, it follows from Theorem 6.2 (1) (b) that X' is connected.

Exactness in the middle: From Lemma 6.9, this amounts to show that $\ker(p) \subset \mathrm{im}(i)$. Let $\phi : X' \to X$ be a connected étale cover and consider the notation:

$$\begin{tikzcd} X' \arrow[r, "\phi"] & X \arrow[r, "f"] \arrow[dl, phantom, "\square"] & S \\ \overline{X}' \arrow[u] \arrow[r, "\overline{\phi}"'] \arrow[ur, phantom, "\square"] & \overline{X} \arrow[u] \arrow[r] & k(\overline{s}). \arrow[u, "\overline{s}"'] \end{tikzcd}$$

Assume that $\overline{\phi} : \overline{X}' \to \overline{X}$ admits a section $\sigma : \overline{X} \to \overline{X}'$. We are to prove that $\phi : X' \to X$ comes, by base-change, from a connected étale cover $S' \to S$.

Since $\phi : X' \to X$ is finite étale and $f : X \to S$ is proper and separable, $g := f \circ \phi : X' \to S$ is also proper and separable. Consider its Stein factorization $X' \xrightarrow{g'} S' \xrightarrow{p} S$. From Theorem 6.2 (2), the morphism $p : S' \to S$ is étale. Furthermore, as X' is connected and $g' : X' \to S'$ is surjective, S' is connected. Consider the following commutative diagram:

$$\begin{tikzcd} & X' \arrow[dl, "\phi"'] \arrow[d, dotted, "\alpha"] \arrow[ddr, bend left, "g'"] & \\ X \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & X'' \arrow[l, "p_X"] \arrow[d, "f'"] & \\ S & S'. \arrow[l, "p"] & \end{tikzcd} \tag{1}$$

Claim: *$\alpha : X \tilde{\to} X''$ is an isomorphism.*

Proof of the claim. As $p : S' \to S$ is an étale cover, its base-change $p_X : X'' \to X$ is an étale cover as well. Since S' is connected, it follows from the exactness on the right that X'' is connected as well hence, from Lemma 5.16 and Corollary 5.9 the morphism $\alpha : X' \to X''$ is an étale cover. So, it only remains to prove that $r(\alpha) = 1$.

For this, consider the base-change of (1) *via* $\overline{s} : \mathrm{spec}(k(\overline{s})) \to S$.

$$\begin{tikzcd}
 & X'_{\overline{s}} \arrow[dl, "\phi_{\overline{s}}"'] \arrow[d, "\alpha_{\overline{s}}"] \arrow[dd, bend left, "g_{\overline{s}}"] \\
X_{\overline{s}} \arrow[d, "f_{\overline{s}}"'] \arrow[ur, bend left, "\sigma"] & X''_{\overline{s}} \arrow[l, "p_{X\overline{s}}"] \arrow[d, "f'_{\overline{s}}"] \arrow[dl, phantom, "\square"] \\
k(\overline{s}) & S'_{\overline{s}}. \arrow[l, "p_{\overline{s}}"]
\end{tikzcd} \tag{2}$$

Since $\alpha_{\overline{s}} : X'_{\overline{s}} \to X''_{\overline{s}}$ is an étale cover, it induces a surjective map $\pi_0(X'_{\overline{s}}) \twoheadrightarrow \pi_0(X''_{\overline{s}})$, where $\pi_0(-)$ denotes the set of connected components. But, as both $g' : X' \to S'$ and $f' : X'' \to S'$ are geometrically connected, $|\pi_0(X'_{\overline{s}})| = |\pi_0(X''_{\overline{s}})|(= r(p))$ hence, actually, the map $\pi_0(X'_{\overline{s}}) \twoheadrightarrow \pi_0(X''_{\overline{s}})$ is bijective. So it is enough to find $X'_{\overline{s}0} \in \pi_0(X'_{\overline{s}})$ such that $\alpha_{\overline{s}} : X'_{\overline{s}} \to X''_{\overline{s}}$ induces an isomorphism from $X'_{\overline{s}0}$ to $\alpha_{\overline{s}}(X'_{\overline{s}0})$. For this, consider $X'_{\overline{s}0} := \sigma(X\overline{s})$ and set $X''_{\overline{s}0} := \alpha_{\overline{s}}(X'_{\overline{s}0})$. Then σ induces an isomorphism from $X_{\overline{s}}$ to $X'_{\overline{s}}$ and, as $p_{X\overline{s}} : X''_{\overline{s}} \to X_{\overline{s}}$ is totally split, it induces an isomorphism from $X''_{\overline{s}0}$ to $X_{\overline{s}}$. Hence the conclusion follows from

$$\sigma|^{X'_{\overline{s}0}} \circ p_{X\overline{s}}|_{X''_{\overline{s}0}} \circ \alpha\overline{s}|^{X''_{\overline{s}0}}_{X'_{\overline{s}0}} = Id_{X''_{\overline{s}0}}.$$

Remark 6.10. The assumption $f_*\mathcal{O}_X = \mathcal{O}_S$ can be omitted and the conclusion of Theorem 9.3 then becomes that the following canonical exact sequence of profinite groups is exact:

$$\pi_1(\overline{X}_1, \overline{x}_1) \xrightarrow{i_1} \pi_1(X, x_{(1)}) \xrightarrow{p_1} \pi_1(S, \overline{s}_1) \to \pi_0(\overline{X}_1) \to \pi_0(X) \to \pi_0(S) \to 1.$$

Theorem 6.4 will also play a crucial part in the construction of the specialization morphism in Section 9.

6.3. Abelian varieties

A main reference for abelian varieties is [Mum70]. See also [Mi86] for a concise introduction.

Let k be an algebraically closed field and A an abelian variety over k. For each $n \geq 1$ let $A[n]$ denote the group of k-points underlying the kernel of the multiplication-by-n morphism:

$$[n_A] : A \to A.$$

For each prime ℓ, the multiplication-by-ℓ morphism induces a projective system structure on the $A[\ell^n]$, $n \geq 0$ and one sets:

$$T_\ell(A) := \varprojlim A[\ell^n].$$

If ℓ is prime to the characteristic of k then $T_\ell(A) \simeq \mathbb{Z}_\ell^{2g}$ whereas if $\ell = p$ is the characteristic of k then $T_p(A) \simeq \mathbb{Z}_p^r$, where g and $r(\leq g)$ denote the dimension and p-rank of A respectively [Mum70, Chap. IV, §18].

Theorem 6.11. *There is a canonical isomorphism:*

$$\pi_1(A;0_A)\tilde{\to}\prod_{\ell:\text{prime}} T_\ell(A).$$

Proof. The proof below was suggested to me by the referee. For another proof based on rigidity, see [Mum70, Chap. IV, §18].

Given a profinite group Π and a prime ℓ, let $\Pi^{(\ell)}$ denote its pro-ℓ completion that is its maximal pro-ℓ quotient, which can also be described as:

$$\Pi^{(\ell)} = \varprojlim \Pi/N,$$

where the projective limit is over all normal open subgroups of index a power of ℓ in Π.

<u>Claim 1:</u> $\pi_1(A;0_A)$ *is abelian. In particular,*

$$\pi_1(A,0) = \prod_{\ell:\text{prime}} \pi_1(A,0)^{(\ell)}.$$

Proof of Claim 1. From Lemma 6.6, the multiplication map $\mu : A\times_k A\to A$ on A induces a morphism of profinite groups:

$$\pi_1(\mu) : \pi_1(A;0_A)\times\pi_1(A;0_A)\to\pi_1(A;0_A).$$

The canonical section $\sigma_1 = A\to A\times_k A$ of the first projection $p_1 : A\times_k A\to A$ induces the morphism of profinite groups:

$$\begin{array}{rccc} \pi_1(\sigma_1): & \pi_1(A;0_A) & \to & \pi_1(A;0_A)\times\pi_1(A;0_A) \\ & \gamma & \mapsto & (\gamma,1) \end{array}$$

and, by functoriality, $\pi_1(\mu)\circ\pi_1(\sigma_1) = Id$. The same holds for the second projection and since σ_1 and σ_2 commute, one gets:

$$\begin{aligned} \pi_1(\mu)(\gamma_1,\gamma_2) &= \pi_1(\mu)(\pi_1(\sigma_1)(\gamma_1)\pi_1(\sigma_2)(\gamma_2)) \\ &= \pi_1(\mu)(\pi_1(\sigma_1)(\gamma_1))\pi_1(\mu)(\pi_1(\sigma_2)(\gamma_2)) = \gamma_1\gamma_2 \\ &= \pi_1(\mu)(\pi_1(\sigma_2)(\gamma_2)\pi_1(\sigma_1)(\gamma_1)) \\ &= \pi_1(\mu)(\pi_1(\sigma_2)(\gamma_2))\pi_1(\mu)(\pi_1(\sigma_1)(\gamma_1)) = \gamma_2\gamma_1. \end{aligned}$$

<u>Claim 2 (Serre-Lang):</u> *Let $\phi : X\to A$ be a connected étale cover. Then X carries a unique structure of abelian variety such that $\phi : X\to A$ becomes a separable isogeny.*

Proof of Claim 2. The idea is to construct first the group structure on one fibre and, then, extend it automatically by the formalism of Galois categories. Let x : $\mathrm{spec}(k)\to X$ such that $\phi(x) = 0_A$. Then the pointed connected étale cover ϕ : $(X;x)\to(A;0_A)$ corresponds to a transitive $\pi_1(A;0_A)$-set M together with a distinguished point $m\in M$. Since $\pi_1(A;0_A)$ is abelian, the map:

$$\begin{array}{rccc} \mu_M: & M\times M & \to & M \\ & (\gamma_1 m,\gamma_2 m) & \mapsto & \gamma_1\gamma_2 m \end{array}$$

is well defined, maps (m, m) to m and is $\pi_1(A; 0_A) \times \pi_1(A; 0_A)$-equivariant if we endow M with the structure of $\pi_1(A; 0_A) \times \pi_1(A; 0_A)$-set induced by $\pi_1(\mu)$ (which corresponds to the étale cover $X \times_{\phi,A,\mu} (A \times_k A) \to A \times_k A$). Hence it corresponds to a morphism $\mu_X^0 : X \times_k X \to X \times_{\phi,A,\mu} (A \times_k A)$ above $A \times_k A$ or, equivalently, to a morphism $\mu_X : X \times_k X \to X$ fitting in:

$$\begin{tikzcd}
X \times_k X \arrow[r, "\mu_X^0"] \arrow[rr, bend left, "\mu_X"] \arrow[d, "\phi \times_k \phi"'] & X \times_{\phi,A,\mu} (A \times_k A) \arrow[r] \arrow[dl] \arrow[d, phantom, "\square"] & X \arrow[d, "\phi"] \\
A \times_k A \arrow[rr, "\mu"'] & & A
\end{tikzcd}$$

and mapping (x, x) to x. By the same arguments, one constructs $i_X : X \to X$ above $[-1_A] : A \to A$ mapping x to x, checks that this endows X with the structure of an algebraic group with unity x (hence, of an abelian variety since X is connected and $\phi : X \to A$ is proper) and such that $\phi : X \to A$ becomes a morphism of algebraic groups (hence a separable isogeny since $\phi : X \to S$ is an étale cover).

Now let $\phi : X \to A$ be a degree n isogeny. Then $\ker(\phi) \subset \ker([n_X])$ hence one has a canonical commutative diagram:

$$\begin{tikzcd}
 & X/\ker(\phi) \arrow[r, equal] \arrow[dl, "\psi"'] & A \\
X & X. \arrow[l, "{[n_X]}"] \arrow[u] \arrow[ur, "\phi"'] &
\end{tikzcd}$$

From the surjectivity of ϕ, one also has $\phi \circ \psi = [n_A]$. When ℓ is a prime different from the characteristic p of k, combining this remark and Claim 2, one gets that $([\ell^n] : A \to A)_{n \geq 0}$ is cofinal among the finite étale covers of A with degree a power of ℓ that is

$$\pi_1(A; 0_A)^{(\ell)} = \varprojlim A[\ell^n] = T_\ell(A).$$

When $\ell = p$, one has to be more careful since, when p divides n, the isogeny $[n_A] : A \to A$ is no longer étale. However, it factors as:

$$\begin{tikzcd}
A \arrow[r, "\psi_n"] \arrow[d, "{[n_A]}"'] & B_n \arrow[dl, "\phi_n"] \\
A, &
\end{tikzcd}$$

where $\phi_n : B_n \to A$ is an étale isogeny and $\psi_n : A \to B_n$ is a purely inseparable isogeny. In particular, one has:

$$\begin{aligned}\mathrm{Aut}(B_n/A) &= \mathrm{Aut}(k(B_n)/k(A)) \\ &= \mathrm{Aut}(k(A) \overset{[n_A]^{\#}}{\hookrightarrow} k(A)) \\ &= A[n](k)\end{aligned}$$

and, if $\phi : X \to A$ is a degree n étale isogeny, one gets a factorization $\phi_n = \phi \circ \psi$. Thus, in that case, $(\phi_{p^n} : B_{p^n} \to A)_{n \geq 0}$ is cofinal among the finite étale covers of A with degree a power of p hence, as $\mathrm{Aut}(B_{p^n}/A) = A[p^n](k)$, one has, again:

$$\pi_1(A; 0_A)^{(p)} = \varprojlim A[p^n](k) = T_p(A). \qquad \square$$

Now, assume that $k = \mathbb{C}$ and that $A = \mathbb{C}^g/\Lambda$, where $\Lambda \subset \mathbb{C}^g$ is a lattice. Then, on the one hand, the universal covering of A is just the quotient map $\mathbb{C}^g \to A$ and has group $\pi_1^{\mathrm{top}}(A(\mathbb{C}); 0_A) \simeq \Lambda$ whereas, on the other hand, for any prime ℓ:

$$\begin{aligned}T_\ell(A) &= \varprojlim A[\ell^n] \\ &= \varprojlim \frac{1}{\ell^n}\Lambda/\Lambda \\ &= \varprojlim \Lambda/\ell^n\Lambda \\ &= \Lambda^{(\ell)},\end{aligned}$$

whence

$$\pi_1(A; 0_A) = \prod_{\ell : prime} T_\ell(A) = \prod_{\ell : prime} \pi_1^{\mathrm{top}}(A(\mathbb{C}); 0_A)^{(\ell)} = \pi_1^{\mathrm{top}}(\widehat{A(\mathbb{C})}; 0_A).$$

This is a special case of the much more general Grauert-Remmert Theorem 8.1 but, basically, the only one where one has a purely algebraically proof of it.

6.4. Normal schemes

Let S be a normal connected (hence integral) scheme.

Lemma 6.12. *Let $k(S) \hookrightarrow L$ be a finite separable field extension. Then the normalization of S in $k(S) \hookrightarrow L$ is finite.*

Proof. Without loss of generality, we may assume that $S = \mathrm{spec}(A)$ is affine that is, we are to prove that given an integrally closed, noetherian ring A with fraction field K and a finite separable field extension $K \hookrightarrow L$, the integral closure B of A in $K \hookrightarrow L$ is a finitely generated A-module. Since $K \hookrightarrow L$ is separable, the trace form:

$$\begin{array}{rccc}\langle -, - \rangle : & L \times L & \to & K \\ & (x, y) & \mapsto & Tr_{L|K}(xy)\end{array}$$

is non-degenerate. Set $n := [L : K]$ and let $b_1, \ldots, b_n \in B$ be a basis of L over K. Let $b_1^*, \ldots, b_n^* \in L$ denote its dual with respect to $\langle -, - \rangle : L \times L \to K$. Then, since

$Tr_{L|K}(B) \subset A$, one has $B \subset \oplus_{i=1}^{n} Ab_i^*$ hence B is a finitely generated A-module as well since A is noetherian. $\square$

When S is normal, we can improve Theorem 5.5 as follows.

Lemma 6.13. *Let A be a noetherian integrally closed local ring with fraction field K and set $S = \mathrm{spec}(A)$. Let $\phi : X \to S$ an unramified (resp. étale) morphism. Then, for any $x \in X$, there exists an open affine neighborhood U of x such that one has a factorization:*

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "\phi"'] & \mathrm{spec}(C), \arrow[dl] \\ S & \end{tikzcd}$$

where $\mathrm{spec}(C) \to S$ is a standard étale morphism where $B = A[T]/PA[T]$ can be chosen in such a way that the monic polynomial $P \in A[T]$ becomes irreducible in $K[T]$ and $U \hookrightarrow \mathrm{spec}(C)$ is an immersion (resp. an open immersion).

Proof. Let $\mathfrak{m}$ denote the maximal ideal of A and, correspondingly, let s denote the closed point of S. From Theorem 5.5, one may assume that $\phi : X \to S$ is induced by an A-algebra of the form $A \to B_b$ with $B = A[T]/PA[T]$ and $b \in B$ such that $P'(t)$ is invertible in B_b. Since A is integrally closed, any monic factor of P in $K[T]$ is in $A[T]$. Let $x \in X_s$ and fix an irreducible monic factor Q of P mapping to 0 in $k(x)$. Write $P = QR$ in $A[T]$. As $\overline{P} \in k(s)[T]$ is separable, $\overline{Q}$ and $\overline{R}$ are coprime in $k(s)[T]$ or, equivalently:

$$\langle \overline{Q}, \overline{R} \rangle = k(s)[T].$$

But, then, as Q is monic $M := A[T]/\langle Q, R \rangle$ is a finitely generated A-module so, from Nakayama, $A[T] = \langle Q, R \rangle$. This, by the Chinese remainder theorem:

$$A[T]/PA[T] = A[T]/QA[T] \times A[T]/RA[T].$$

Set $B_1 := A[T]/QA[T]$ and let b_1 denote the image of b in B_1. Then the open subscheme $U_1 := \mathrm{spec}(B_{1b_1}) \hookrightarrow X$ contains x and:

$$U_1 := \mathrm{spec}(B_{1b_1}) \hookrightarrow X \to S$$

is a standard morphism of the required form. $\square$

Lemma 6.14. *Let $\phi : X \to S$ be an étale cover. Then X is also normal and, in particular, it can be written as the coproduct of its (finitely many) irreducible components. Furthermore, given a connected component X_0 of X, the induced étale cover $X_0 \to S$ is the normalization of S in $k(S) \hookrightarrow k(X_0)$.*

Proof. We first prove the assertion when $S = \mathrm{spec}(A)$ with A a noetherian integrally closed local ring and $\phi : X \to S$ is a standard morphism as in Lemma 6.13. Let $K(= k(S))$ denote the fraction field of A. By assumption, $L := C \otimes_A K = K[T]/PK[T]$ is a finite separable field extension of K. Let A^c denote the integral closure of A in $K \hookrightarrow L$. Since B is integral over A, one has $A \subset B \subset A^c \subset L$ hence

$B_b \subset (A^c)_b = ((A^c)_b)^c \subset L$. So, to show that C is integrally closed in $K \hookrightarrow L$, it is enough to show that $A^c \subset B_b$. So let $\alpha \in A^c$ and write:

$$\alpha = \sum_{i=0}^{n-1} a_i t^i,$$

with $a_i \in K$, $i = 1, \ldots, n$ and $n = \deg(P)$. As $K \hookrightarrow L$ is separable of degree n, there are exactly n distinct morphisms of K-algebras:

$$\phi_i : L \hookrightarrow \overline{K}$$

Let $V_n(t) := V(\phi_1(t), \ldots, \phi_n(t))$ denote the Vandermonde matrix associated with $\phi_1(t), \ldots, \phi_n(t)$. Then one has:

$$|V_n(t)|(a_i)_{0 \leq i \leq n-1} = {}^tCom(V_n(t))(\phi_i(\alpha))_{1 \leq i \leq n}$$

(where ${}^tCom(-)$ denotes the transpose of the comatrix and $|-|$ the determinant). Hence, as the $\phi_i(t)$ and the $\phi_i(\alpha)$ are all integral over A, the $|V_n(t)|a_i$ are also all integral over A. By assumption, the a_i are in K and $|V_n(t)|$ is in K since it is symmetric in the $\phi_i(t)$. So, as A is integrally closed, the $|V_n(t)|a_i$ are in A, from which the conclusion follows since $|V_n(t)|$ is a unit in C (recall that $P'(t)$ is invertible in C).

We now turn to the general case. From Lemma 6.13, the above already shows that X is normal and, in particular, it can be written as the coproduct of its (finitely many) irreducible components. So, without loss of generality we may assume that X is a normal connected hence integral scheme. But then, for any open subscheme $U \subset S$, the ring $\mathcal{O}_X(\phi^{-1}(U))$ is integral ring and its local rings are all integrally closed so $\mathcal{O}_X(\phi^{-1}(U))$ is integrally closed as well and, since it is also integral over $\mathcal{O}_S(U)$, it is the integral closure of $\mathcal{O}_S(U)$ in $k(S) \hookrightarrow k(X)$. $\square$

The following provides a converse to Lemma 6.14:

Lemma 6.15. *Let $k(S) \hookrightarrow L$ be a finite separable field extension which is unramified over S. Then the normalization $\phi : X \to S$ of S in $k(S) \hookrightarrow L$ is an étale cover.*

Proof. Since S is locally noetherian, $\phi : X \to S$ is finite by Lemma 6.12; it is also surjective [AM69, Thm. 5.10] and, by construction it is unramified. So we are only to prove that $\phi : X \to S$ is flat, namely that $\mathcal{O}_{S,\phi(x)} \hookrightarrow \mathcal{O}_{X,x}$ is a flat algebra, $x \in X$. One has a commutative diagram:

$$\begin{tikzcd} \mathcal{O}_{X,x} & C \arrow[l, two heads] \\ \mathcal{O}_{S,\phi(x)} \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

where $\mathcal{O}_{S,\phi(x)} \to C$ is a standard algebra as in Lemma 6.13, $C \twoheadrightarrow \mathcal{O}_{X,x}$ is surjective and, as $\phi : X \to S$ is surjective, $\mathcal{O}_{S,\phi(x)} \hookrightarrow \mathcal{O}_{X,x}$. In particular,

$$\mathcal{O}_{S,\phi(x)} \otimes_{\mathcal{O}_{S,\phi(x)}} k(S) \hookrightarrow \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,\phi(x)}} k(S)$$

is injective as well hence:

$$C \otimes_{\mathcal{O}_{S,\phi(x)}} k(S) \to \mathcal{O}_{X,x} \otimes_{\mathcal{O}_{S,\phi(x)}} k(S)$$

is non-zero. But, as $C \otimes_{\mathcal{O}_{S,\phi(x)}} k(S)$ is a field, the above morphism is actually injective and, as $\mathcal{O}_{S,\phi(x)} \to k(S)$ is faithfully flat, this implies that $C \twoheadrightarrow \mathcal{O}_{X,x}$ is injective hence bijective. □

Lemma 6.14 shows that there is a well-defined functor:

$$\begin{array}{llll} R: & \mathcal{C}_S & \to & (FEAlg/k(S))^{op} \\ & X \to S & \mapsto & k(S) \hookrightarrow R(X) := \prod_{X_0 \in \pi_0(X)} k(X_0). \end{array}$$

Let $FEAlg/k(S)/S \subset FEAlg/k(S)$ denote the full subcategory of finite étale algebras $k(S) \hookrightarrow R$ which are unramified over S. Lemmas 6.14 and 6.15 show:

Theorem 6.16. *The functor $R : \mathcal{C}_S \to FEAlg/k(S)$ is fully faithful and induces an equivalence of categories $R : \mathcal{C}_S \to FEAlg/k(S)/S$ with pseudo-inverse the normalization functor.*

Let $\eta \in S$ denote the generic point of S hence $k(\eta) = k(S)$. Let $k(\eta) \hookrightarrow \Omega$ be an algebraically closed field extension defining geometric points $\overline{s}_\eta : \mathrm{spec}(\Omega) \to \mathrm{spec}(k(\eta))$ and $\overline{\eta} : \mathrm{spec}(\Omega) \to S$. From Theorem 6.16, the base-change functor

$$\eta^* : \mathcal{C}_S \to \mathcal{C}_{\mathrm{spec}(k(\eta))}$$

is fully faithful hence, from Proposition 4.3 (1), induces an epimorphism of profinite groups:

$$\pi_1(\eta) : \pi_1(\mathrm{spec}(k(\eta); \overline{s}_\eta) \twoheadrightarrow \pi_1(S; \overline{s})$$

whose kernel is the absolute Galois group of the maximal algebraic extension $k(\eta) \hookrightarrow M_{k(S),S}$ of $k(\eta)$ in Ω which is unramified over S.

Example 6.17. Let S be a curve, smooth and geometrically connected over a field k and let $S \hookrightarrow S^{cpt}$ be the smooth compactification of S. Write $S^{cpt} \setminus S = \{P_1, \ldots, P_r\}$. Then the extension $k(S) \hookrightarrow M_{k(S),S}$ is just the maximal algebraic extension of $k(S)$ in Ω unramified outside the places $P_1, \ldots, P_r$.

7. Geometrically connected schemes of finite type

Let S be a scheme geometrically connected and of finite type over a field k. Fix a geometric point $\overline{s} : \mathrm{spec}(k(\overline{s})) \to S_{k^s}$ with image again denoted by $\overline{s}$ in S and $\mathrm{spec}(k)$.

Proposition 7.1. *The morphisms $(S_{k^s}, \overline{s}) \to (S, \overline{s}) \to (\mathrm{spec}(k), \overline{s})$ induce a canonical short exact sequence of profinite groups:*

$$1 \to \pi_1(S_{k^s}; \overline{s}) \xrightarrow{i} \pi_1(S; \overline{s}) \xrightarrow{p} \pi_1(\mathrm{spec}(k); \overline{s}) \to 1.$$

Example 7.2. Assume furthermore that S is normal. Then the assumption that S is geometrically connected over k is equivalent to the assumption that $\overline{k} \cap k(S) = k$ and, with the notation of Subsection 6.4, the short exact sequence above is just the one obtained from usual Galois theory:

$$1 \to \mathrm{Gal}(M_{k(S),S}|k^s(S)) \to \mathrm{Gal}(M_{k(S),S}|k(S)) \to \Gamma_k \to 1.$$

Proof. We use, again, the criteria of Proposition 4.3.

Exactness on the right: As S is geometrically connected over k, the scheme S_K is also connected for any finite separable field extension $k \hookrightarrow K$.

Exactness on the left: For any étale cover $f : X \to S_{k^s}$ we are to prove that there exists an étale cover $f : \tilde{X} \to S$ such that $f_{k(s)}$ dominates f. From Lemma 6.7, there exists a finite separable field extension $k \hookrightarrow K$ and an étale cover $\tilde{f} : \tilde{X} \to S_K$ which is a model of $f : X \to S_{k^s}$ over S_K. But then, the composite $f : \tilde{X} \to S_K \to S$ is again an étale cover whose base-change *via* $S_{k^s} \to S$ is the coproduct of $[K : k]$ copies of f hence, in particular, dominates f.

Exactness in the middle: From Lemma 6.9, this amounts to show that $\ker(p) \subset \mathrm{im}(i)$. For any connected étale cover $\phi : X \to S$ such that $\phi_{k^s} : X_{k^s} \to S_{k^s}$ admits a section, say $\sigma : S_{k^s} \hookrightarrow X_{k^s}$, we are to prove that there exists a finite separable field extension $k \hookrightarrow K$ such that the base change of $\mathrm{spec}(K) \to \mathrm{spec}(k)$ *via* $S \to \mathrm{spec}(k)$ dominates $\phi : X \to S$. So, let $k \hookrightarrow K$ be a finite separable field extension over which $\sigma : S_{k^s} \hookrightarrow X_{k^s}$ admits a model $\sigma_K : S_K \hookrightarrow X_K$. This defines a morphism from S_K to X over S by composing $\sigma_K : S_K \hookrightarrow X_K$ with $X_K \to X$. □

Proposition 7.1 shows that the fundamental group $\pi_1(S)$ of a scheme S geometrically connected and of finite type over a field k can be canonically decomposed into a geometric part $\pi_1(S_{k^s})$ and an arithmetic part Γ_k. This raises several problems:

1. Determine the geometric part $\pi_1(S_{k^s})$;
2. Describe the sections of $\pi_1(S) \twoheadrightarrow \Gamma_k$;
3. Describe the outer representation $\rho : \Gamma_k \to \mathrm{Out}(\pi_1(S_{k^s}))$.

In the end of these notes, we are going to explain how problem (1) can be solved (fully in characteristic 0 and partly in positive characteristic). Basically, this is done in three steps (one step in characteristic 0):

(a) G.A.G.A. theorems (see Section 8), which show that the étale fundamental group of a connected scheme locally of finite type over $\mathbb{C}$ is the profinite completion of the topological fundamental of its underline topological space. The latter can often be explicitly computed by methods from algebraic topology. From the invariance of fundamental groups under algebraically closed field extensions (see Subsection 6.2), this yields the determination of most of the étale fundamental groups of connected schemes locally of finite type over algebraically closed field in characteristic 0.

(b) Specialization theory (see Section 9), which says that if $f : X \to S$ is a proper separable morphism with geometrically connected fibres and $s_0,\ s_1 \in S$ are such that s_0 is a specialization of s_1, there is an epimorphism of profinite groups:

$$sp : \pi_1(X_{\overline{s}_1}) \twoheadrightarrow \pi_1(X_{\overline{s}_0}).$$

(c) The Zariski-Nagata purity theorem (see Section 10.1), which yields information about the kernel of the above specialization epimorphism when $f : X \to S$ is furthermore assumed to be smooth and, in particular, shows that it induces an isomorphism on the prime-to-p completions, where p denotes the residue characteristic of s_0. Note that, however, to understand the prime-to-p completion of the étale fundamental group in positive characteristic $p > 0$ by this method, one has to face the deep problem of lifting schemes from characteristic p to characteristic 0; we will give an illustration of this in the proof of Theorem 11.1. Concerning the pro-p completion and the determination of the full étale fundamental groups of curves in positive characteristic $p > 0$, see Remarks 6.8 and 11.5.

Problems (2) and (3) are still widely open.

The section conjecture provides a conjectural answer to problem (2) when k is a finitely generated field of characteristic 0 and S is a smooth, separated, geometrically connected hyperbolic curve over k. More precisely, let $S \hookrightarrow S^{cpt}$ denote the smooth compactification of S. Any $s \in S(k)$ induces a ($\pi_1(S_{k^s})$-conjugacy class of) section(s) $s : \Gamma_k \to \pi_1(S)$. More generally, given a point $\tilde{s} \in S^{cpt}(k)$, if $I(\tilde{s})$ and $D(\tilde{s})$ denote the inertia and decomposition group of $\tilde{s}$ in $\Gamma_{k(S^{cpt})}$ respectively, then the short exact sequence:

$$1 \to I(\tilde{s}) \to D(\tilde{s}) \to \Gamma_k \to 1$$

always splits but this splitting is not unique up to inner conjugation by elements of $\Gamma_{\overline{k}(S^{cpt})}$ hence, any point $\tilde{s} \in S^{cpt}(k) \setminus S(k)$ gives rise to several ($\pi_1(S_{k^s})$-conjugacy class of) sections. A section $s : \Gamma_k \to \pi_1(S)$ is said to be *geometric* if it raises from a point $\tilde{s} \in S^{cpt}(k)$ and is said to be *unbranched* if $s(\Gamma_k)$ is contained in no decomposition group of a point $\tilde{s} \in S^{cpt}(k) \setminus S(k)$ in $\pi_1(S)$. Let $\Sigma(S)$ denote the set of conjugacy classes of sections of $\pi_1(S) \twoheadrightarrow \Gamma_k$. A basic form of the section conjecture can thus be formulated as follows:

Conjecture 7.3. (Section conjecture) *For any smooth, separated and geometrically connected curve S over a finitely generated field k of characteristic* 0 *the canonical map $S(k) \to \Sigma(S)$ is injective and induces a bijection onto the set of $\pi_1(S_{k^s})$-conjugacy classes of unbranched sections. Furthermore, any section is a geometric section.*

The injectivity part of the section conjecture was already known to A. Grothendieck (basically as a consequence of Lang-Néron theorem with some technical adjustments in the non-proper case); it is the surjectivity part which is difficult. It easily follows from the formalism of Galois categories, Mordell conjecture and

Uchida's theorem [U77] that the section conjecture (for all hyperbolic curves over k) is equivalent to:

Conjecture 7.4. (Section conjecture – reformulation) *For any smooth, separated and geometrically connected curve S over a finitely generated field k of characteristic 0 one has $S(k) \neq \emptyset$ if and only if $\Sigma(S) \neq \emptyset$.*

One can formulate a pro-p variant of the section conjecture. Let $K^{(p)}$ denote the kernel of the pro-p completion $\pi_1(S_{k^s}) \twoheadrightarrow \pi_1(S_{k^s})^{(p)}$; by definition $K^{(p)}$ is characteristic in $\pi_1(S_{k^s})$ hence normal in $\pi_1(S)$. So, defining $\pi_1(S)^{[p]} := \pi_1(S)/K^{(p)}$, one gets a short exact sequence of profinite groups:

$$1 \to \pi_1(S_{k^s})^{(p)} \to \pi_1(S)^{[p]} \to \Gamma_k \to 1$$

Let $\Sigma^{(p)}(S)$ denote the set of conjugacy classes of sections of $\pi_1(S)^{[p]} \twoheadrightarrow \Gamma_k$ and consider the composite map:

$$S(k) \to \Sigma(S) \to \Sigma^{(p)}(S).$$

Then, S. Mochizuki showed that this remains injective [Mo99] but Y. Hoshi showed that it is no longer surjective [Ho10b].

One can also formulate a birational variant of the section conjecture, where the short exact sequence of profinite group:

$$1 \to \pi_1(S_{k^s}) \to \pi_1(S) \to \Gamma_k \to 1$$

is replaced by the usual short exact sequence from Galois theory of field extensions:

$$1 \to \Gamma_{k^s(S)} \to \Gamma_{k(S)} \to \Gamma_k \to 1$$

In that case, there are some examples where the answer is known to be positive [St07] and the birational section conjecture itself was proved by J. Koenigsmann when k is replaced by a p-adic field [K05].

As for problem (3), it leads to a whole bunch of questions and conjectures usually gathered under the common denomination of *anabelian geometry*. Among those problems one can mention, for instance:

- Is $\rho : \Gamma_k \to \mathrm{Out}(\pi_1(S_{k^s}))$ injective? The answer is known to be positive for smooth, separated, geometrically connected hyperbolic curves over sub-p-adic fields (*i.e.*, subfields of finitely generated extensions of $\mathbb{Q}_p$). The affine case when k is a number field was proved by M. Matsumoto [M96], the general case was completed by Y. Hoshi and S. Mochizuki when k is a sub-p-adic field [HoMo10].
- Given a prime ℓ, up to what extend does the kernel of the outer pro-ℓ representation $\rho^{(\ell)} : \Gamma_k \to \mathrm{Out}(\pi_1(S_{k^s})^{(\ell)})$ determine the isomorphism class of S? Under some technical conditions Y. Hoshi [Ho10a] and S. Mochizuki [Mo03] obtained partial results for affine hyperbolic curves of genus ≤ 1.
- Up to what extend does the outer (resp. the outer pro-ℓ) representation $\rho : \Gamma_k \to \mathrm{Out}(\pi_1(S_{k^s})$ (resp. $\rho^{(\ell)} : \Gamma_k \to \mathrm{Out}(\pi_1(S_{k^s})^{(\ell)})$) determine S? When S is assumed to be an hyperbolic curve, this rather vague question is

often referred to as *Grothendieck's anabelian conjecture*. One motivation for it is Tate conjecture for abelian varieties. Indeed, given two proper hyperbolic curves S_1, S_2 over a finitely generated field k of characteristic 0 then, or any prime ℓ if the outer pro-ℓ abelianized representations:

$$\rho_i^{(\ell),ab} : \Gamma_k \to \mathrm{Out}(\pi_1(S_{i\overline{k}})^{(\ell),ab}) = \mathrm{Aut}(T_\ell(J_{S_i|k}))$$

coincide for $i = 1, 2$ then, $J_{S_1|k}$ and $J_{S_2|k}$ are isogenous. In particular, from the isogeny theorem, there are only finitely many isomorphism classes of proper hyperbolic curves X with the same outer pro-ℓ abelianized representation. It is thus reasonable to expect that taking into account the whole outer pro-ℓ representation or, even more, the whole outer representation, will determine entirely the isomorphism classes of hyperbolic curves. Note that the assumption that S is hyperbolic implies that $\pi_1(S_{\overline{k}})$ has trivial center hence that:

$$\pi_1(S) = \mathrm{Aut}(\pi_1(S_{\overline{k}})) \times_{\mathrm{Out}(\pi_1(S_{\overline{k}})),\rho} \Gamma_k$$

so $\pi_1(S) \twoheadrightarrow \Gamma_k$ can be recovered from $\rho : \Gamma_k \to \mathrm{Out}(\pi_1(S_{\overline{k}}))$. More precisely, one can formulate Grothendieck's anabelian conjecture for hyperbolic curves as follows. Let Pro_k^{open} denote the category of profinite groups G equipped with an epimorphism $p : G \twoheadrightarrow \Gamma_k$ and where morphisms from $p_1 : G_1 \twoheadrightarrow \Gamma_k$ to $p_2 : G_2 \twoheadrightarrow \Gamma_k$ are morphisms from G_1 to G_2 in Pro with representatives $\phi : G_1 \to G_2$ such that:

(i) $\rho_2 \circ \phi = \rho_1$ modulo inner conjugation by elements of Γ_k;
(ii) $\mathrm{im}(\phi)$ is open in G_2.

Conjecture 7.5. (Grothendieck's anabelian conjecture for hyperbolic curves) *Let k be a finitely generated field of characteristic 0. Then the functor $\pi_1(-)$ from the category of smooth, separated, geometrically hyperbolic curves over k with dominant morphisms to Pro_k^{open} is fully faithful.*

After works of K. Uchida [U77], A. Tamagawa proved Conjecture 7.5 for affine hyperbolic curves [T97]. Using techniques from p-adic Hodge theory, S. Mochizuki then proved the general form of Conjecture 7.5 (and, more generally, its pro-ℓ-variant for k a sub-ℓ-adic field) [Mo99]. For an introduction to this subject, see [NMoT01]. For more elaborate surveys, see [Sz00], [H00] and the Bourbaki lecture by G. Faltings [F98].

One can formulate birational, higher-dimensional variants, variants over finite fields or function fields of Conjecture 7.5. These questions are currently intensively studied. For more recent results, see the works of Y. Hoshi, S. Mochizuki, H. Nakamura, F. Pop, M. Saïdi, J. Stix, A. Tamagawa *etc.*

8. G.A.G.A. theorems

In this section, we review implications of the so-called G.A.G.A. theorems (named after J.-P. Serre's fundamental paper [S56] *Géométrie algébrique et géométrie analytique*) to the description of étale fundamental groups of schemes locally of finite

type over $\mathbb{C}$. The main result is Theorem 8.1, which states that this is nothing but the profinite completion of the topological fundamental group of the underlying topological space. However, the definition of what is meant by "underlying topological space" is not so clear *a priori* and the definition – as well of the proof – goes through the complex analytic space X^{an} which can canonically be associated to any scheme X locally of finite type over $\mathbb{C}$. In Subsection 8.1, we give the definition of complex analytic spaces, sketch the construction of the analytification functor $X \mapsto X^{an}$ and provide a partial dictionary of properties which it preserves. In Subsection 8.2, we state the main G.A.G.A. theorem alluded to above. The proof of this theorem is beyond the scope of these notes. For a clear exposition based on [S56] and [Hi64], we refer to [SGA1, Chap. XII, §5].

8.1. Complex analytic spaces

As schemes over $\mathbb{C}$ are obtained by gluing affine schemes over $\mathbb{C}$ in the category $LR/\mathbb{C}$ of locally-ringed spaces in $\mathbb{C}$-algebras, complex analytic spaces are obtained by gluing "affine" complex analytic spaces in $LR/\mathbb{C}$.

Affine complex analytic spaces are defined as follows. Let $U \subset \mathbb{C}^n$ denote the polydisc of all $\underline{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n$ such that $|z_i| < 1$, $i = 1, \ldots, n$ and, given analytic functions $f_1, \ldots, f_r : U \to \mathbb{C}$, let $\mathfrak{U}(f_1, \ldots, f_r)$ denote the locally ringed space in $\mathbb{C}$-algebra whose underlying topological space the closed subset:

$$\bigcap_{i=1}^{r} f_i^{-1}(0) \subset U$$

endowed with the topology inherited from the transcendent topology on U and whose structural sheaf is:

$$\mathcal{O}_U / \langle f_1, \ldots, f_r \rangle,$$

where $\mathcal{O}_U$ is the sheaf of germs of analytic functions on U.

The category $An_{\mathbb{C}}$ of complex analytic spaces is then the full subcategory of $LR/\mathbb{C}$ whose objects $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ are locally isomorphic to affine complex analytic spaces.

Now, let X be a scheme locally of finite type over $\mathbb{C}$

<u>Claim:</u> *The functor* $\mathrm{Hom}_{LR/\mathbb{C}}(-, X) : An_{\mathbb{C}}^{op} \to Sets$ *is representable that is there exists a complex analytic space* X^{an} *and a morphism* $\phi_X : X^{an} \to X$ *in* $LR/\mathbb{C}$ *inducing a functor isomorphism*

$$\phi_X \circ : \mathrm{Hom}_{An_{\mathbb{C}}}(-, X^{an}) \tilde{\to} \mathrm{Hom}_{LR_{\mathbb{C}-Alg}}(-, X)|_{An_{\mathbb{C}}^{op}}.$$

Furthermore, for any $x \in X^{an}$, *the canonical morphism induced on completions of local rings* $\hat{\mathcal{O}}_{X,\phi_X(x)} \tilde{\to} \hat{\mathcal{O}}_{X^{an},x}$ *is an isomorphism.*

Proof (*sketch of*)

1. Assume that X^{an} exists for a given scheme X, locally of finite type over $\mathbb{C}$. Then:
   (a) U^{an} exists for any open subscheme $U \hookrightarrow X$ ($U^{an} = \phi_X^{-1}(U)$ with the structure of complex analytic space induced from the one of X);

(b) Z^{an} exists for any closed subscheme $Z \hookrightarrow X$ (if $\mathcal{I}_Z$ denotes the coherent sheaf of ideals of $\mathcal{O}_X$ defining Z then $\phi_X^{an}\mathcal{I}_Z =: \mathcal{I}_Z^{an}$ is again a coherent sheaf of ideals of $\mathcal{O}_{X^{an}}$ hence defines a closed analytic subspace $Z^{an} \hookrightarrow X^{an}$).

2. Assume that X_i^{an} exists for a given scheme X_i, locally of finite type over $\mathbb{C}$, $i = 1, 2$. Then $(X_1 \times_{\mathbb{C}} X_2)^{an}$ exists and is $X_1^{an} \times X_2^{an}$.
3. $(\mathbb{A}^1_{\mathbb{C}})^{an}$ exists $(= \mathbb{A}^1(\mathbb{C}))$ hence it follows from (2) that$(\mathbb{A}^n_{\mathbb{C}})^{an}$ exists for $n \geq 1$. Then, it follows from (1) (b) that X^{an} exists for any affine scheme, locally of finite type over $\mathbb{C}$.
4. Now, given any scheme X locally of finite type over $\mathbb{C}$, consider a covering of X by open affine subschemes $X_i \hookrightarrow X$, $i \in I$ and set $X_{i,j} := X_i \cap X_j$, $i,\ j \in I$. From (3) and (1) (a), one knows that X_i^{an} and $X_{i,j}^{an}$ exist, $i, j \in I$. Then the analytic space X^{an} obtained by gluing the X_i^{an} along the $X_{i,j}^{an}$ satisfies the required universal property. □

The morphism $\phi_X : X^{an} \to X$ is unique up to a unique X-isomorphism and is called the *complex analytic space associated with* X or the *analytification of* X. In particular, given a $\mathbb{C}$-morphism $f : X \to Y$ of schemes locally of finite type over $\mathbb{C}$, it follows from the universal property of $\phi_Y : Y^{an} \to Y$ that there exists a unique morphism $f^{an} : X^{an} \to Y^{an}$ in $An_{\mathbb{C}}$ such that $\phi_Y \circ f^{an} = f \circ \phi_X$. One readily checks that this gives rise to a functor:

$$(-)^{an} : Sch^{LFT}/\mathbb{C} \to An_{\mathbb{C}},$$

where $Sch^{LFT}/\mathbb{C}$ denotes the category of schemes locally of finite type over $\mathbb{C}$.

There is a nice dictionary between the properties of X (resp. $X \to Y$) and those of X^{an} (resp. $X^{an} \to Y^{an}$). Morally, all those which are encoded in the completion of the local rings are preserved. For instance:

1. Let P be the property of being connected, irreducible, regular, normal, reduced, of dimension d. Then X has P if and only if X^{an} has P;
2. Let P be the property of being surjective, dominant, a closed immersion, finite, an isomorphism, a monomorphism, an open immersion, flat, unramified, étale, smooth. Then $X \to Y$ has P if and only if $X^{an} \to Y^{an}$ has P.

Concerning the categories $\mathrm{Mod}(X)$ and $\mathrm{Mod}(X^{an})$ of $\mathcal{O}_X$-modules and $\mathcal{O}_{X^{an}}$ respectively, one can easily show that the functor:

$$\phi_X^* : \mathrm{Mod}(X) \to \mathrm{Mod}(X^{an})$$

is exact, faithful, conservative and sends coherent $\mathcal{O}_X$-modules to coherent $\mathcal{O}_{X^{an}}$-modules.

8.2. Main G.A.G.A. theorem

The most important result of [S56] is that, when X is assumed to be projective over $\mathbb{C}$, the functor $\phi_X^* : \mathrm{Mod}(X) \to \mathrm{Mod}(X^{an})$ induces an equivalence of categories from coherent $\mathcal{O}_X$-modules to coherent $\mathcal{O}_{X^{an}}$-modules. By technical arguments such as Chow's lemma, this can be extended to schemes proper over $\mathbb{C}$. From the equivalence of categories between finite morphisms $Y \to X$ (resp. $Y^{an} \to X^{an}$)

and coherent $\mathcal{O}_X$-algebras (resp. coherent $\mathcal{O}_{X^{an}}$-algebras), one easily deduces that for a proper schemes X over $\mathbb{C}$ the categories of finite étale covers of X and X^{an} are equivalent. Working more, one gets:

Theorem 8.1. ([SGA1, XII, Thm. 5.1]) *For any scheme X locally of finite type over $\mathbb{C}$, the functor $(-)^{an} : Sch^{LFT}/\mathbb{C} \to An_{\mathbb{C}}$ induces an equivalence from the category of étale covers of X to the category of étale covers of X^{an}.*

The category of étale covers of X^{an} is equivalent to the category of finite topological covers of the underlying transcendent topological space X^{top} of X^{an}. Indeed, observe that if $f : Y \to X^{\mathrm{top}}$ is a finite topological cover then the local trivializations endow Y with a unique structure of analytic space (induced from X^{an}) and such that, with this structure, $f : Y \to X^{\mathrm{top}}$ becomes an analytic cover. Conversely, if $f : Y \to X^{an}$ is an étale cover then, from Theorem 5.5, for any $y \in Y$ one can find open affine neighborhoods $V = \mathrm{spec}(B)$ of y and $U = \mathrm{spec}(A)$ of $f(y)$ such that $f(V) \subset U$, $B = A[X]/\langle f\rangle$ and $(\frac{\partial f}{\partial X})_y \in \mathcal{O}_{Y,y}^{\times}$ hence the local inversion theorem gives local trivializations. So, for any $x \in X$ one has a canonical isomorphism of profinite groups :

$$\widehat{\pi_1^{\mathrm{top}}(X^{\mathrm{top}}, x)} \simeq \pi_1(X, x).$$

Example 8.2. Let X be a smooth connected curve over $\mathbb{C}$ of type (g, r) (that is the smooth compactification $\tilde{X}$ of X has genus g and $|\tilde{X} \setminus X| = r$). Then, for any $x \in X$ one has a canonical profinite group isomorphism $\widehat{\Gamma}_{g,r} \simeq \pi_1(X, x)$, where $\Gamma_{g,r}$ denotes the group defined by the generators $a_1, \ldots, a_g, b_1, \ldots, b_g, \gamma_1, \ldots, \gamma_r$ with the single relation $[a_1, b_1] \cdots [a_g, b_g]\gamma_1 \cdots \gamma_r = 1$. From Section 6.4, $\pi_1(X, x)$ can also be described as the Galois group $\mathrm{Gal}(M_{\mathbb{C}(X),X}|\mathbb{C}(X))$ of the maximal algebraic extension $M_{\mathbb{C}(X),X}$ of $\mathbb{C}(X)$ in $\overline{\mathbb{C}(X)}$ étale over X.

In particular, if $g = 0$ then $\pi_1(X, x)$ is the pro-free group on $r-1$ generators, so, any finite group G generated by $\leq r-1$ elements is a quotient of $\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{t_1, \ldots, t_r\}, x)$ or, equivalently, appears as the Galois group of a Galois extension $\mathbb{C}(T) \hookrightarrow K$ unramified everywhere except over $t_1, \ldots, t_r$. This solves the inverse Galois problem over $\mathbb{C}(T)$.

Exercise 8.3. Show that the étale fundamental group of an algebraic group over an algebraically closed field of characteristic 0 is commutative.

9. Specialization

9.1. Statements

Let S be a connected scheme and $f : X \to S$ a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_S$ (so, in particular, $f : X \to S$ is surjective, geometrically connected and X is connected). Fix $s_0,\ s_1 \in S$ with $s_0 \in \overline{\{s_1\}}$ and geometric points $\overline{x}_i : \mathrm{spec}(\Omega_i) \to X_{\overline{s}_i}$, $i = 0, 1$. Denote again by $\overline{x}_i$ the images of $\overline{x}_i$ in X_{s_i} and X_i and by $\overline{s}_i$ the image of $\overline{x}_i$ in S, $i = 0, 1$.

The theory of specialization of fundamental groups consists, essentially, in comparing $\pi_1(X_{\overline{s}_1};\overline{x}_1)$ and $\pi_1(X_{\overline{s}_0};\overline{x}_0)$. The main result is the following.

Theorem 9.1. (Semi-continuity of fundamental groups) *There exists a morphism of profinite groups*

$$sp : \pi_1(X_{\overline{s}_1};\overline{x}_1) \to \pi_1(X_{\overline{s}_0};\overline{x}_0),$$

canonically defined up to inner automorphisms of $\pi_1(\overline{X}_0,\overline{x}_0)$*. If, furthermore,* $f : X \to S$ *is separable, then* $sp : \pi_1(X_{\overline{s}_1};\overline{x}_1) \twoheadrightarrow \pi_1(X_{\overline{s}_0};\overline{x}_0)$ *is an epimorphism.*

The morphism $sp : \pi_1(X_{\overline{s}_1},\overline{x}_1) \to \pi_1(X_{\overline{s}_0},\overline{x}_0)$ is called the *specialization morphism from* s_1 *to* s_0.

The proof of Theorem 9.1 relies on the first homotopy sequence, already studied in Subsection 6.2 but that we restate below with our notation.

Theorem 9.2. (First homotopy sequence) *Consider the canonical sequence of profinite groups induced by* $(X_{\overline{s}_1},\overline{x}_1) \to (X,\overline{x}_1) \to (S,\overline{s}_1)$*:*

$$\pi_1(X_{\overline{s}_1};\overline{x}_1) \xrightarrow{i_1} \pi_1(X;\overline{x}_1) \xrightarrow{p_1} \pi_1(S;\overline{s}_1). \tag{3}$$

Then $p_1 : \pi_1(X;\overline{x}_1) \twoheadrightarrow \pi_1(S;\overline{s}_1)$ *is an epimorphism and* $\mathrm{im}(i_1) \subset \ker(p_1)$*. If, furthermore,* $f : X \to S$ *is separable then* $\mathrm{im}(i_1) = \ker(p_1)$*.*

and the second homotopy sequence:

Theorem 9.3. (Second homotopy sequence) *Assume that* $S = \mathrm{Spec}(A)$ *with* A *a local complete noetherian ring and that* s_0 *is the closed point of* S*. Then, the canonical sequence of profinite groups induced by* $(X_{\overline{s}_0},\overline{x}_0) \to (X,\overline{x}_0) \to (S,\overline{s}_0)$*:*

$$1 \to \pi_1(X_{\overline{s}_0};\overline{x}_0) \xrightarrow{i_0} \pi_1(X;\overline{x}_0) \xrightarrow{p_0} \pi_1(S;\overline{s}_0) \to 1 \tag{4}$$

is exact and the canonical morphism $\Gamma_{k(s_0)} \tilde{\to} \pi_1(S;\overline{s}_0)$ *is an isomorphism. In particular, the canonical morphism* $\pi_1(X_{s_0};\overline{x}_0) \tilde{\to} \pi_1(X;\overline{x}_0)$ *is an isomorphism and if* $x_0 \in X(k(s_0))$ *then the above short exact sequence splits.*

9.2. Construction of the specialization morphism

Assume first that $S = \mathrm{Spec}(A)$ with A a local complete noetherian ring and that s_0 is the closed point of S, $s_1 \in S$ is any point of S. Then, one has the following canonical diagram of profinite groups, which commutes up to inner automorphisms:

$$\begin{tikzcd}
(4) & 1 \arrow[r] & \pi_1(X_{\overline{s}_0};\overline{x}_0) \arrow[r, "i_0"] & \pi_1(X;\overline{x}_0) \arrow[r, "p_0"] & \pi_1(S;\overline{s}_0) \arrow[r] & 1 \\
(3) & & \pi_1(X_{\overline{s}_1};\overline{x}_1) \arrow[r, "i_1"] \arrow[u, dotted, "\exists!\ sp"] & \pi_1(X;x\overline{x}_1) \arrow[r, "p_1"] \arrow[u, "\alpha_X"] & \pi_1(S;\overline{s}_1) \arrow[r] \arrow[u, "\alpha_S"] & 1,
\end{tikzcd}$$

where the vertical arrows $\alpha_X : \pi_1(X;\overline{x}_1) \tilde{\to} \pi_1(X;\overline{x}_0)$ and $\alpha_S : \pi_1(S;\overline{s}_1) \tilde{\to} \pi_1(S;\overline{s}_0)$ are the canonical (up to inner automorphisms) isomorphisms of Theorem 2.8.

Now, since $p_0 \circ \alpha_X \circ i_1$ " = " $\alpha_S \circ p_1 \circ i_1 \overset{(*)}{=} 0$ (here " = " means equal up to inner automorphisms and equality $(*)$ comes from Theorem 9.3), it follows from Theorem 9.2 that:

$$\mathrm{im}(\alpha_X \circ i_1) \subset \ker(p_0) = \mathrm{im}(i_0)$$

and, hence, there exists a morphism of profinite groups:

$$sp : \pi_1(X_{\overline{s}_1}; \overline{x}_1) \to \pi_1(X_{\overline{s}_0}; \overline{x}_0),$$

unique up to inner automorphisms and such that $\alpha_X \circ p_1$ "=" $i_0 \circ sp$.

If, furthermore, $\mathrm{im}(i_1) = \ker(p_1)$, a straightforward diagram chasing shows that:

$$sp : \pi_1(X_{\overline{s}_1}; \overline{x}_1) \twoheadrightarrow \pi_1(X_{\overline{s}_0}; \overline{x}_0)$$

is an epimorphism.

We come back to the case where S is any locally noetherian scheme and $s_0,\ s_1 \in S$ with $s_0 \in \overline{\{s_1\}}$. One then has a commutative diagram (where we abbreviate $\mathrm{spec}(K)$ by K when K is a field):

$$\begin{tikzcd}
\overline{k(s_1)} \arrow[r] & k(s_1) \arrow[rr, "s_1"] \arrow[dr] & & S & & k(s_0) \arrow[ll, "s_0"'] \arrow[dl] \arrow[d, equal] & \overline{k(s_0)} \arrow[l] \arrow[d, equal] \\
& & \mathrm{spec}(\mathcal{O}_{S,s_1}) \arrow[r] & \mathrm{spec}(\mathcal{O}_{S,s_0}) \arrow[u] & & & \\
\overline{k(\hat{s}_1)} \arrow[r] \arrow[uu] & k(\hat{s}_1) \arrow[rr, "\hat{s}_1"'] \arrow[uu] & & \mathrm{spec}(\widehat{\mathcal{O}}_{S,s_0}) \arrow[u] & & k(\hat{s}_0) \arrow[ll, "\hat{s}_0"] \arrow[u, equal] & \overline{k(\hat{s}_0)}, \arrow[l] \arrow[u, equal]
\end{tikzcd}$$

where the existence of $\hat{s}_1$ is ensured by the fact that $\mathcal{O}_{S,s_0} \to \widehat{\mathcal{O}}_{S,s_0}$ is faithfully (flat). Choose a geometric point $\overline{\hat{x}}_1$ of $\overline{X}_{\overline{s}_1} := X_{\overline{s}_1} \times_{\overline{k(s_1)}} \overline{k(\hat{s}_1)}$ over $\overline{x}_1$. Since $X_{\widehat{\mathcal{O}}_{S,s_0}} \to \mathrm{spec}(\widehat{\mathcal{O}}_{S,s_0})$ is proper (and separable as soon as $f : X \to S$ is), it follows from (1) that one has a canonical specialization morphism:

$$(*) \quad sp : \pi_1(\overline{X}_{\overline{s}_1}; \overline{\hat{x}}_1) \to \pi_1(X_{\overline{s}_0}; \overline{x}_0)$$

and, from Corollary 6.6, the canonical morphism:

$$(**) \quad \pi_1(\overline{X}_{\overline{s}_1}; \overline{\hat{x}}_1) \tilde{\to} \pi_1(X_{\overline{s}_1}; \overline{x}_1)$$

is an isomorphism. Thus the specialization isomorphism is obtained by composing the inverse of $(**)$ with $(*)$.

9.3. Proof of Theorem 9.2

The proof resorts to difficult results from [EGA3]; we will only sketch it but give references for the missing details. See also [I05] for a more detailed treatment.

Claim 1: *If A is a local artinian ring, the conclusions of Theorem* 9.2 *hold.*

Proof of Claim 1. Recall that, in an Artin ring, any prime ideal is maximal hence the nilradical and the Jacobson radical coincide. In particular, if A is local, the nilpotent elements of A are precisely those of its maximal ideal. From Theorem A.2, one may thus assume that $A = k(s_0)$ and, then, the conclusion $\pi_1(S, \overline{s}_0) \simeq \Gamma_{k(s_0)}$ is straightforward. Let $k(s_0)^i$ denote the inseparable closure of $k(s_0)$ in $\overline{k(s_0)}$ and write $X^i_{s_0} := X \times_S k(s_0)^i$. Then the cartesian diagram:

$$\begin{tikzcd} X_{\overline{s}_0} \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\square"] & X \arrow[r] & S \\ X_{\overline{s}_0} \arrow[r] & X^i_{s_0} \arrow[r] \arrow[u] & \mathrm{Spec}(k(s_0)^i) \arrow[u] \end{tikzcd} \tag{5}$$

induces a commutative diagram of morphisms of profinite groups:

$$\begin{tikzcd} \pi_1(X_{\overline{s}_0}; \overline{x}_0) \arrow[r] & \pi_1(X; x_{(0)}) \arrow[r] & \pi_1(S; \overline{s}_0) \\ \pi_1(X_{\overline{s}_0}; \overline{x}_0) \arrow[r] \arrow[u] & \pi_1(X^i_0; x^i_{(0)}) \arrow[r] \arrow[u] & \pi_1(\mathrm{Spec}(k(s_0)^i); \overline{s}^i_0) \arrow[u] \end{tikzcd} \tag{6}$$

Now, since each of the vertical arrows in (5) is faithfully flat, quasi-compact and radicial, it follows from Corollary A.4 that the vertical arrows in (6) are isomorphisms of profinite groups. Hence it is enough to prove that the bottom line of (6) is exact that is one may assume that $k(s_0)$ is perfect.

But, then, $\overline{k(s_0)}$ can be written as the inductive limit of its finite Galois subextensions $k(s_0) \hookrightarrow k_i \hookrightarrow \overline{k(s_0)}$, $i \in I$ hence, writing again $\overline{x}_0$ for the image of $\overline{x}_0$ in X_{k_i}, it follows from Lemma 6.7 that the morphism:

$$X_{\overline{s}_0} \to \varinjlim X_{ki}$$

induces an isomorphism of profinite groups:

$$\pi_1(X_{\overline{s}_0}; \overline{x}_0) \xrightarrow{\sim} \varprojlim \pi_1(X_{k_i}; \overline{x}_0).$$

But, for each $i \in I$, the étale cover $X_{k_i} \to X$ is Galois with group $\mathrm{Aut}_{Alg/k(s_0)}(k_i)$ so, from Proposition 4.4 one has a short exact sequence of profinite groups:

$$1 \to \pi_1(X_{k_i}; \overline{x}_0) \to \pi_1(X; \overline{x}_0) \to \mathrm{Aut}_{Alg/k(s_0)}(k_i) \to 1.$$

Using that the projective limit functor is exact in the category of profinite groups, we thus get the expected short exact sequence of profinite groups:

$$1 \to \varprojlim \pi_1(X_{k_i}; \overline{x}_0) \to \pi_1(X; \overline{x}_0) \to \Gamma_{k(s_0)} \to 1.$$

<u>Claim 2:</u> *The closed immersion* $i_{X_{s_0}} : X_{s_0} \hookrightarrow X$ *induces an equivalence of categories* $\mathcal{C}_X \to \mathcal{C}_{X_{s_0}}$ *hence, in particular, an isomorphism of profinite groups:*

$$\pi_1(X_{s_0}; \overline{x}_0) \tilde{\to} \pi_1(X; \overline{x}_0).$$

Proof of Claim 2. One has to prove:

1. For any étale covers $p : Y \to X$, $p' : Y' \to X$ the canonical map
$$\mathrm{Hom}_{\mathcal{C}_X}(p, p') \to \mathrm{Hom}_{\mathcal{C}_{X_{s_0}}}(p \times_X X_{s_0}, p' \times_X X_{s_0})$$
is bijective;
2. For any étale cover $p_0 : Y_0 \to X_{s_0}$ there exists an étale cover $p : Y \to X$ which is a model of $p_0 : Y_0 \to X_{s_0}$ over X.

The proof of these two assertions is based on Grothendieck's Comparison and Existence theorems in algebraic-formal geometry. We first state simplified versions of these theorems.

Let S be a noetherian scheme and let $p : X \to S$ be a proper morphism. Let $\mathcal{I} \subset \mathcal{O}_S$ be a coherent sheaf of ideals. Then the descending chains $\cdots \subset \mathcal{I}^{n+1} \subset \mathcal{I}^n \subset \cdots \subset \mathcal{I}$ corresponds to a chain of closed subschemes $S_0 \hookrightarrow S_1 \hookrightarrow \cdots \hookrightarrow S_n \hookrightarrow \cdots \hookrightarrow S$. We will use the notation in the diagram below:

$$\begin{tikzcd}
S & S_n \arrow[l, hook'] & \cdots \arrow[l, hook'] & S_1 \arrow[l, hook'] & S_0 \arrow[l, hook'] \\
X \arrow[u, "p"] \arrow[ur, phantom, "\square"] & X_n \arrow[l, hook'] \arrow[u, "p_n"] \arrow[ur, phantom, "\square"] & \cdots \arrow[l, hook'] & X_1 \arrow[l, hook'] \arrow[u, "p_1"] \arrow[ur, phantom, "\square"] & X_0 \arrow[l, hook'] \arrow[u, "p_0"]
\end{tikzcd}$$

and write $i_n : X_n \hookrightarrow X$, $n \geq 0$. For any coherent $\mathcal{O}_X$-module $\mathcal{F}$, set $\mathcal{F}_n := i_n^* \mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}$, $n \geq 0$. Then $\mathcal{F}_n$ is a coherent $\mathcal{O}_{X_n}$-module and the canonical morphism of $\mathcal{O}_X$-modules $\mathcal{F} \to \mathcal{F}_n$ induces morphism of $\mathcal{O}_S$-modules $\mathrm{R}^q p_* \mathcal{F} \to \mathrm{R}^q p_* \mathcal{F}_n$, $q \geq 0$ hence morphism of $\mathcal{O}_{S_n}$-modules:

$$(\mathrm{R}^q p_* \mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_n} \to \mathrm{R}^q p_* \mathcal{F}_n, \ q \geq 0$$

and, taking projective limit, canonical morphisms:

$$\varprojlim((\mathrm{R}^q p_* \mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_n}) \to \varprojlim \mathrm{R}^q p_* \mathcal{F}_n, \ q \geq 0.$$

When $S = \mathrm{spec}(A)$ is affine and $I \subset A$ is the ideal corresponding to $\mathcal{I} \subset \mathcal{O}_S$, the above isomorphism becomes:

$$\mathrm{H}^q(X, \mathcal{F}) \otimes_A \widehat{A} \tilde{\to} \varprojlim \mathrm{H}^q(X_n, \mathcal{F}_n), \ q \geq 0,$$

where $\widehat{A}$ denotes the completion of A with respect to the I-adic topology.

Theorem 9.4. (Comparison theorem [EGA3, (4.1.5)]) *The canonical morphisms:*

$$\varprojlim((\mathrm{R}^q p_* \mathcal{F}) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_n}) \tilde{\to} \varprojlim \mathrm{R}^q p_* \mathcal{F}_n, \ q \geq 0$$

are isomorphisms.

Theorem 9.5. (Existence theorem [EGA3, (5.1.4)]) *Assume, furthermore that* $S = \mathrm{spec}(A)$ *is affine and that* A *is complete with respect to the* I*-adic topology. Let* $\mathcal{F}_n$, $n \geq 0$ *be coherent* $\mathcal{O}_{X_n}$*-modules such that* $\mathcal{F}_{n+1} \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n} \tilde{\to} \mathcal{F}_n$, $n \geq 0$. *Then there exists a coherent* $\mathcal{O}_X$*-module* $\mathcal{F}$ *such that* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n} \tilde{\to} \mathcal{F}_n$, $n \geq 0$.

Also, for any étale cover $p : Y \to X$, observe that $\mathcal{A}(p) := p_* \mathcal{O}_Y$ is a locally free $\mathcal{O}_X$-algebra of finite rank and that, denoting by $FLFAlg/\mathcal{O}_X$ the category of locally free $\mathcal{O}_X$-algebra of finite rank the functor:

$$\begin{array}{rlcl} \mathcal{A}: & \mathcal{C}_X & \to & FLFAlg/\mathcal{O}_X \\ & p : Y \to X & \to & \mathcal{A}(p) \end{array}$$

is fully faithful.

<u>Proof of (1):</u> One has canonical functorial isomorphisms:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}_X}(p, p') & \tilde{\to} & \mathrm{H}^0(X, \underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_X}(\mathcal{A}(p'), \mathcal{A}(p))) \\ & \tilde{\to} & \varprojlim \mathrm{H}^0(X_n, \underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_X}(\mathcal{A}(p'), \mathcal{A}(p)) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}), \end{array}$$

where the first isomorphism comes from the fact that $\mathcal{A}$ is fully faithful and the second isomorphism is just the comparison theorem applied to $q = 0$, $\mathcal{F} = \underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_X}(\mathcal{A}(p'), \mathcal{A}(p))$ and I the maximal ideal of A, observing that, since A is complete with respect to the I-adic topology, $A = \widehat{A}$.

Furthermore, as $\mathcal{A}(p)$, $\mathcal{A}(p')$ are locally free $\mathcal{O}_X$-module, one has canonical isomorphisms:

$$\underline{\mathrm{Hom}}_{\mathrm{Mod}(X)}(\mathcal{A}(p'), \mathcal{A}(p)) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n} \tilde{\to} \underline{\mathrm{Hom}}_{\mathrm{Mod}(X_n)}(\mathcal{A}(p'_n), \mathcal{A}(p_n))$$

But these preserve the structure of $\mathcal{O}_X$-algebra morphisms hence one also gets, by restriction:

$$\underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_X}(\mathcal{A}(p'), \mathcal{A}(p)) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n} \tilde{\to} \underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_{X_n}}(\mathcal{A}(p'_n), \mathcal{A}(p_n)).$$

Whence,

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}_X}(p, p') & \tilde{\to} & \varprojlim \mathrm{H}^0(X_n, \underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_X}(\mathcal{A}(p'), \mathcal{A}(p)) \otimes_{\mathcal{O}_X} \mathcal{O}_{X_n}) \\ & \tilde{\to} & \varprojlim \mathrm{H}^0(X_n, \underline{\mathrm{Hom}}_{FLFAlg/\mathcal{O}_{X_n}}(\mathcal{A}(p'_n), \mathcal{A}(p_n))) \\ & \tilde{\to} & \varprojlim \mathrm{Hom}_{\mathcal{C}_{X_n}}(p_n, p'_n) \\ & \tilde{\to} & \varprojlim \mathrm{Hom}_{\mathcal{C}_{X_{s_0}}}(p_0, p'_0), \end{array}$$

where the last isomorphism comes from the fact $\mathrm{Hom}_{\mathcal{C}_{X_n}}(p_n, p'_n) \tilde{\to} \mathrm{Hom}_{\mathcal{C}_{X_{s_0}}}(p_0, p'_0)$, $n \geq 0$ by Theorem A.2.

<u>Proof of (2):</u> By Theorem A.2, there exist étale covers $p_n : Y_n \to X_n$, $n \geq 0$ such that $p_n \tilde{\to} p_{n+1} \times_{X_{n+1}} X_n$, or, equivalently, $\mathcal{A}(p_{n+1}) \otimes_{\mathcal{O}_{X_{n+1}}} \mathcal{O}_{X_n} \tilde{\to} \mathcal{A}(p_n)$, $n \geq 0$. So, by the Existence theorem, there exists a locally free $\mathcal{O}_X$-algebra of finite rank $\mathcal{A}$ such that $\mathcal{A} \otimes_{\mathcal{O}_{X_n}} \mathcal{O}_X \tilde{\to} \mathcal{A}(p_n)$, $n \geq 0$ hence, setting $p : Y = \mathrm{spec}\ (\mathcal{A}) \to X$ one has $p \times_X X_{s_0} \tilde{\to} p_0$.

It remains to show that $p : Y = \text{spec}\ (\mathcal{A}) \to X$ is an étale cover. For this, see [Mur67, pp. 159–161].

One can now conclude the proof. From Claim 1 applied to $A = k(s_0)$, $X = X_{s_0}$, one gets the short exact sequence of profinite groups:

$$1 \to \pi_1(X_{\overline{s}_0}; \overline{x}_0) \to \pi_1(X_{s_0}; \overline{x}_0) \to \Gamma_{k(s_0)} \to 1.$$

Now, from Claim 2 one has the canonical profinite group isomorphisms $\pi_1(X; \overline{x}_0) \tilde{\to} \pi_1(X_{s_0}; \overline{x}_0)$ and (for $X = S$) $\pi_1(S; \overline{s}_0) \tilde{\to} \Gamma_{k(s_0)}$, which yields the required short exact sequence.

Eventually, for the last assertion of Theorem 9.2, just observe that, as above, one can assume that $A = k(s_0)$ thus, if $x \in X(k(s_0))$, it produces a section $x : S \to X$ of $f : X \to S$ such that $x \circ s_0 = x$ thus a section $\Gamma_{k(s_0)} \to \pi_1(X; \overline{x}_0)$ of (4). □

10. Purity and applications

In this section, we use Zariski-Nagata purity theorem to prove that the étale fundamental group is a birational invariant in the category of proper regular schemes over a field and to determine the kernel of the specialization epimorphism constructed in Section 9.

Theorem 10.1. (Zariski-Nagata purity theorem [SGA2, Chap. X, Thm. 3.4]) *Let X, Y be integral schemes with X normal and Y regular. Let $f : X \to Y$ be a quasi-finite dominant morphism and let $Z_f \subset X$ denote the closed subset of all $x \in X$ such that $f : X \to Y$ is not étale at x. Then, either $Z_f = X$ or Z_f is pure of codimension* 1 *(that is, for any generic point $\eta \in Z_f$, one has* $\dim(\mathcal{O}_{X,\eta}) = 1$*).*

10.1. Birational invariance of the étale fundamental group

Corollary 10.2. *Let X be a connected, regular scheme and let $i_U : U \hookrightarrow X$ be an open subscheme such that $X \setminus U$ has codimension ≥ 2 in X. Then $i_U : U \hookrightarrow X$ induces an equivalence of categories:*

$$i_U^* : \mathcal{C}_X \to \mathcal{C}_U$$

hence an isomorphism of profinite groups:

$$\pi_1(i_U) : \pi_1(U) \tilde{\to} \pi_1(X).$$

Proof. As X is connected, locally noetherian and regular (hence with integral local rings), X is irreducible. Since X is normal and $X \setminus U \subset X$ is a closed subset of codimension ≥ 2, the functor $i_U^* : \mathcal{C}_X \to \mathcal{C}_U$ is fully faithful [L00, Thm. 4.1.14] hence, one only has to prove that it is also essentially surjective that is, for any étale cover $p_U : V \to U$ there exists a (necessarily unique by the above) étale cover $p : Y \to X$ such that $p_U : V \to U$ is the base-change of $p : Y \to X$ *via* $i_U := U \hookrightarrow X$. One may assume that V is connected hence, it follows from Lemma 6.14 that V is the normalization of U in $k(X) = k(U) \hookrightarrow k(V)$. Let $p : Y \to X$ be the normalization of X in $k(X) \hookrightarrow k(V)$. Then, on the one hand,

it follows from the universal property of normalization that $p_U : V \to U$ is the base-change of $p : Y \to X$ *via* $i_U := U \hookrightarrow X$ as expected. On the other hand, since X is normal and $k(X) \hookrightarrow k(V)$ is a finite separable field extension, $p : Y \to X$ is finite, dominant and, from Lemma 6.15, étale on:

$$p^{-1}(U) = V = Y \setminus p^{-1}(X \setminus U).$$

But $X \setminus U$ has codimension ≥ 2 in X hence, since $p : Y \to X$ is finite, $p^{-1}(X \setminus U)$ has codimension ≥ 2 in Y as well. Thus, it follows from Theorem 10.1 that $p : Y \to X$ is étale. $\square$

Let X be a connected, regular scheme, Y a connected scheme and $f : X \rightsquigarrow Y$ be a rational map. Write $U_f \subset X$ for the maximal open subset on which $f : X \rightsquigarrow Y$ is defined and assume that $X \setminus U_f$ has codimension ≥ 2 in X. Then, corresponding to the sequence of base-change functors:

$$\mathcal{C}_Y \overset{f|_{U_f}^*}{\to} \mathcal{C}_{U_f} \overset{i_{U_f}^*}{\leftarrow} \mathcal{C}_X$$

one has, for any geometric point $x \in U_f$, the sequence of morphisms of profinite groups:

$$\pi_1(X; x) \overset{\pi_1(i_{U_f})}{\leftarrow} \pi_1(U_f; x) \overset{\pi_1(f|_{U_f})}{\to} \pi_1(Y; f(x)).$$

So, if $\mathcal{C}$ denotes the category of all connected, regular schemes pointed by geometric points in codimension 1 together with dominant rational maps defined on an open subscheme whose complement has codimension ≥ 2 one gets a well-defined functor $\pi_1(-)$ from $\mathcal{C}$ to the category of profinite groups. In particular, let k be a field, X, Y two schemes proper over k, connected and regular and $f : X \leftrightsquigarrow Y$ a birational map of schemes over k. Then f is always defined over an open subscheme $i_{U_f} : U_f \hookrightarrow X$ such that $X \setminus U_f$ has codimension ≥ 2 in X and the same holds for f^{-1}. So, from Corollary 10.2, one gets a sequence of isomorphisms of profinite groups:

$$\pi_1(X) \overset{\pi_1(i_{U_f})^{-1}}{\tilde{\to}} \pi_1(U_f) \overset{\pi_1(f|_{U_f}^{U_{f^{-1}}})}{\tilde{\to}} \pi_1(U_{f^{-1}}) \overset{\pi_1(i_{U_f^{-1}})}{\tilde{\to}} \pi_1(Y).$$

Example 10.3. Let k be any field and consider the blowing-up $f : B_x \to \mathbb{P}^2_k$ of $\mathbb{P}^2_k$ at any point $x \in \mathbb{P}^2_k$. Then for any geometric point $b \in B_x$:

$$\pi_1(B_x; b) \tilde{\to} \pi_1(\mathbb{P}^2_k; f(b)).$$

However, B_x and $\mathbb{P}^2_k$ are not k-isomorphic (any two curves in $\mathbb{P}^2_k$ intersects whereas the exceptional divisor E in B_x does not intersect the inverse images of the curves in $\mathbb{P}^2_k$ passing away from x). This shows that one has to be careful when formulating higher-dimensional variants of Conjecture 7.5.

10.2. Kernel of the specialization morphism

We retain the notation of §9. Let S be a locally noetherian scheme and $X \to S$ a *smooth*, proper, geometrically connected morphism. The aim of this section is to determine the kernel of the specialization epimorphism:

$$sp : \pi_1(X_{\overline{s}_1}; \overline{x}_1) \twoheadrightarrow \pi_1(X_{\overline{s}_0}; \overline{x}_0)$$

constructed in Section 9 namely, to prove:

Theorem 10.4. *For any finite group G of order prime to the residue characteristic p of S at s_0 and for any profinite group epimorphism $\phi : \pi_1(X_{\overline{s}_1}; \overline{x}_1) \twoheadrightarrow G$ there exists an epimorphism of profinite groups $\phi_0 : \pi_1(X_{\overline{s}_0}; \overline{x}_0) \twoheadrightarrow G$ such that $\phi_0 \circ sp = \phi$. In particular, sp induces an isomorphism of profinite groups:*

$$sp^{(p)'} : \pi_1(X_{\overline{s}_1}; \overline{x}_1)^{(p)'} \xrightarrow{\sim} \pi_1(X_{\overline{s}_0}; \overline{x}_0)^{(p)'},$$

where $(-)^{(p)'}$ denotes the prime-to-p profinite completion.

Proof. After reducing to the case where $S = \mathrm{spec}(\mathcal{O})$ with $\mathcal{O}$ a complete discrete valuation ring with algebraically closed residue field, the proof of Theorem 10.4 amounts to showing the following. Given an étale cover $Y \to X_{s_1}$ Galois with group G of prime-to-p order n, there exists a finite field subextension $K \hookrightarrow L \hookrightarrow K^s$ such that the extension $k(X).L \hookrightarrow k(Y).L$ be unramified over $X \times_S S^L$, where $S^L := \mathrm{spec}(\mathcal{O}^L)$. Zariski-Nagata purity theorem actually shows that it is enough to construct $K \hookrightarrow L$ in such a way that $k(X).L \hookrightarrow k(Y).L$ be unramified only over the points above the generic point of the closed fibre of X. Such a L can be constructed by Abhyankar's lemma.

<u>Claim 1:</u> *One may assume that $S = \mathrm{spec}(\mathcal{O})$, with $\mathcal{O}$ a complete discrete valuation ring with algebraically closed residue field.*

Proof of Claim 1. Let $s_0 = t_0, t_1, \ldots, t_r = s_1 \in S$ such that $t_i \in \overline{\{t_{i+1}\}}$ and $\mathcal{O}_{\overline{\{t_{i+1}\}}, t_i}$ has dimension 1, $i = 0, \ldots, r-1$. Then, one has the sequence of specialization epimorphisms:

$$\pi_1(X_{\overline{s}_1}) \twoheadrightarrow \pi_1(X_{\overline{t}_{r-1}}) \twoheadrightarrow \cdots \twoheadrightarrow \pi_1(X_{\overline{t}_1}) \twoheadrightarrow \pi_1(X_{\overline{s}_0}).$$

Thus, without loss of generality, we may assume that $\dim(\mathcal{O}_{\overline{\{s_1\}}, s_0}) = 1$. Next, let R denote the strict henselianization of the integral closure of $\mathcal{O}_{\overline{\{s_1\}}, s_0}$ and let $R \hookrightarrow \hat{R}$ denotes its completion. Then $\hat{R}$ is a complete discrete valuation ring with separably closed residue field and the canonical morphism $\mathrm{spec}(\hat{R}) \to S$ maps the generic point of $\mathrm{spec}(\hat{R})$ to s_1 and the closed point of $\mathrm{spec}(\hat{R})$ to s_0.

We will use the following notation for $\mathcal{O}$. Given a finite Galois extension L/K we will write $\mathcal{O}^L$ for the integral closure of $\mathcal{O}$ in L and $e_{L/K}(\mathcal{O})$ for the order of the inertia group of $\mathcal{O}$ in L/K. Now fix an algebraic closure $K \hookrightarrow \overline{K}$ of the fraction field K of $\mathcal{O}$ and let $K \hookrightarrow K^s$ be the separable closure of K in $\overline{K}$. For simplicity, we remove the reference to the base point in the notation below.

From Theorem 9.2, and the construction of the specialization morphism, one has the following situation:

$$\begin{tikzcd}
\pi_1(X_{\overline{s}_1}) \arrow[r] \arrow[dr, "sp"'] & \pi_1(X) & \pi_1(X_{s_0}) \arrow[l, "\simeq"'] \\
& & \pi_1(X_{\overline{s}_1}) \arrow[u, hook]
\end{tikzcd}$$

which shows that:

$$\ker(sp) = \ker(\pi_1(X_{\overline{s}_1}) \to \pi_1(X)).$$

Consider the following factorization of $\overline{s}_1 : \mathrm{spec}(\overline{K}) \to S$:

$$\begin{tikzcd}
\mathrm{spec}(\overline{K}) \arrow[r] \arrow[d] \arrow[rr, bend left, "\overline{s}_1"] & \mathrm{spec}(K) \arrow[r, "s_0"] & S \\
\mathrm{spec}(K^s). \arrow[urr, "s_1^s"'] & &
\end{tikzcd}$$

Since $\mathrm{spec}(\overline{K}) \to \mathrm{spec}(K^s)$ is faithfully flat, quasi-compact and radicial, it follows from Corollary A.4 that the morphism of profinite groups:

$$\pi_1(X_{\overline{s}_1}) \tilde{\to} \pi_1(X_{s_1^s})$$

is an isomorphism. Hence:

$$\ker(sp) = \ker(\pi_1(X_{s_1^s}) \to \pi_1(X)).$$

Let $K \hookrightarrow L$ be a finite field extension. Then $\mathcal{O}^L$ is again a complete discrete valuation ring. Set $S^L := \mathrm{spec}(\mathcal{O}^L)$ and write $s_{L,1}$, $s_{L,0}$ for its generic and closed points respectively. Note that $k(s_0) = k(s_{L,0}) = k$ since k is algebraically closed.

<u>Claim 2:</u> *The morphism of profinite groups:*

$$\pi_1(X \times_S S^L) \tilde{\to} \pi_1(X)$$

induced by $X \times_S S^L \to X$ *is an isomorphism.*

Proof of Claim 2. From Theorem 9.2, one has the following commutative diagram with exact row:

$$\begin{tikzcd}
1 \arrow[r] & \pi_1((X \times_S S^L)_{s_{L,0}}) \arrow[r] \arrow[d] & \pi_1(X \times_S S^L) \arrow[r] \arrow[d] & \pi_1(S^L) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \pi_1(X_{s_0}) \arrow[r] & \pi_1(X) \arrow[r] & \pi_1(S) \arrow[r] & 1.
\end{tikzcd}$$

But since $k(s_0) = k(s_{L,0}) = k$ is algebraically closed one has $\pi_1(S) = \Gamma_{k(s_0)} = 1$, $\pi_1(S^L) = \Gamma_{k(s_{L,0})} = 1$ and $X_{s_0} = (X \times_S S^L)_{s_{L,0}}$, whence the conclusion.

So, one can replace freely K by any finite separable field extension.

From Lemma 4.2 (2), the assertion of Theorem 10.4 amounts to showing that for any étale cover $Y \to X_{s_1^s}$ Galois with group G of prime-to-p order n,

there exists a finite separable field subextension $K \hookrightarrow L \hookrightarrow K^s$ and an étale cover $Y^L \to X \times_S S^L$ Galois with group G which is a model of $Y \to X_{s_1^s}$ over $X \times_S S^L$.

Since K^s is the inductive limit of the finite extensions of K contained in K^s, by the argument of the proof of Proposition 6.7, there exists a finite separable extension $K \hookrightarrow L$ and an étale cover $Y^{0L} \to X_L$ Galois with group G which is a model of $Y \to X_{s_1^s}$ over X_L. Thus, from Claim 2, we are to prove:

<u>Claim 3:</u> *For any étale cover $Y \to X_{s_1}$ Galois with group G of prime-to-p order n, there exists a finite field subextension $K \hookrightarrow L \hookrightarrow K^s$ and an étale cover $Y^L \to X \times_S S^L$ Galois with group G which is a model of $Y_L \to X_L$ over $X \times_S S^L$.*

Proof of Claim 3. Observe first that, for any finite separable subextension $K \hookrightarrow L \hookrightarrow K^s$, as S^L is regular and $X \times_S S^L \to S^L$ is smooth then $X \times_S S^L$ is regular as well (hence, in particular, normal). Also, since $X \times_S S^L \to S^L$ is closed (since proper), surjective and with connected fibres an since S^L is connected, $X \times_S S^L$ is connected as well hence being noetherian and normal, it is irreducible. So, one can consider the normalization $Y^L \to X \times_S S^L$ of $X \times_S S^L$ in

$$k(X \times_S S^L) = k(X_L) \hookrightarrow k(Y_L).$$

From the universal property of normalization, $Y^L \to X \times_S S^L$ is a model of $Y_L \to X_L$ over $X \times_S S^L$). From Theorem 6.16, it only remains to show that $K \hookrightarrow L$ can be chosen in such a way that $k(X_L) \hookrightarrow k(Y_L)$ be unramified over $X \times_S S^L$. Since $X \times_S S^L$ is regular, from the Zariski-Nagata purity Theorem 10.1, we are only to to show that $K \hookrightarrow L$ can be chosen in such a way that $k(X_L) \hookrightarrow k(Y_L)$ be unramified over the codimension 1 points of $X \times_S S^L$. But as all the codimension 1 points of X are either contained in the generic fibre X_{s_1} or the generic point ζ of the closed fibre X_{s_0}, we are only to to show that $K \hookrightarrow L$ can be chosen in such a way that $k(X_L) \hookrightarrow k(Y_L)$ be unramified over the points of $X \times_S S^L$ lying over ζ in $S \times_S S^L \to X$.

For this, let π be a uniformizing parameter of $\mathcal{O}$; it is also a uniformizing parameter of $\mathcal{O}_{X,\zeta}$. Set $L := K[T]/\langle T^n - \pi\rangle$. Then, $k(X_L) = k(X) \cdot L = k(X)[T]/\langle T^n - \pi\rangle$ is a degree n extension of $k(X)$, tamely ramified over $\mathcal{O}_{X,\zeta}$ with inertia group of order n by Kummer theory. Now, apply Lemma 10.5 below to the extensions $k(Y)/k(X)$ and $k(X^L)/k(X)$ to obtain that the compositum $k(Y)\dot{k}(X^L)$ is unramified over $\mathcal{O}_{X \times_S S^L, \zeta^L}$ for any point ζ^L in $X \times_S S^L$ above ζ. □

Lemma 10.5. (Abhyankar's lemma) *Let L/K and M/K be two finite Galois extensions tamely ramified over $\mathcal{O}$ and assume that $e_{M|K}(\mathcal{O})$ divides $e_{L|K}(\mathcal{O})$. Then, for any maximal ideal $\mathfrak{m}_L$ of $\mathcal{O}^L$, the compositum $L.M$ is unramified over $\mathcal{O}^L_{\mathfrak{m}_L}$.*

11. Proper schemes over algebraically closed fields

In this last section, we would like to prove the following:

Theorem 11.1. *The étale fundamental group of a proper connected scheme over an algebraically closed field is topologically finitely generated.*

A striking consequence of this theorem is that a proper connected scheme over an algebraically closed field has only finitely many isomorphism classes of étale covers of bounded degree.

Proof. We proceed by induction on the dimension d to reduce to the case of curves. However, to make the induction step work, we need the two intermediary Claims 1 and 2 below.

Claim 1: *Fix an integer $d \geq 0$ and assume that Theorem* 11.1 *holds for all projective normal connected and d-dimensional schemes over an algebraically closed field k. Then Theorem* 11.1 *holds for all proper connected and d-dimensional schemes over k.*

Proof of Claim 1. Let X be a proper connected and d-dimensional scheme over an algebraically closed field k. The first ingredient is:

Theorem 11.2 (Chow's lemma [EGA2, Cor. 5.6.2]). *Let S be a noetherian scheme. Then, for any $X \to S$ proper there exists $X' \to S$ projective and a surjective birational morphism $X' \to X$ over S.*

Applying Chow's lemma to the structural morphism $X \to \operatorname{spec}(k)$, one obtains a scheme X' projective over k and a surjective birational morphism $X' \to X$ over k, which is automatically proper since both X' and X are proper over k. Then, from Theorem A.5 and Corollary A.7, the profinite group $\pi_1(X)$ is topologically finitely generated as soon as $\pi_1(X'_0)$ is for each connected component $X'_0 \in \pi_0(X')$.

Assume that X' is connected. The underlying reduced closed subscheme $X'^{\mathrm{red}} \hookrightarrow X'$ is projective over k since X' is. Also, as X'^{red} is of finite type over k, its normalization $\tilde{X}'^{\mathrm{red}} \to X'^{\mathrm{red}}$ is a finite and, in particular, $\tilde{X}'^{\mathrm{red}}$ is projective over k as well. And, from Theorem A.5 and Corollary A.7, $\pi_1(X')$ is topologically finitely generated as soon as $\pi_1(\tilde{X}'^{\mathrm{red}}_0)$ is for each connected component $\tilde{X}'^{\mathrm{red}}_0$ of $\tilde{X}'^{\mathrm{red}}$.

Claim 2: *Let X be projective, normal connected and d-dimensional scheme over an algebraically closed field k. Then there exists a proper, connected and $d-1$-dimensional scheme Y over k and an epimorphism of profinite groups:*

$$\pi_1(Y) \twoheadrightarrow \pi_1(X).$$

Proof of Claim 2. Let $i : X \hookrightarrow \mathbb{P}^n_k$ be a closed immersion and let $H \hookrightarrow \mathbb{P}^n_k$ be an hyperplane such that $X \not\subset H$ then the corresponding hyperplane section $X.H$ (regarded as a scheme with the induced reduced scheme structure) has dimension $\leq d-1$. The fact that $Y := X.H$ has the required properties results from the following application of Bertini theorem and the Stein factorization theorem:

Theorem 11.3 ([J83, Thm. 7.1]). *Let X be a proper scheme over k, let $f : X \to \mathbb{P}^n_k$ be a morphism over k and $L \hookrightarrow \mathbb{P}^n_k$ a linear projective subscheme. Assume that:*

(i) *X is irreducible;* (ii) $\dim(f(X)) + \dim(L) > n$.

Then $f^{-1}(L)$ is connected and non-empty.

Since X is connected, noetherian with integral local ring, X is irreducible and one can apply Theorem 11.3 to the closed immersion $i : X \hookrightarrow \mathbb{P}^n_k$ to obtain that $X \cdot H$ is (projective) and connected over k. It remains to prove that the morphism of profinite groups $\pi_1(X \cdot H) \to \pi_1(X)$ induced by the closed immersion $X \cdot H \hookrightarrow X$ is an epimorphism. But this follows again from Theorem 11.3. Indeed, for any connected étale cover $Y \to X$, the scheme Y is again connected, noetherian with integral local ring (Y is normal since X is) hence irreducible and, from Theorem 11.3 applied to $Y \to X \overset{i}{\hookrightarrow} \mathbb{P}^n_k$, one gets that $Y \times_X (X \cdot H)$ is connected.

Combining Claims 1 and 2, one reduce by induction on the dimension d to the case of 0 and 1-dimensional projective normal connected schemes over k. (First apply Claim 1 to show that Theorem 11.1 for d-dimensional proper connected schemes over k is equivalent to Theorem 11.1 for d-dimensional projective normal connected schemes over k, then apply Claim 2 to show that Theorem 11.1 for d-dimensional projective normal connected schemes over k is implied by Theorem 11.1 for $d-1$-dimensional proper connected schemes over k and so on.)

If $d = 0$ then $X = \mathrm{spec}(k)$ and $\pi_1(X) = \Gamma_k = \{1\}$. So, let X be a projective, smooth, connected curve of genus say g.

Write Q for the prime field of k. Since X is of finite type over k, there exists a subextension $Q \hookrightarrow k_0 \hookrightarrow k$ of finite transcendence degree over Q and a model X_0 of X over k_0.

Assume first that Q has characteristic 0. Since k_0 is of finite transcendence degree over Q, one can find a field embedding $k_0 \hookrightarrow \mathbb{C}$ hence, from Lemma 6.5, one has the following isomorphism of profinite groups:

$$\pi_1(X) = \pi_1(X_0 \times_{k_0} k) = \pi_1(X_0 \times_{k_0} \overline{k}_0) = \pi_1(X_0 \times_{k_0} \mathbb{C}).$$

So, one can assume that $k = \mathbb{C}$. It then follows from Example 8.2 that one has an isomorphism of profinite groups:

$$\pi_1(X) \tilde{\to} \widehat{\Gamma}_{g,0}.$$

Assume now that Q has characteristic $p > 0$. The key ingredients here are the specialization theorem and the following consequence of Grothendieck's existence theorem for lifting smooth projective curves from characteristic > 0 to characteristic 0:

Theorem 11.4 ([SGA1, III, Cor. 7.3]). *Let $S := \mathrm{spec}(A)$ with A a complete local noetherian ring with residue field k and closed point $s_0 \in S$. For any smooth and projective scheme X_1 over k, if:*

$$\mathrm{H}^2(X_1, (\Omega^1_{X_1|k})^\vee) = \mathrm{H}^2(X_1, \mathcal{O}_{X_1}) = 0$$

then X_1 has a smooth and projective model $X \to S$ over S.

By Grothendieck's vanishing theorem for cohomology [Hart77, Chap. III, Thm. 2.7], the hypotheses of Theorem 11.4 are always satisfied when X is a smooth projective curve. So, write A for the ring $W(k)$ of Witt vectors over k; it is a complete discrete valuation ring with residue field k and fraction field K of characteristic 0. Set $S := \mathrm{spec}(A)$ and let s_0, s_1 denote the generic and closed point of S respectively. From Theorem 11.4, there exists a smooth projective curve $\mathcal{X} \to S$ such that:

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d] \\ k \arrow[r, "s_1"'] & S. \end{tikzcd}$$

Since $\mathcal{X} \to S$ is proper and smooth (hence separable), it follows from Theorem 9.1 that the specialization morphism is an epimorphism:

$$sp : \pi_1(\mathcal{X}_{\overline{s}_1}) \twoheadrightarrow \pi_1(\mathcal{X}_{\overline{s}_0} = X).$$

Hence the conclusion follows from $\pi_1(\mathcal{X}_{\overline{s}_1}) = \widehat{\Gamma}_{g,0}$. $\square$

Remark 11.5. Let S be a smooth, separated and geometrically connected curve over an algebraically closed field k of characteristic $p > 0$, let g denote the genus of its smooth compactification $S \hookrightarrow S^{cpt}$ and r the degree of $S \smallsetminus S^{cpt}$. From Remark 6.8, the pro-p-completion $\pi_1(S)^{(p)}$ of $\pi_1(S)$ is known and, from Theorem 10.4 and the proof of Theorem 11.1, the prime-to-p completion $\pi_1(S)^{(p)'}$ of $\pi_1(S)$ is known as well (and equal to $\widehat{\Gamma_{g,r}}^{(p)'}$). But this does not determine $\pi_1(S)$ entirely (except when $(g,r) = (0,i)$, $i = 0,1,2$ or $(g,r) = (1,0)$). However, in direction of a more precise determination of $\pi_1(S)$ one had the following conjecture:

Conjecture 11.6 (Abhyankar's conjecture). *With the above notation, any finite group G such that $G^{(p)'}$ is quotient of $\pi_1(S)^{(p)'} = \widehat{\Gamma_{g,r}}^{(p)'}$ (or, equivalently, is generated by $\leq 2g + r - 1$ elements) is a quotient of $\pi_1(S)$.*

Abhyankar's conjecture for $S = \mathbb{A}^1_k$ was proved by M. Raynaud [R94] and the general case was proved by D. Harbater, by reducing it to the case of the affine line [Harb94]. Note that, in the affine case, $\pi_1(S)$ is not topologically finitely generated so the knowledge of its finite quotients does not determine its isomorphism class.

Appendix
Digest of descent theory for étale fundamental groups

A.1. The formalism of descent

We recall briefly the formalism of descent. Let S be a scheme and $\mathcal{C}_S$ a subcategory of the category of S-schemes closed under fibre product. A *fibred category over* $\mathcal{C}_S$ is a pseudofunctor $\mathfrak{X} : \mathcal{C}_S \to Cat$ that is the data of:

- for any $U \in \mathcal{C}_S$, a category $\mathfrak{X}_U$ (sometimes called the fibre of $\mathfrak{X}$ over $U \to S$);
- for any morphism $\phi : V \to U$ in $\mathcal{C}_S$, a base change functor $\phi^\star : \mathfrak{X}_U \to \mathfrak{X}_V$;
- for any morphisms $W \xrightarrow{\chi} V \xrightarrow{\phi} U$ in $\mathcal{C}_S$, a functor isomorphism $\alpha_{\chi,\phi} : \chi^\star\phi^\star \tilde{\to} (\phi \circ \chi)^\star$ satisfying the usual cocycle relations that is, for any morphisms $X \xrightarrow{\psi} W \xrightarrow{\chi} V \xrightarrow{\phi} U$ in $\mathcal{C}_S$, the following diagrams are commutative:

$$\begin{tikzcd}
\psi^\star\chi^\star\phi^\star \arrow[r, "\psi^\star(\alpha_{\chi,\phi})"] \arrow[d, "\alpha_{\psi,\chi}(\phi^\star)"'] & \psi^\star(\phi\circ\chi)^\star \arrow[d, "\alpha_{\psi,\phi\circ\chi}"] \\
(\chi\circ\psi)^\star\phi^\star \arrow[r, "\alpha_{\chi\circ\psi,\phi}"'] & (\phi\circ\chi\circ\psi)^\star.
\end{tikzcd}$$

Given a morphism $\phi : U' \to U$ in $\mathcal{C}_S$, write

$$U'' := U' \times_U U', \quad U''' := U' \times_U U' \times_U U', \quad p_i : U'' \to U', \quad i = 1, 2,$$

$$p_{i,j} : U''' \to U'', \quad 1 \le i < j \le 3,$$

$$u_i : U''' \to U', \quad i = 1, 2, 3$$

for the canonical projections.

A morphism $\phi : U' \to U$ in $\mathcal{C}_S$ is said to be a *morphism of descent for* $\mathfrak{X}$ if for any $x, y \in \mathfrak{X}_U$ and any morphism $f' : \phi^\star x \to \phi^\star y$ in $\mathfrak{X}_{U'}$ such that the following diagram commute:

$$\begin{tikzcd}
 & p_1^\star\phi^\star(x) \arrow[r, "p_1^\star f'"] \arrow[dl, "\alpha_{p_1,\phi}(x)"'] & p_1^\star y \arrow[dr, "\alpha_{p_1,\phi}(y)"] & \\
\phi'^\star(x) \arrow[rrr, dotted, "p_1^\star f'", "p_2^\star f'"'] \arrow[dr, "\alpha_{p_2,\phi}(x)"'] & & & \phi'^\star(y) \arrow[dl, "\alpha_{p_2,\phi}(y)"] \\
 & p_2^\star\phi^\star(x) \arrow[r, "p_2^\star f'"] & p_1^\star y &
\end{tikzcd}$$

there exists a unique morphism $f : x \to y$ in $\mathfrak{X}_U$ such that $\phi^\star f = f'$.

A morphism $\phi : U' \to U$ in $\mathcal{C}_S$ is said to be a *morphism of effective descent for* $\mathfrak{X}$ if $\phi : U' \to U$ is a morphism of descent for $\mathfrak{X}$ and if for any $x' \in \mathfrak{X}_{U'}$ and any

isomorphism $u : p_1^\star(x') \tilde{\to} p_2^\star(x')$ in $\mathfrak{X}_{U''}$ such that the following diagram commute

$$\begin{tikzcd}
 & p_{1,3}^\star p_1^\star(x') \arrow[r, "p_{1,3}^\star u"] \arrow[dl, "\alpha_{p_{1,3},p_1}(x')"'] & p_{1,3}^\star p_2^\star(x') \arrow[dr, "\alpha_{p_{1,3},p_2}(x')"] & \\
u_1^\star(x') \arrow[rrr, dotted, "p_{1,3}^\star u"] \arrow[ddd, dotted, "p_{1,2}^\star u"] & & & u_3^\star(x') \\
p_{1,2}^\star p_1^\star(x') \arrow[u, "\alpha_{p_{1,2},p_1}(x')"] \arrow[d, "p_{1,2}^\star u"'] & & & p_{2,3}^\star p_2^\star(x') \arrow[u, "\alpha_{p_{2,3},p_2}(x')"'] \\
p_{1,2}^\star p_2^\star(x') \arrow[dr, "\alpha_{p_{1,2},p_2}(x')"'] & & & p_{2,3}^\star p_1^\star(x') \arrow[u, "p_{2,3}^\star u"'] \arrow[dl, "\alpha_{p_{2,3},p_1}(x')"] \\
 & u_2^\star(x') \arrow[rr, equal] & & u_2^\star(x') \arrow[uuu, dotted, "p_{2,3}^\star u"]
\end{tikzcd}$$

there is a (necessarily unique since $\phi : U' \to U$ is a morphism of descent for $\mathfrak{X}$) $x \in \mathfrak{X}_U$ and an isomorphism $f' : \phi^\star(x) \tilde{\to} x'$ in $\mathfrak{X}_{U'}$ such that the following diagram commute

$$\begin{tikzcd}
 & p_1^\star \phi^\star(x) \arrow[r, "p_1^\star f'"] \arrow[dl, "\alpha_{p_1,\phi}(x)"'] & p_1^\star(x') \arrow[dd, "u"] \\
\phi'^\star(x) \arrow[ur, dotted, "p_1^\star f'"] \arrow[dr, dotted, "p_2^\star f'"] & & \\
 & p_2^\star \phi^\star(x) \arrow[ul, "\alpha_{p_2,\phi}(x)"] \arrow[r, "p_2^\star f'"] & p_2^\star(x').
\end{tikzcd}$$

The pair $\{x',\ u : p_1^\star(x') \tilde{\to} p_2^\star(x')\}$ is called a descent datum for $\mathfrak{X}$ relatively to $\phi : U' \to U$. Denoting by $\mathfrak{D}(\phi)$ the category of descent data for $\mathfrak{X}$ relatively to $\phi : U' \to U$, saying that $\phi : U' \to U$ is a morphism of descent for $\mathfrak{X}$ is equivalent to saying that the canonical functor $\mathfrak{X}_U \to \mathfrak{D}(\phi)$ is fully faithful and saying that $\phi : U' \to U$ is a morphism of effective descent for $\mathfrak{X}$ is equivalent to saying that the canonical functor $\mathfrak{X}_U \to \mathfrak{D}(\phi)$ is an equivalence of category.

Example A.1. The basic example is that any faithfully flat and quasi-compact morphism $\phi : U' \to U$ is a morphism of effective descent for the fibered category of quasi-coherent modules. See for instance [V05] for a comprehensive introduction to descent techniques.

A.2. Selected results

The fibred categories we will now focus our attention on are the categories of finite étale covers. We only mention results that are used in these notes. For the proofs, we refer to [SGA1, Chap. VIII and IX].

Theorem A.2. *Let X be a scheme and $i : X^{\mathrm{red}} \hookrightarrow X$ be the underlying reduced closed subscheme. Then the functor $i^\star : \mathcal{C}_X \to \mathcal{C}_{X^{\mathrm{red}}}$ is an equivalence of categories. In particular, if X is connected, it induces an isomorphism of profinite groups:*

$$\pi_1(i) : \pi_1(X^{\mathrm{red}}) \tilde{\to} \pi_1(X).$$

Theorem A.3. *Let S be a scheme and let $f : S' \to S$ be a morphism which is either:*

- *finite and surjective or*
- *faithfully flat and quasi-compact.*

Then $f : S' \to S$ is a morphism of effective descent for the fibred category of étale, separated schemes of finite type.

Corollary A.4. *Let S be a scheme and let $f : S' \to S$ be a morphism which is either:*

- *finite, radicial and surjective or*
- *faithfully flat, quasi-compact and radicial.*

Then $f : S' \to S$ induces an equivalence of categories $\mathcal{C}_S \to \mathcal{C}_{S'}$.

Theorem A.5. *Let S be a scheme and let $f : S' \to S$ be a proper and surjective morphism. Then $f : S' \to S$ is a morphism of effective descent for the fibre category of étale covers.*

A.3. Comparison of fundamental groups for morphism of effective descent

Assume that $f : S' \to S$ is a morphism of effective descent for the fibre category of étale covers. Our aim is to interpret this in terms of fundamental groups.

Consider the usual notation S'', S''' and:

$$\begin{aligned} &p_i : S'' \to S',\ i = 1, 2,\\ &p_{i,j} : S''' \to S'',\ 1 \le i < j \le 3,\\ &u_i : S''' \to S',\ = 1, 2, 3. \end{aligned}$$

Assume that S, S', S'', S''' are disjoint union of connected schemes, then, with $E' := \pi_0(S')$, $E'' := \pi_0(S'')$, $E''' := \pi_0(S''')$, also set:

$$\begin{aligned} &q_i = \pi_0(p_i) : E'' \to E',\ i = 1, 2,\\ &q_{i,j} = \pi_0(p_{i,j}) : E''' \to E'',\ 1 \le i < j \le 3,\\ &v_i = \pi_0(u_i) : E''' \to E',\ i = 1, 2, 3. \end{aligned}$$

Write $\mathcal{C} := \mathcal{C}_S$, $\mathcal{C}' := \mathcal{C}_{S'}$, $\mathcal{C}'' := \mathcal{C}_{S''}$, $\mathcal{C}''' := \mathcal{C}_{S'''}$. We assume that S is connected.

Fix $s'_0 \in E'$ and for each $s' \in E'$, fix an element $\overline{s'} \in E''$ such that

$$q_1(\overline{s'}) = s'_0 \text{ and } q_2(\overline{s'}) = s'.$$

Also, for any $s' \in E'$ (resp. $s'' \in E''$, $s''' \in E'''$) fix a geometric point $\underline{s}' \in s'$ (resp. $\underline{s}'' \in s''$, $\underline{s}'' \in s''$) and write $\pi_{s'} := \pi_1(s'; \underline{s}')$ (resp. $\pi_{s''} := \pi_1(s''; \underline{s}'')$, $\pi_{s'''} := \pi_1(s'''; \underline{s}''')$) for the corresponding fundamental group.

Since for any $s'' \in E''$ $p_i(\underline{s}'')$ and $\underline{q_i(s'')}$ lie in the same connected component of S', one gets étale paths $\alpha_i^{s''} : F''_{\underline{s''}} \circ p_i^\star = F'_{p_i(\underline{s}'')} \tilde{\to} F'_{\underline{q_i(s'')}}$, hence profinite group morphisms:

$$q_i^{s''} : \pi_{s''} \to \pi_1(q_i(s''), p_i(\underline{s}'')) \simeq \pi_{q_i(s'')},\ i = 1, 2.$$

Similarly, one gets étale paths $\alpha_{i,j}^{s'''} : F'''_{\underline{s'''}} \circ p_{i,j}^\star = F''_{p_{i,j}(\underline{s}''')} \tilde{\to} F''_{\underline{q_{i,j}(s''')}}$ and profinite group morphisms:

$$q_{i,j}^{s'''} : \pi_{s'''} \to \pi_1(q_{i,j}(s'''), p_i(\underline{s}''')) \simeq \pi_{q_{i,j}(s''')},\ 1 \le i < j \le 3.$$

Eventually, from the étale paths

$$F'''_{s'''} \circ p_{1,2}^\star \circ p_1^\star \tilde{\to} F_{\underline{v_1(s''')}} \tilde{\leftarrow} F'''_{s'''} \circ p_{1,3}^\star \circ p_1^\star;$$
$$F'''_{s'''} \circ p_{1,2}^\star \circ p_2^\star \tilde{\to} F_{\underline{v_2(s''')}} \tilde{\leftarrow} F'''_{s'''} \circ p_{2,3}^\star \circ p_1^\star;$$
$$F'''_{s'''} \circ p_{1,3}^\star \circ p_2^\star \tilde{\to} F_{\underline{v_3(s''')}} \tilde{\leftarrow} F'''_{s'''} \circ p_{2,3}^\star \circ p_2^\star;$$

one gets $a_i^{s'''} \in \pi_{v_i(s''')}$, $i = 1, 2, 3$ such that

$$q_1^{q_{1,2}(s''')} \circ q_{1,2}^{s'''} = \mathrm{int}(a_1^{s'''}) \circ q_1^{q_{1,3}(s''')} \circ q_{1,3}^{s'''};$$
$$q_2^{q_{1,2}(s''')} \circ q_{1,2}^{s'''} = \mathrm{int}(a_2^{s'''}) \circ q_1^{q_{2,3}(s''')} \circ q_{2,3}^{s'''};$$
$$q_2^{q_{1,3}(s''')} \circ q_{1,2}^{s'''} = \mathrm{int}(a_3^{s'''}) \circ q_2^{q_{2,3}(s''')} \circ q_{2,3}^{s'''}.$$

Since $f : S' \to S$ is a morphism of effective descent, the above data allows us to recover $\mathcal{C}$ from $\mathcal{C}'$, $\mathcal{C}''$, $\mathcal{C}'''$ up to an equivalence of category hence to reconstruct $\pi_1(S, \underline{p(s'_0)})$ from the $\pi_{s'}$, $\pi_{s''}$, $\pi_{s'''}$.

More precisely, the category $\mathcal{C}'$ with descent data for $f : S' \to S$ is equivalent to the category $\mathcal{C}(\{\pi_{s'}\}_{s' \in E'})$ together with a collection of functor automorphisms $g_{s''} : Id \tilde{\to} Id$, $s'' \in E''$ satisfying the following relations:

(1) $g_{s''} q_1^{s''}(\gamma'') = q_1^{s''}(\gamma'') g_{s''}$, $s'' \in E''$;

(2) $g_{\overline{s'}} = g_{\overline{s'_0}}$, $s' \in E'$;

(3) $a_3^{s'''} g_{q_{1,3}(s''')} a_1^{s'''} = g_{q_{2,3}(s''')} a_2^{s'''} g_{q_{1,2}(s''')}$, $s''' \in E'''$.

So, set

$$\Phi := \bigsqcup_{s' \in S'} \pi_{s'} \bigsqcup_{s'' \in E''} \hat{\mathbb{Z}} g_{s''} / \langle (1), (2), (3) \rangle,$$

where $\coprod$ stands for the free product in the category of profinite groups and let $\mathcal{N}$ be the class of all normal subgroups $N \lhd \Phi$ such that $[\Phi : N]$ and $[\pi_{s'} : i_{s'}^{-1}(N)]$ are finite (here $i_s : \pi_s \hookrightarrow \coprod_{s' \in S'} \pi_{s'} \coprod_{s'' \in E''} \hat{\mathbb{Z}} g_{s''} \twoheadrightarrow \Phi$ denotes the canonical morphism). Then writing

$$\pi := \varprojlim_{N \in \mathcal{N}} \Phi / N$$

one gets that the category $\mathcal{C}'$ with descent data for $f : S' \to S$ is also equivalent to the category $\mathcal{C}(\pi)$. Whence:

Theorem A.6. *With the above assumptions and notation, one has a canonical profinite group isomorphism*

$$\pi_1(S, p(s'_0)) \tilde{\to} \pi.$$

Corollary A.7. *With the above assumptions and notation, if E' and E'' are finite and if the $\pi_{s'}$, $s' \in E'$ are topologically of finite type then so is $\pi_1(S, p(s'_0))$.*

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Fundamental Groupoid Scheme

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Abstract. This article is an overview of the original construction by Nori of the fundamental group scheme as the Galois group of some Tannaka category $EF(X)$ (the category of essentially finite vector bundles) with a special stress on the correspondence between fiber functors and torsors. Basic definitions and duality theorem in Tannaka categories are stated. A paragraph is devoted to the characteristic 0 case and to a reformulation of Grothendieck's section conjecture in terms of fiber functors on $EF(X)$.

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1. Introduction

The aim of this text is to give an account of the construction by Nori of the fundamental group scheme. In his article [18], Nori develops two points of view. The first using the machinery of tannakian categories is the one which is developed here. The second one closer to Galois category point of view defines the fundamental group scheme of a scheme defined over a field k as the projective limit of finite groups of torsors on X under finite k-group schemes. This point of view allowed Gasbarri to extend the definition of the fundamental group scheme to relative schemes over a Dedekind scheme [11]. But we will not go in these developments in this article.

Before introducing the fundamental group scheme, we will look in Section 2 over a few classical facts on the topological and the algebraic fundamental groups. We recall in particular the classical correspondence on a compact Riemann surface X between vector bundles, finite in the sense of Weil and representations of the fundamental group of X which factor through a finite quotient. This will give a natural introduction to the idea developed by Nori.

The purpose of Paragraphs 3 and 4 is to introduce or recall the Tannaka duality theory, which we will use in Paragraph 5 to define the fundamental group

scheme of a proper reduced scheme X over a field k as the Tannaka Galois group of the tannakian category of *essentially finite* vector bundles on X. We state different properties of the fundamental group scheme and of the related universal torsor. A natural question is to ask, if a finite morphism $f : Y \to X$ is given over k such that f_*O_Y is essentially finite, whether Y has a fundamental group scheme and to compare the universal torsors of X and Y. This problem is not discussed in these notes; we refer the reader to [10] and [1].

A few examples are given in Paragraph 7. The case of positive characteristic is of course the most interesting as there are torsors under finite local group schemes. But even in the case when the base field has characteristic 0 studied in Paragraph 6, the fundamental group scheme has some interest. It is in fact more or less equivalent to the data of the short exact sequence linking the geometric and the arithmetic fundamental groups. This leads for instance to an interpretation of the sections of this exact sequence as fiber functors of the category of essentially finite vector bundles on the scheme X, and to a reformulation of Grothendieck's section conjecture, which has at least the advantage to get rid of the base point.

We limited ourselves to the case where X is proper over k. There has been recent developments in the case of an affine curve for instance. We refer the interested reader to [3], where the author proposes a theory of tame fundamental group scheme using the tannakian category of essentially finite vector bundles on some stack of roots of the divisor at infinity of X. In an other direction the category of finite vector bundles with connection is used in [8] to define the fundamental group scheme, but the method is limited to the characteristic 0 case.

We assume that the reader is familiar with the theory of étale fundamental group and more generally with Galois categories, as well as with the definition of stacks. The reader can complete his knowledge on stacks in [2]. In the same way we will use freely the notion of Grothendieck topology and the descent theory, in particular étale topology and $fpqc$-topology, for which we refer the reader to [16] and [27]. In order to be self contained definitions on groupoids, gerbes, tannakian categories are given in Sections 3 and 4 as well as main theorems on tannakian duality, with a special stress on the correspondence between fiber functors and torsors in Section 5.1. The reader is invited to consult classical literature on the subject to complete his information.

2. Topological and algebraic fundamental groupoid

In this section we very quickly recall some facts about the category of covers (resp. algebraic covers) of a topological space (resp. of a scheme), and we compare different points of view on this category. We report the reader to [30], [24] or [4] for an account of the theory of Galois categories and the theory of étale fundamental group.

The analogy between local systems of finite sets and local systems of vectors spaces for the étale topology will lead us from Galois categories to Tannaka categories and from the Grothendieck étale fundamental group to the Nori fundamental group scheme.

2.1. Several descriptions of a topological cover

Let X be a locally path connected locally simply connected topological space. Recall that a topological cover of X is the data of a topological space Y together with a continuous map $f : Y \to X$ which is locally trivial: there is a covering of X by open set U_i, $i \in I$, such that for any $i \in I$, $f^{-1}(U_i) \simeq U_i \times F$, where F is some set endowed with discrete topology. A morphism h between two coverings of X, $f : Y \to X$ and $g : Z \to X$ is a continuous map $Y \to Z$ such that $g \circ h = f$.

It is rather obvious from the definition that, for any couple of points $a, b \in X$ and any point $y \in f^{-1}(a)$, any path γ in X with origin a and extremity b lifts uniquely to a path in Y with origin y. The extremity z of this path in Y, which lies in the fiber of b, depends only on the homotopy class of γ and will be denoted $z = \gamma.y$.

One may define the *fundamental groupoid* $\pi_1^{\text{top}}(X)$ of X as a category whose objects are points of X and isomorphisms from a point a to a point b are homotopy classes of path from a to b. From the discussion above we conclude that a topological cover $f : Y \to X$ gives rise to a *representation* of the fundamental groupoid of X in the category of sets, in other words a covariant functor from the fundamental groupoid $\pi_1^{\text{top}}(X)$ to the category of sets. It maps a point a of X to the set $Y_a = f^{-1}(a)$ and a class of homotopy of path γ with origin a and extremity b to the bijection $Y_a \to Y_b$ given by $y \to \gamma.y$.

In particular, when $a = b$, class of homotopy of loops in X based at a form a group $\pi_1^{\text{top}}(X, a)$ which acts on the fiber $Y_a = f^{-1}(a)$ of a. A topological cover $f : Y \to X$ gives rise to a morphism $\pi_1^{\text{top}}(X, a) \to S_{Y_a}$, where S_{Y_a} denotes the group of bijection of the fiber Y_a.

The basic result about coverings is the following theorem (see for instance [7]):

Theorem 2.1. *The map which associates to a covering $f : Y \to X$ the corresponding representation of $\pi_1(X)$ is an equivalence of categories. Any fixed point $a \in X$ induces a functor from the category of topological covers of X to the category of sets and an equivalence of categories from the category of topological covers of X to the category of $\pi_1(X, a)$-sets.*

Moreover there is an universal cover $\tilde{X}^a \to X$ with a point $\tilde{a}$ in the fiber at a, which satisfies the following universal property: for any cover $f : Y \to X$ endowed with a point y in Y_a, there is a unique morphism $h : \tilde{X}^a \to Y$ such that $h(\tilde{a}) = y$.

Points of X determine fibers functors from the category of topological covers of X to the category of sets. This theorem says in particular that natural transformations between two fiber functors at a and b come from path from a to b.

A cover of X a called a *Galois cover* when the corresponding action of $\pi_1^{\text{top}}(X, a)$ on the fiber Y_a is transitive, and for points $y \in Y_a$ the stabilizers of y depend only on a. In this case the *monodromy group* of the cover, which is the image of $\pi_1^{\text{top}}(X, a)$ in S_{Y_a}, is isomorphic to $G = \pi_1^{\text{top}}(X, a)/Fix(y)$, where $Fix(y)$ is the stabilizer of some point $y \in Y_a$, and the cover $Y \to X$ is determined by the morphism $\pi_1^{\text{top}}(X, a) \to G$.

A slightly different point of view on covers is the point of view of *local systems* of sets.

Definition 2.1. A local system L of sets on the topological space X is a locally trivial sheaf of sets on X: for any open set $U \subset X$, $L(U)$ is a set and for any open subsets $V \subset U \subset X$ there is a restriction map $R_{V,U} : L(U) \to L(V)$ satisfying the following relation: if $W \subset V \subset U \subset X$

$$R_{W,V} \circ R_{V,U} = R_{W,U}.$$

Moreover the sheaf condition has the following expression: suppose $\mathcal{U} = \{U_i, i \in I\}$ is a covering of an open set U with open subsets; we will denote by $U_{i,j} = U_i \cap U_j$ the intersection of U_i and U_j; from the restriction maps, we get a map

$$R_{\mathcal{U}} : L(U) \to \coprod_i L(U_i)$$

then $R_{\mathcal{U}}$ induces a bijection from $L(U)$ to the subset of $(s_i) \in \coprod_i L(U_i)$ satisfying $R_{U_{i,j},U_i}(s_i) = R_{U_{i,j},U_j}(s_j)$ for all i, j.

Finally we require that L is locally trivial: any point $a \in X$ has an open neighborhood V such that the restriction of L to V is isomorphic to the trivial sheaf; or equivalently the restriction of L to any simply connected open set is trivial.

Morphisms between local systems of sets are morphisms of sheaves. With this notion one can state the following:

Theorem 2.2. *The map which associates to a cover $f : Y \to X$ the sheaf L defined by posing $L(U)$ to be the set of continuous sections of $f^{-1}(U) \to U$, for any open set $U \subset X$, is an equivalence of categories.*

Proof. The sheaves conditions are easy to check. The local triviality of L comes from the local triviality of f.

In the other direction, from a local system, one defines for any covering $\mathcal{U} = \{U_i, i \in I\}$ of X by simply connected open subsets, a family of trivial covers $Y_i = U_i \times L(U_i)$ and $Y_{ij} = U_{ij} \times L(U_{ij})$. The bijections $R_{U_{ij},U_i} : L(U_i) \to L(U_{ij})$ induce isomorphisms $r_{i,j} : Y_{i|U_{ij}} \to Y_{ij}$ and finally isomorphisms

$$\alpha_{i,j} : r_{j,i}^{-1} \circ r_{i,j} : Y_{i|U_{ij}} \simeq Y_{j|U_{ij}}$$

obviously satisfying the relation $\alpha_{k,j} \circ \alpha_{i,j} = \alpha_{k,i}$. One can paste together the trivial covers Y_i using the isomorphisms $\alpha_{i,j}$ and get a topological cover $Y \to X$. $\square$

To summarize the equivalences of categories stated in the above theorems, one can say that a topological cover $f : Y \to X$ has the following equivalent descriptions:

1. a morphism from the topological fundamental group $\pi_1^{\mathrm{top}}(X, a)$ based at a point $a \in X$ to the permutation group S_{Y_a} of the fiber at a.
2. a representation of the fundamental groupoid $\pi_1^{\mathrm{top}}(X)$ in the category of sets;
3. a local system of sets, i.e., a locally trivial sheaf of sets on X.

2.2. Algebraic fundamental group versus topological fundamental group

Let X be a locally noetherian connected scheme. In the context of algebraic geometry the equivalent of a locally trivial continuous map is an étale morphism ([16]).

Definition 2.2. A cover of X is a finite étale morphism $Y \to X$.

Grothendieck developed the theory of Galois categories to study étale covers [30]. A Galois category $\mathcal{C}$ is endowed with fiber functors. And for any fiber functor $F : \mathcal{C} \to \mathcal{S}$, where $\mathcal{S}$ denotes the category of finite sets, the fundamental group based at F is by definition $\pi_1(\mathcal{C}, F) = \mathrm{Aut}(F)$. It is a profinite group, and the basic result about Galois categories is that, for any fiber functor $F : \mathcal{C} \to \mathcal{S}$, the Galois category $\mathcal{C}$ is equivalent to the category of $\pi_1(\mathcal{C}, F)$-finite sets, i.e., the category of finite sets endowed with a continuous action of $\pi_1(\mathcal{C}, \mathcal{S})$.

One can also define the *fundamental groupoid* of $\mathcal{C}$, whose objects are the fiber functors of $\mathcal{C}$, and morphisms are isomorphisms between fiber functors. We will denote it by $\pi_1(\mathcal{C})$. The equivalence stated above can be reformulated using the fundamental groupoid: the Galois category $\mathcal{C}$ is equivalent to the category of continuous representations of the fundamental groupoid $\pi_1(\mathcal{C})$ on finite sets.

Let $\mathcal{C}$ be a Galois category and F a fiber functor of $\mathcal{C}$. There exists a universal pro-object $\hat{C}$ (projective limit of objects of $\mathcal{C}$) with a pro-point $\hat{a}$ in its fiber at F (projective limit of points of $F(C)$ for C running among objects of $\hat{C}$), satisfying a universal property similar to that of the universal covering: for any couple (D, d) where D is an object of $\mathcal{C}$ and $d \in F(D)$, there is a unique morphism $h : \hat{C} \to D$ such that $h(\hat{a}) = d$.

Grothendieck showed that the category of finite étale covers of a scheme X is indeed a Galois category. Geometric points a on X define fiber functors on this category. The fundamental groupoid and fundamental groups of the category of finite étale covers of X will be called étale fundamental groupoid and étale fundamental group and denoted in this case $\pi_1(X)$ and $\pi_1(X, a)$ (or more generally $\pi_1(X, F)$ where F is any fiber functor on the category of finite étale covers of X). From the general theory of Galois categories, one gets the following basic theorem, similar to Theorem 2.1 (see for instance [30]):

Theorem 2.3. *The category $Revet_X$ of finite étale covers of X is a Galois category. It is equivalent to the category of continuous representations of the étale fundamental groupoid $\pi_1(X)$ on finite sets. Any fibre functor F from $Revet_X$ to the category $\mathcal{S}$ of finite sets induces an equivalence of categories $\tilde{F} : Revet_X \to \pi_1(X, F)$-finite sets, where $\pi_1(X, F) = \mathrm{Aut}(F)$ is the étale fundamental group of X based at F.*

Moreover there exists a pro-universal object $\hat{X}^F$ based at F with a point $\hat{a}$ in the fiber at F satisfying the following universal property: for any finite étale cover $Y \to X$ with a point y in the fiber at F, there exists a unique morphism of covers $h : \hat{X}^F \to Y$ such that the image of $\hat{a}$ by $F(h)$ is y.

As in the topological setting a finite étale Galois cover of X can be described by a surjective morphism of groups $\pi_1(X, a) \to G$. We will see in Section 5.4

(Proposition 5.1, 5, (e)) a very similar description for G-torsors, where G is a finite k-group scheme instead of an abstract finite group and the étale fundamental group is replaced by Nori's fundamental group scheme.

To develop the point of view of local systems, one need a notion of local triviality. We don't have at our disposal the usual topology. Instead *étale topology* will fit our needs (cf. [16]). This is a fact that for any étale cover $f : Y \to X$, there exists an étale finite map $g : Z \to X$ such that $g^*f : g^*Y \to Z$ is trivial, i.e., isomorphic as a cover to $Z \times F$, where F is a finite set. One can give the definition of a local system for the étale topology similar to that of 2.1, étale topology replacing usual topology. If $U_i \to X$ and $U_j \to X$ are two étale open sets of X, the "intersection" U_{ij} is by definition $U_{ij} = U_i \times_X U_j$.

Definition 2.3. A local system L of finite sets on X is a locally trivial sheaf of finite sets on X for the étale topology: for any étale open set $u : U \to X$, $L(U)$ is a finite set and for any commutative diagram

$$\begin{tikzcd} V \arrow[r, "r_{V,U}"] \arrow[dr] & U \arrow[d] \\ & X \end{tikzcd}$$

there is a restriction map $R_{V,U} : L(U) \to L(V)$ satisfying the following relation; for any commutative diagram

$$\begin{tikzcd} W \arrow[r, "r_{W,V}"] \arrow[dr] & V \arrow[r, "r_{V,U}"] \arrow[d] & U \arrow[dl] \\ & X & \end{tikzcd}$$

$$R_{W,V} \circ R_{V,U} = R_{W,U}.$$

Moreover the sheaf condition has the following expression: suppose $u_i : U_i \to U$ is an étale covering $\mathcal{U}$ of U; we will denote by $U_{i,j} = U_i \times_U U_j$ the "intersection" of U_i and U_j; from the restriction maps, we get a map

$$R_{\mathcal{U}} : L(U) \to \coprod_i L(U_i);$$

then $R_{\mathcal{U}}$ induces a bijection from $L(U)$ to the subset of $(s_i) \in \coprod_i L(U_i)$ satisfying $R_{U_{i,j},U_i}(s_i) = R_{U_{i,j},U_j}(s_j)$ for all i, j.

Finally there is an étale covering $\mathcal{U} = \{u_i : U_i \to X\}_{i \in I}$ of X such that the restriction of L to any U_i is trivial.

Theorem 2.4. *The category of algebraic finite étale covers of X is equivalent to the category of local system of finite sets for the étale topology.*

Sketch of the proof. In one direction, starting from an étale cover $f : Y \to X$, for any étale open $u : U \to X$, one defines $L(U)$ as the set of sections of $u^\star f : u^\star Y \to U$.

As there is an étale and surjective morphism $u : U \to X$ such that $u^\star f : u^\star Y \to U$ is trivial, the local system is trivial on the connected components of U and thus locally trivial.

In the other direction the fact that one can paste together trivial covers $U_i \times L(U_i)$ is given by the descent theory (see for instance [27]). □

To summarize the equivalences of categories stated in the above theorems, one can say that a finite étale cover $f : Y \to X$ has the following equivalent descriptions:

1. a continuous morphism from the étale fundamental group $\pi_1(X, a)$ based at a geometric point $a \in X$ to the permutation group S_{Y_a} of the fiber of a.
2. a representation of the fundamental groupoid $\pi_1(X)$ in the category of finite sets;
3. a local system of finite sets for the étale topology on X.

Let k be a field. A finite étale cover of $\mathrm{Spec}(k)$ is of the form $\mathrm{Spec}(L)$, where L is an finite étale algebra over k. The choice of a separable closure $\bar{k}$ of k defines a fiber functor which associates to any finite étale algebra L over k the set of k-embedding from L into $\bar{k}$. And the étale fundamental group based at this fiber functor is identified to $\mathrm{Gal}(\bar{k}/k)$.

Suppose we are given a k-scheme $X \to \mathrm{Spec}(k)$. For any geometric point $a \in X(\bar{k})$, the pro-universal cover $\hat{X}^a \to X$ factors through the arithmetic part $X_{\bar{k}} \to X$ and $\hat{X}^a \simeq \hat{X}^a_{\bar{k}}$. One has the following short exact sequence:

$$1 \to \pi_1(X_{\bar{k}}, a) \to \pi_1(X, a) \to \mathrm{Gal}(\bar{k}/k) \to 1$$

quoted as the fundamental short exact sequence.

Algebraic covers over $\mathbf{C}$

Let X be a proper smooth algebraic variety over $\mathbf{C}$. One can consider the associated analytic variety X^{an}. Riemann's existence theorem for projective curves or more generally GAGA principle establishes an equivalence of categories between algebraic finite étale covers of X and finite topological cover of X^{an}. As a consequence, we get the following theorem.

Theorem 2.5. *Let $a \in X(\mathbf{C})$ be a point of X. Then there is a canonical isomorphism*

$$\pi_1(X, a) \simeq \widehat{\pi_1^{\mathrm{top}}(X^{an}, a)}$$

where $\widehat{\pi_1^{\mathrm{top}}(X^{an}, a)}$ denotes the profinite completion of the group $\pi_1^{\mathrm{top}}(X^{an}, a)$.

Moreover Grothendieck showed that if $K \subset L$ are two characteristic 0 algebraically closed fields, and X is a K-scheme, then for any geometric point a, $\pi_1(X, a) \simeq \pi_1(X_L, a)$ ([30]). As an application one sees that if X is a $\bar{Q}$-scheme, then for any geometric point a, $\pi_1(X, a) \simeq \pi_1(X_{\mathbf{C}}, a) \simeq \widehat{\pi_1^{\mathrm{top}}(X^{an}, a)}$. This means in particular that any finite topological cover of X^{an} has an unique algebraic model defined over $\bar{Q}$.

3. Gerbes and groupoids and their representations

3.1. Gerbes and groupoids

In the description of topological covers of a topological space and of étale covers of a scheme, we encountered the notion of groupoid. More generally *a groupoid is a category whose all morphisms are isomorphisms*. In the context of Nori's theory, one will encounter k-groupoids acting on k-schemes S, where k is a fixed field.

As for a k-group scheme a k-groupoid scheme is a k-scheme which can be defined by its functor of points. The objects of the category are k-morphisms $u : U \to S$, and for any $u, t : U \to S$, the set of morphism from u to t defined over U is some $G_U(u,t)$ satisfying a list of axioms that we will omit here. The translation in schematic terms leads to the following definition:

Definition 3.1. A k-groupoid G acting on a k-scheme S is a k-scheme G given with a k-morphism $(t,s) : G \to S \times_k S$ (target and source) and a product morphism

$$m : G \times_{s_S t} G \to G$$

over $S \times_k S$, a unit element morphism $e : S \to G$ over the diagonal $S \to S \times_k S$, and an inverse element morphism $i : G \to G$ over the morphism $S \times_k S \to S \times_k S$ which maps (s_1, s_2) to (s_2, s_1); these morphism must satisfy the commutativity of the following diagrams:

- associativity

$$\begin{tikzcd}
 & G \times_{s_S t} G \arrow[dr, "m"] & \\
G \times_{s_S t} G \times_{s_S t} G \arrow[ur, "m \times 1"] \arrow[dr, "1 \times m"'] & & G \\
 & G \times_{s_S t} G \arrow[ur, "m"'] &
\end{tikzcd}$$

- identity

$$\begin{tikzcd}
 & G \times_{s_S t} G \arrow[dr, "m"] & \\
G = G \times_{s_S} S = S \times_{S^t} G \arrow[ur, "e \times 1"] \arrow[dr, "1 \times e"'] & & G \\
 & G \times_{s_S t} G \arrow[ur, "m"'] &
\end{tikzcd}$$

- inverse

$$\begin{tikzcd}
G \arrow[r, "i \times 1"] \arrow[d, "s"'] & G \times_{s_S t} G \arrow[d, "m"] \\
S \arrow[r, "e"] & G
\end{tikzcd}$$

and

$$\begin{tikzcd} G \arrow[r, "1\times i"] \arrow[d, "t"'] & G\times_{s\,S^t} G \arrow[d, "m"] \\ S \arrow[r, "e"] & G. \end{tikzcd}$$

A k groupoid gives rise to a category $\mathcal{G}_0$ whose objects are k-morphisms $a : T \to S$ and morphisms between two objects $a : T \to S$ and $b : T \to S$ the T-points $G_{a,b}(T)$, where $G_{a,b}$ is defined in the following manner:

$$G_{a,b} = (a, b)^\star G \to T$$

the morphism m inducing the composition. This category is a groupoid, i.e., every morphism is an isomorphism.

If $(t, s) : G \to S \times_k S$ is a k-groupoid acting on S, and $f : T \to S$ is a k-morphism of k-schemes, the pull back G_T of G is a k-groupoid acting on T:

$$\begin{tikzcd} G_T \arrow[r] \arrow[d] & G \arrow[d, "s,t"] \\ T \times_k T \arrow[r, "f,f"] & S \times_k S. \end{tikzcd}$$

Remark 3.1. In the case where $s = t$, the morphism $s, t : G \to S \times_k S$ factors through $\Delta : S \to S \times_k S$, and the k groupoid acting on S is a S-group scheme.

Definition 3.2. The k-groupoid $G \to S \times_k S$ acts *transitively* on S if there is a *fpqc*-covering $T \to S \times_k S$ such that $G_T(T) \neq \emptyset$.

Equivalently in the category $\mathcal{G}_0$ any two objects $a : U \to S$ and $b : U \to S$ are locally isomorphic for the *fpqc*-topology.

Definition 3.3. A gerbe $\mathcal{G}$ over S for the *fpqc*-topology is a stack over S for the *fpqc*-topology such that

1. $\mathcal{G}$ is locally non-empty: there is a covering of S by $(U_i)_i$ such that $\mathcal{G}(U_i) \neq \emptyset$
2. any two objects are locally isomorphic: if ξ and ξ' are objects of $\mathcal{G}(T)$, where $T \to S$, there is a covering $(T_j)_j$ of T such that, for all j, $\xi_{|T_j} \simeq \xi'_{|T_j}$.

There is a correspondence between k-groupoids acting transitively on a scheme S and gerbes over S which is described in [5]. Given a k-groupoid acting transitively on a scheme S, the associated gerbe is the stack attached to the pre-stack $\mathcal{G}_0$ defined above. The fact that the action is transitive implies that this stack is indeed a gerbe. In the other direction let $\mathcal{G}$ be a gerbe over a k-scheme S. Assume that for any $u : T \to S$ and ω_1 and ω_2 two sections of $\mathcal{G}$ over T, the functor $\underline{\mathrm{Isom}}_T(\omega_1, \omega_2)$ is representable. One defines for any section $\omega \in \mathcal{G}(X)$ over a k-scheme X the k-groupoid $\Gamma_{X,\mathcal{G},\omega} = \underline{\mathrm{Aut}}(\omega)$ representing the functor which associates to any morphism $(b, a) : T \to X \times_k X$, $\mathrm{Isom}_T(a^*\omega, b^*\omega)$.

3.2. Representations

Let $s, t : G \to S \times_k S$ be a k groupoid acting on the k-scheme S.

Definition 3.4. A representation of the k-groupoid G is a quasi-coherent O_S module V together with an action of G on V: for any k-scheme $T \to S$ and any element $g \in G(T)$ is given a morphism $\rho(g) : s(g)^\star V \to t(g)^\star V$ (as O_T-modules) and these morphisms are compatible with base change, composition, and if $s \circ g = t \circ g = u : T \to S$ and $g = 1_u$, $\rho(g) = Id_{u^\star V}$.

We denote by $\mathrm{Rep}_S(G)$ the category of representations of the k-groupoid scheme G acting transitively on the k-scheme S.

Remark 3.2. If G is a S-group, representations of G as group scheme and as groupoid scheme are the same.

One can define also the representations of a gerbe.

Definition 3.5. Let $\mathcal{G}$ be a gerbe over a scheme S. A representation of $\mathcal{G}$ is a functor over the category Sch_S of schemes over S from $\mathcal{G}$ to the category of quasi-coherent modules over varying schemes $T \to S$ compatible with base changes.

We will call $\mathcal{G}$-mod the category of representations of the gerbe $\mathcal{G}$.

The correspondence between gerbes and groupoids is compatible with representations (see [5], Section 3):

Proposition 3.1. *Let G be a groupoid acting transitively on a k-scheme S and $\mathcal{G}$ be the gerbe over* $\mathrm{Spec}(k)$ *corresponding to G as explained in Section* 3.1*, then the category* $\mathrm{Rep}(\mathcal{G})$ *is equivalent to the category* $\mathrm{Rep}_k(G)$*.*

4. Tannakian categories

4.1. Definitions

In what follows we fix a field k. Let S be a k-scheme. We will denote by S-mod, the category of coherent O_S-modules. We collect here for the convenience of the reader a few definitions and facts about tannakian categories. We report for more details to [5], [6], [22], [24], [26] or the appendix in the original article of Nori [18].

Definition 4.1. A symmetric tensor category is an *abelian* k-linear category $\mathcal{T}$ endowed with a tensor product $\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$ satisfying

- k-bilinearity on the Hom: for any objects A, B, C of $\mathcal{T}$, composition
$$\mathrm{Hom}(B, C) \times \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, C)$$
is bilinear;
- associativity constraints: for any objects A, B, C of $\mathcal{T}$, there is a natural isomorphism $\alpha_{A,B,C} : (A \otimes B) \otimes C \simeq A \otimes (B \otimes C)$;
- commutativity constraints: for any objects A, B of $\mathcal{T}$, there is a natural isomorphism $\beta_{A,B} : A \otimes B \simeq B \otimes A$;

- the existence of an unit element 1 with natural isomorphisms for any object A of $\mathcal{T}$, $A \simeq 1 \otimes A \simeq A \otimes 1$;
- the existence of a dual $A^\star$ for any object A with natural morphisms $\epsilon_A : A \otimes A^\star \to 1$ and $\delta_A : 1 \to A^\star \otimes A$;
- a fixed isomorphism $\mathrm{End}(1) \simeq k$,

all these natural morphisms fitting in commutative diagrams we omit here.

Definition 4.2. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two symmetric tensor categories. A tensor functor $T : \mathcal{T}_1 \to \mathcal{T}_2$ is a functor compatible with the tensor product (i.e., there are functorial isomorphisms $T(X) \otimes_{\mathcal{T}_2} T(Y) \simeq T(X \otimes_{\mathcal{T}_1} Y)$ compatible with associativity, commutativity, and unity constraints).

Definition 4.3. A *fibre functor* of the tensor category $\mathcal{T}$ over S is an exact k-linear tensor functor $F : \mathcal{T} \to S$-mod. Let $u : T \to S$ be a k-morphism, one defines $u^\star F : \mathcal{T} \to T$-mod in an obvious manner.

It is a fact that a fibre functor takes its values in the category of finitely generated locally free O_S-modules (see [5], 1.9). The fact that any descent data for the $fpqc$-topology is effective in the category of coherent sheaves on an affine scheme implies the same property in the category of fiber functors of a symmetric tensor category.

Definition 4.4. A *tannakian category* over k is a symmetric tensor category over the field k which has a fibre functor over some k-scheme $S \neq \emptyset$.

Remark that if $\mathcal{T}$ is endowed with a fiber functor ω on the k-scheme S, any point $x : \mathrm{Spec}(K) \to S$ over some field extension K of k gives rise to a fibre functor $x^*\omega$ over the field K.

Definition 4.5. A neutral tannakian category is a tannakian category for which there exists a fibre functor over the base field k.

Definition 4.6. Let ω_1 and ω_2 be two fibre functors of the tannakian category $\mathcal{T}$ on S. Following Deligne [5] we denote by $\underline{\mathrm{Isom}}_S^{\otimes}(\omega_1, \omega_2)$ the functor which send $u : T \to S$ to the set of natural isomorphisms of tensor functors between $u^\star\omega_1$ and $u^\star\omega_2$. It is representable by an affine scheme over S ([5], 1.11).

If ω_1 and ω_2 are two fibre functors of the tannakian category $\mathcal{T}$ over S_1 and S_2, we denote by $\underline{\mathrm{Isom}}_k^{\otimes}(\omega_2, \omega_1) = \underline{\mathrm{Isom}}_{S_1 \times_k S_2}^{\otimes}(pr_2^\star\omega_2, pr_1^\star\omega_1)$.

If ω is a fibre functor over S, we define

$$\underline{\mathrm{Aut}}_k^{\otimes}(\omega) = \underline{\mathrm{Isom}}_k^{\otimes}(\omega, \omega) = \underline{\mathrm{Isom}}_{S \times_k S}^{\otimes}(pr_2^\star\omega, pr_1^\star\omega)$$

this means that for any k-morphism $(b, a) : T \to S \times_k S$, then $\underline{\mathrm{Aut}}_k^{\otimes}(\omega)(T) = \mathrm{Isom}^{\otimes}(a^*\omega, b^*\omega)$.

4.2. Fundamental example

As we will see in the next section, the following example of tannakian category describes in fact the general situation.

Theorem 4.1 ([5], Theorem 1.12). *Let G be a k-groupoid acting transitively on a k-scheme S. Then the category* $\mathrm{Rep}_S(G)$ *is a tannakian category over k and the forgetful functor* $\mathrm{forget} : \mathrm{Rep}_S(G) \to S$*-mod is a fibre functor. Moreover*

$$G \simeq \underline{\mathrm{Aut}}_k^{\otimes}(\mathrm{forget}).$$

In particular when G is a k-group scheme, $S = \mathrm{Spec}(k)$, one gets the following bijective correspondence:

Corollary 4.1. *Let G and H be two k-group schemes. Any morphism of k-groups $\varphi : G \to H$ induces a tensor functor $\tilde{\varphi} : \mathrm{Rep}_k(H) \to \mathrm{Rep}_k(G)$ and the correspondence $\varphi \to \tilde{\varphi}$ is a bijection between morphisms of k-groups $\varphi : G \to H$ and tensor functors $\tilde{\varphi} : \mathrm{Rep}_k(H) \to \mathrm{Rep}_k(G)$ satisfying* $\mathrm{forget}_{kG} \circ \tilde{\varphi} = \mathrm{forget}_{kH}$.

Proof. The first assertion is clear. In the other direction let $F : \mathrm{Rep}_k(H) \to \mathrm{Rep}_k(G)$ be a tensor functor satisfying $\mathrm{forget}_{kG} \circ F = \mathrm{forget}_{kH}$. One gets a morphism $\mathrm{Aut}^{\otimes}(\mathrm{forget}_{kG}) \to \mathrm{Aut}^{\otimes}(\mathrm{forget}_{kH})$ defined by $\alpha \to \alpha \circ 1_F$. According to Theorem 4.1 $\mathrm{Aut}^{\otimes}(\mathrm{forget}_{kH}) \simeq H$ and $\mathrm{Aut}^{\otimes}(\mathrm{forget}_{kG}) \simeq G$. So the functor F induces a morphism $\varphi : G \to H$. One checks easily that these correspondences are inverse of each other. □

We have the following result whose proof relies on the correspondence between gerbes and groupoids:

Proposition 4.1 (see [5], Section 3, 3.5.1). *Let $T \to S$ be a morphism of k-schemes. Then there is an equivalence of tannakian categories*

$$\mathrm{Rep}_S(G) \equiv \mathrm{Rep}_T(G_T).$$

Example 4.1 (A trivial one). Take $S = \mathrm{Spec}(k)$ where k is a field, and $G = \{1\}$ is the trivial group. And let $T = \mathrm{Spec}(L)$ where L is a finite Galois extension of k. Then $\mathrm{Rep}_k(G) = k$-mod the category of finite-dimensional k-vector spaces. The groupoid G_T is $\mathrm{Spec}(L) \times_k \mathrm{Spec}(L)$, and the category $\mathrm{Rep}_L(G_L)$ is the category of finite-dimensional L-vector spaces endowed with descent data from L to k.

Corollary 4.2. *Suppose that the k-scheme S has a k-rational point x. Then the category* $\mathrm{Rep}_S(G)$ *is equivalent to the category* $\mathrm{Rep}_{\mathrm{Spec}(k)}(x^{\star}(G))$ *of representations of the k-group scheme x^*G.*

4.3. Tannakian duality

The following theorem states that the example given in Section 4.2 is the general situation for any tannakian category.

Theorem 4.2 ([5], Theorem 1.12).

1. *For any fibre functor ω of the tannakian category $\mathcal{T}$ over a k-scheme S,* $\mathrm{Aut}_k^{\otimes}(\omega)$ *is a k-groupoid acting transitively on S.*

2. *Two fibre functors are locally isomorphic for the fpqc-topology.*
3. *ω induces a tensor equivalence $\tilde{\omega} : \mathcal{T} \to \mathrm{Rep}_S(\mathrm{Aut}_k^{\otimes}(\omega))$.*

As we have seen in the preceding section descent data for the *fpqc*-topology are effective in the category of fiber functors of some tannakian category. Thus the category of fiber functors on a tannakian category over k is a stack over k. Points 1 and 2 of Theorem 4.2 imply the following result.

Corollary 4.3. *Let $\mathcal{T}$ be a tannakian category over k. The category of fiber functors over k-schemes is a gerbe over k.*

We call this gerbe the *fundamental gerbe* of the tannakian category $\mathcal{T}$.

Let $\mathcal{G}$ be the gerbe of fiber functors of some tannakian category $\mathcal{T}$ over k. And let ω be a fiber functor over some k-scheme S. Then the k-groupoid $\Gamma_{S,\mathcal{G},\omega}$ constructed in Section 3.1 is precisely the k-groupoid $\mathrm{Aut}_k^{\otimes}(\omega)$ introduced in Theorem 4.2.

Corollary 4.3 is a translation of parts 1 and 2 of Theorem 4.2. Part 3 can be reformulated as follows:

Theorem 4.3. *The correspondence which associates to an object T of the tannakian category $\mathcal{T}$ the representation of the fundamental gerbe $\mathcal{G}_{\mathcal{T}}$ of $\mathcal{T}$ given by*

$$\omega \to \omega(T)$$

is an equivalence of tannakian categories

$$\mathcal{T} \equiv \mathrm{Rep}(\mathcal{G}_{\mathcal{T}}).$$

In the case of a neutral tannakian category – which means that the gerbe of fiber functors is neutral, in other words there exists a fiber functor over k – the duality theorem has the following expression:

Theorem 4.4.

1. *For any fibre functor ω of the tannakian category $\mathcal{T}$ over k, $\mathrm{Aut}_k^{\otimes}(\omega)$ is a faithfully flat affine k-group scheme.*
2. *Two fibre functors are locally isomorphic for the fpqc-topology.*
3. *ω induces a tensor equivalence $\tilde{\omega} : \mathcal{T} \to \mathrm{Rep}_k(\mathrm{Aut}_k^{\otimes}(\omega))$.*

Let $\mathcal{T}_1, \mathcal{T}_2$ be two neutral tannakian categories endowed with neutral fiber functors ω_1 and ω_2 and $F : \mathcal{T}_1 \to \mathcal{T}_2$ a tensor functor such that $\omega_2 \circ F \simeq \omega_1$. Then F induces a morphism $\varphi : G_2 = \mathrm{Aut}^{\otimes}(\omega_2) \to G_1 = \mathrm{Aut}^{\otimes}(\omega_1)$ between the associated group schemes. In the other direction morphisms $\varphi : G_2 \to G_1$ between k-group schemes give rise to tensor functors $F : \mathrm{Rep}_k(G_1) \to \mathrm{Rep}_k(G_2)$ satisfying the formula $\omega_2 \circ F \simeq \omega_1$, where ω_i, $1 \leq i \leq 2$, is the forgetful functor. Modulo Theorem 4.4, these correspondences are inverses from each other.

In the situation of neutral tannakian categories, the following proposition states the link between properties of the morphism φ and properties of the functor $\tilde{\varphi}$ (see [18], Appendix, Proposition 3).

Proposition 4.2. *Let $\varphi : G \to H$ be a morphism of affine group schemes over a field k, and $\tilde{\varphi}$ the corresponding functor* $\mathrm{Rep}_k(H) \to \mathrm{Rep}_k(G)$.

1. *φ is faithfully flat if and only if $\tilde{\varphi}$ is fully faithful and every subobject of an object of the image of $\tilde{\varphi}$ is in the essential image of $\tilde{\varphi}$.*
2. *φ is a closed immersion if and only if every object of* $\mathrm{Rep}_k(G)$ *is isomorphic to a subquotient of an object of the essential image of $\tilde{\varphi}$.*

4.4. Fiber functors and torsors

Let G be an affine k-group scheme. In what follows *BG will denote the gerbe of right G-torsors over k-schemes.* It is a neutral gerbe with the trivial torsor G_d (G acting on it-self by right multiplication) being a section over $\mathrm{Spec}(k)$. Let V be a representation of G and ξ be a G-torsor on some k-scheme S. One can define the twisted sheaf of V by the torsor ξ which is a quasi coherent sheaf on S in the following way. The torsor $\xi : S \to BG$ corresponds to some cocycle c_{ij} with values in G with respect to some $fpqc$-covering S_i, $i \in I$, of S. This cocycle gives gluing data between the objects $V \times_S S_i$ and $V \times_S S_j$ over the intersection $S_i \times_S S_j$. As descent data with respect to $fpqc$-topology is effective for quasi-coherent sheaves over S, one gets a quasi-coherent sheaf that we denote following [22]

$$\xi \times^G V.$$

One can check that this construction does not depend on the $fpqc$-covering trivializing the torsor ξ and that one gets a bifunctor

$$BG \times \mathrm{Rep}_k\, G \to Coh$$

where Coh denotes the category of coherent sheaves on k-schemes, which is compatible with base changes. For instance a morphism $\alpha : \xi \to \xi'$ between two torsors on S given by the cocycles c_{ij} and c'_{ij} is given by a collection of elements $g_i \in G(S_i)$ satisfying relations $g_i c_{ij} = c'_{ij} g_j$ on S_{ij}, and thus morphisms $g_i : V \times_{\mathrm{Spec}(k)} S_i \simeq V \times_{\mathrm{Spec}(k)} S_i$ are compatibles with the gluing data given on S_{ij} by the cocycles c_{ij} and c'_{ij} and finally give a morphism $\xi \times^G V \to \xi' \times^G V$ on S[1]. This construction is a particular case of a twisting by a torsor operation which is explained in the appendix at the end of the paper.

Lemma 4.1. *The functor*

$$\begin{aligned} \mathrm{Rep}_k\, G &\xrightarrow{U} BG\text{-mod} \\ V &\longrightarrow (\xi \to \xi \times^G V) \end{aligned}$$

[1]There is a different description of $\xi \times^G V$ (see for instance [18], 2.2): suppose that the G-torsor ξ is given by $\xi = \{\pi : T \to X\}$ and V is a representation of G. Then for any open set $U \subset X$, $(\xi \times^G V)(U) \simeq (V \otimes_k O_{\pi^{-1}(U)})^G$ where G acts diagonally.

is an equivalence of categories whose a quasi inverse is given by the functor

$$BG\text{-mod} \xrightarrow{W} \mathrm{Rep}_k\, G$$
$$F \longrightarrow F(G_d)$$

where G_d denotes the trivial torsor (G acting on itself by right multiplication).

Proof. The fact that U and W are quasi-inverse of each other boils down to the following natural isomorphisms

$$G_d \times^G V \simeq V \tag{1}$$
$$\xi \times^G F(G_d) \simeq F(\xi) \tag{2}$$

Formula (1) is an immediate consequence of the definition. For the proof of formula (2), see the appendix at the end of this article.

The fact that W is compatible with tensor product is clear and as a consequence the same is true for U. □

Let $\mathcal{T}$ be a tannakian category over the field k. Suppose we are given a fiber functor $\omega : \mathcal{T} \to S$-mod where S is some k-scheme. Let $G = \mathrm{Aut}_S^{\otimes}(\omega)$ which is an affine group scheme over S. Then for any object T of $\mathcal{T}$, G acts naturally on $\omega(T)$ which becomes an object of $\mathrm{Rep}(G)$ over S. Then ω factors as $\omega = \mathrm{forget} \circ \tilde{\omega}$ where $\tilde{\omega} : \mathcal{T} \to \mathrm{Rep}_S(G)$. As we already mentioned descent data with respect to $fpqc$-topology is effective for fiber functors. So the operation of twisting by a G-torsor defined in the introduction to Lemma 4.1 makes sense for fiber functors: if ξ is a G-torsor over S, with $G = \mathrm{Aut}^{\otimes}(\omega)$, then $\xi \times^G \omega$ will denote the fiber functor

$$\mathcal{T} \to S\text{-mod}$$
$$T \to \xi \times^G \tilde{\omega}(T)$$

With this definition one can state the following proposition:

Proposition 4.3. *Let $\mathcal{T}$ be a tannakian category over the field k and $\omega : \mathcal{T} \to S$-mod a fibre functor over S. Let $G = \mathrm{Aut}_S^{\otimes}(\omega)$ which is a group scheme over S. Then the correspondence*

$$\mathcal{G}_{\mathcal{T}|S} \to BG_S$$

which associates to any fiber functor ω' over some S-scheme $u : S' \to S$ the u^*G-torsor $\mathrm{Isom}_{S'}^{\otimes}(u^*\omega, \omega')$ is an equivalence of gerbes over S.

A quasi-inverse is given by the functor

$$BG_S \to \mathcal{G}_{\mathcal{T}|S}$$

whose description over $u : S' \to S$ is

$$\xi' \to \xi' \times^{u^*G} u^*\omega.$$

Sketch of the proof. The fact that $\mathrm{Isom}^{\otimes}_{S'}(u^*\omega, \omega')$ is a torsor for the $fpqc$-topology expresses the fact that two fibre functors are locally isomorphic. In the other direction, a G-torsor over S' considered as a 1-cocycle for the $fpqc$-topology, gives descent data for the restrictions ω_i of ω to $S_i \to S'$ where $(S_i \to S')_{i\in I}$ is some $fpqc$-covering of S'. As descent data with respect to a $fpqc$-covering are effective for the fibre functors, one gets from ω and the G-torsor over S' a new fibre functor ω' over S'. □

An example of the situation described by Proposition 4.3 is given by $\mathcal{T} = \mathrm{Rep}_k G$ where G is a profinite k-group scheme, and $\omega : \mathrm{Rep}_k G \to k\text{-mod}$ the forgetful functor. One gets an equivalence between the gerbe of fiber functors on $\mathrm{Rep}_k G$ and BG.

Corollary 4.4. *Any G-torsor $\xi : S \to BG$ on some k-scheme $u : S \to \mathrm{Spec}(k)$ defines by composition a fiber functor ξ^* over S*

$$\xi^* : BG\text{-mod} \to S\text{-mod}.$$

Moreover the correspondence $\xi \to \xi^* \circ U$ is an equivalence of gerbes between BG and the gerbe of fiber functors on $\mathrm{Rep}_k G$ *(where U has been defined in Lemma* 4.1*). One has natural transformations*

$$\xi^* \circ U(-) \simeq \xi \times^G u^*(-)$$
$$\xi \simeq \mathrm{Isom}^{\otimes}(u^* \circ \mathrm{forget}_{kG}, \xi^* \circ U).$$

Proof. In view of Proposition 4.3 the only thing to check is that there is a natural isomorphism $\xi^* \circ U(-) \simeq \xi \times^G u^*(-)$. This is also an immediate consequence of definitions as the following diagram shows:

$$\begin{array}{ccccc} \mathrm{Rep}_k G & \xrightarrow{U} & BG\text{-mod} & \xrightarrow{\xi^*} & S\text{-mod} \\ V & \longrightarrow & (\alpha \to \alpha \times^G V) & \longrightarrow & \xi \times^G u^*V. \end{array}$$

□

Let $\mathcal{G}$ be a gerbe over k. If it is the gerbe of fiber functors of some tannakian category, there exists a k-scheme S and a section ω of $\mathcal{G}$ over S such that the groupoid $\mathrm{Aut}^{\otimes}_k(\omega)$ is representable by a faithfully flat scheme over $S \times_k S$.

Consider the 2-category Gtann of gerbes satisfying this property, where morphisms between gerbes are morphisms of gerbes over Sch_k. On the other hand, consider the 2-category Tann of tannakian categories over k, where morphisms between tannakian categories are exact tensor functors. Following [22] we will denote $\mathrm{Fib} : \mathrm{Tann} \to \mathrm{Gtann}$ the 2-functor which associates to a tannakian category $\mathcal{T}$ the gerbe of fiber functors on $\mathcal{T}$. In the opposite direction denote $\mathrm{Rep} : \mathrm{Gtann} \to \mathrm{Tann}$ the 2-functor which associates to a gerbe $\mathcal{G}$ in *Gtann* the category $\mathrm{Rep}(\mathcal{G})$.

Theorem 4.5 (see [22], 2.3.2). *The* 2*-functors* Fib *and* Rep *are equivalences quasi-inverse of each other.*

Proof. For any object $\mathcal{G}$ of Gtann, define

$$\alpha_{\mathcal{G}} : \mathcal{G} \to \mathrm{Fib}(\mathrm{Rep}(\mathcal{G}))$$
$$\forall \rho \quad \forall F \quad \alpha_{\mathcal{G}}(\rho)(F) = F(\rho)$$

where ρ denotes a section of $\mathcal{G}$ over some k-scheme S, and F an object of $\mathrm{Rep}(\mathcal{G})$. If $f : \rho_1 \to \rho_2$ is a morphism in $\mathcal{G}$ over some k-scheme S, define $\alpha_{\mathcal{G}}(f)(F) = F(f) : \alpha_{\mathcal{G}}(\rho_1)(F) \to \alpha_{\mathcal{G}}(\rho_2)(F)$.

Similarly, for any tannakian category $\mathcal{T}$ in Tann, one defines

$$\beta_{\mathcal{T}} : \mathcal{T} \to \mathrm{Rep}(\mathrm{Fib}(\mathcal{T}))$$
$$\forall T \in \mathcal{T} \quad \forall \rho \quad \beta_{\mathcal{T}}(\rho) = \rho(T)$$

where ρ denotes a fiber functor of $\mathcal{T}$ over some k-scheme S. If $\lambda : T_1 \to T_2$ is a morphism in $\mathcal{T}$, define $\beta_{\mathcal{T}}(\lambda)(\rho) = \rho(\lambda)$.

The fact that $\beta_{\mathcal{T}}$ is an equivalence of tannakian categories is given by Theorem 4.3. To show that $\alpha_{\mathcal{C}}$ is an equivalence, it is enough to check it locally, in which case $\mathcal{G} \simeq BG$ for some affine group G on some k-scheme S. In this case with the notation introduced in Corollary 4.4, $\alpha_{BG}(\xi) = \xi^*$ and the claim reduces to the statement of Corollary 4.4. □

Corollary 4.5. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two gerbes in* Gtann. *Then* Rep *defines an equivalence*

$$\mathrm{Hom}(\mathcal{G}_1, \mathcal{G}_2) \simeq \mathrm{Hom}(\mathrm{Rep}(\mathcal{G}_2), \mathrm{Rep}(\mathcal{G}_1))$$

compatible with base change.

Proof. This is a consequence of Theorem 4.5 together with the commutativity of the following diagrams

$$\begin{tikzcd} \mathcal{G}_1 \arrow[r, "\alpha_{\mathcal{G}_1}"] \arrow[d, "a"'] & \mathrm{Fib}(\mathrm{Rep}(\mathcal{G}_1)) \arrow[d, "\mathrm{Fib}(\mathrm{Rep}(a))"] \\ \mathcal{G}_2 \arrow[r, "\alpha_{\mathcal{G}_2}"'] & \mathrm{Fib}(\mathrm{Rep}(\mathcal{G}_2)) \end{tikzcd}, \qquad \begin{tikzcd} \mathcal{T}_1 \arrow[r, "\beta_{\mathcal{T}_1}"] \arrow[d, "b"'] & \mathrm{Rep}(\mathrm{Fib}(\mathcal{T}_1)) \arrow[d, "\mathrm{Rep}(\mathrm{Fib}(b))"] \\ \mathcal{T}_2 \arrow[r, "\beta_{\mathcal{T}_2}"'] & \mathrm{Rep}(\mathrm{Fib}(\mathcal{T}_2)). \end{tikzcd}$$

□

5. Nori fundamental group scheme

5.1. Introduction

We return to the topological setting. Let X be a locally path connected locally simply connected topological space. We have already considered in Section 2 local systems of finite sets on the topological space X, that we have seen to be equivalent to finite topological covers of X. Consider instead now the category $\mathrm{Loc}(X)$ of local systems of $\mathbf{C}$-vector spaces of finite dimension on X. It is not difficult to see that $\mathrm{Loc}(X)$ is equivalent to the category $\mathrm{Rep}(\pi_1^{\mathrm{top}}(X))$ of finite-dimensional representations of the topological fundamental group $\pi_1^{\mathrm{top}}(X, x)$ (or equivalently of the finite-dimensional representations of the topological fundamental groupoid

$\pi_1^{\mathrm{top}}(X)$). This is a neutral tannakian category, every point x of X giving rise to a fibre functor $x^\star$

In the case X is a compact connected analytical variety, to a local system L of finite-dimensional $\mathbf{C}$-vector spaces corresponds a connection on the locally free module $L \otimes_{\mathbf{C}} O_X$, where O_X denotes the sheaf of analytic functions on X.

$$\nabla : M = O_X \otimes_{\mathbf{C}} L \to \Omega_X \otimes_{O_X} M$$

defined by $\nabla(f \otimes a) = df \otimes a$. The local system can be recovered from the connection as the sheaf of horizontal sections. In other words the sheaf of local solutions of a system of differential equations attached to the connection ∇.

We will restrict ourselves to the category $\mathrm{FLoc}(X)$ of *finite* local systems on a compact connected analytical variety X, that is local systems globally trivialised by a finite étale cover $Y \to X$. The corresponding representation of $\pi_1^{\mathrm{top}}(X, x)$ factors through a finite quotient. The starting point of Nori's construction is the following fact observed by Weil in [29]: if V is a finite-dimensional representation of a finite group on a characteristic 0 field, then there are polynomials $p, q \in \mathbf{N}[X]$, $p \neq q$ such that $p(V) \simeq q(V)$ (product is the tensor product of representations, and sum is the direct sum of representations).

Definition 5.1. An object of a tensor category $\mathcal{T}$ is finite if there are polynomials $p, q \in \mathbf{N}[X]$, $p \neq q$ such that $p(V) \simeq q(V)$ (product is the tensor product and sum is the direct sum in the category $\mathcal{T}$).

As the equivalence between local systems of vector spaces, representations of the fundamental group and vector bundles with connection commute with tensor product and direct sum, one deduces that vector bundles with connection corresponding to finite local systems of vector spaces are finite in the sense of Definition 5.1. We will see in the opposite direction that vector bundles which are finite in the sense of Definition 5.1 are trivialized by an étale finite Galois cover $Y \to X$, giving rise to a finite representation of the fundamental group of X and thus to a local system of vector spaces on X (Corollary 6.1). The following statement summarize the situation.

Theorem 5.1. *The equivalence between local systems of finite-dimensional* $\mathbf{C}$*-vector spaces on a compact connected analytical variety* X *and vector bundles with connection on* X *induce an equivalence between finite local systems and finite vector bundles. In particular finite vector bundles are endowed with a canonical connection.*

5.2. Nori tannakian category

We limit ourselves here to the case considered by Nori, when he introduced the fundamental group scheme. We are given a field k and a proper reduced k-scheme X and we assume that $H^0(X, O_X) = k$.

A vector bundle is said to be *finite* if it is finite in the sense of Definition 5.1. The category of finite vector bundles will be denoted by $F(X)$. Contrary to the case of characteristic 0 where the category $F(X)$ is tannakian, in positive

characteristic $F(X)$ is not in general an abelian category. But Nori introduced an abelian category, that of semi-stable vector bundles, and shows that $F(X)$ is a sub-category of that category: finite vector bundles are semi-stable in the sense of Nori ([18], Corollary 3.5) and the category of Nori's semi-stable bundles is abelian ([18], Lemma 3.6). This led Nori to define a larger category: the category $EF(X)$ of essentially finite vector bundles, as the "abelian hull" of $F(X)$ in the category of semi-stable vector bundles. The following statement was proved by Nori [18] (see also [24]):

Theorem 5.2. *The category* $(EF(X), \otimes, O_X, {}^{\mathrm{v}})$ *is a tannakian category.*

The category $EF(X)$ has a tautological fibre functor over X: the inclusion $i_X : EF(X) \subset X$-mod is obviously a fibre functor, where X-mod stands for the category of coherent O_X-modules. So one can use this particular fibre functor to define the fundamental groupoid.

Definition 5.2. The fundamental groupoid scheme $\pi_1(X)$ is the k-groupoid associated to the tannakian category $EF(X)$: $\pi_1(X) = \mathrm{Aut}_k^{\otimes}(i_X)$.

The tannakian duality ensures that $EF(X)$ is equivalent to the category of representations $\mathrm{Rep}_X(\pi_1(X))$. If X has a k-rational point x, this category reduces to $\mathrm{Rep}_{\mathrm{Spec}(k)}(\pi_1(X,x))$, where $\pi_1(X,x) = x^*\pi_1(X)$ is the Nori fundamental group scheme of X based at x.

5.3. Nori fundamental group scheme

In this paragraph, we assume that $X(k) \neq \emptyset$ and we choose a k-rational point $x \in X(k)$. Denote by p the structural morphism $p : X \to \mathrm{Spec}(k)$. Then x^* is a neutral fibre functor from $EF(X)$ to the category k-mod of k-vector spaces of finite dimension. The duality theorem on neutral tannakian categories has in this case the following expression:

Theorem 5.3. *The functor* x^* *factors through an equivalence of category*

$$\tilde{x} : EF(X) \to \mathrm{Rep}_k(\pi_1(X,x))$$

making the following diagram commutative

$$\begin{tikzcd} EF(X) \arrow[r, "\tilde{x}"] \arrow[dr, "x^*"'] & \mathrm{Rep}_k(\pi_1(X,x)) \arrow[d, "\mathrm{forget}_{k\pi_1(X,x)}"] \\ & k\text{-mod}. \end{tikzcd}$$

If one pulls the fundamental groupoid scheme

$$\pi_1(X) = \mathrm{Aut}^{\otimes}(i_X) \to X \times_k X$$

by $(x \circ p, 1_X) : X \to X \times_k X$ one gets $\widehat{X}_x = \mathrm{Isom}^{\otimes}(p^\star x^*, i_X)$:

$$\begin{tikzcd} \mathrm{Isom}^{\otimes}(p^\star x^*, i_X) \arrow[r] \arrow[d] & \pi_1(X) \arrow[d] \\ X \arrow[r, "x\circ p, 1_X"] & X \times_k X. \end{tikzcd}$$

Definition 5.3. $\widehat{X}_x = \mathrm{Isom}^{\otimes}(p^\star x^*, i_X)$ is the universal torsor based at x.

Lemma 5.1. *$\widehat{X}_x$ is a torsor on X under $p^\star \pi_1(X, x)$ and has a rational point $\hat{x}$ above x.*

Proof. The only point to check is that $j : \widehat{X}_x \to X$ is locally isomorphic to $X \times \pi_1(X, x)$. In other words that $\mathrm{Isom}^{\otimes}(p^\star x^*, i_X)$ is locally trivial. This is due to the general fact that two fiber functors on a tannakian category are locally isomorphic which we apply to the two fiber functors $p^* x^*$ and i_X.

Finally $x^* \widehat{X}_x \simeq \mathrm{Isom}^{\otimes}(x^\star p^* x^\star, x^\star) = \pi_1(X, x)$ has a k-rational point $\hat{x}$ corresponding to $1 \in \pi_1(X, x)$. □

The main result of this section is the following theorem.

Theorem 5.4. *The Nori fundamental group scheme is the projective limit of the family of finite k-group schemes G occurring as structural groups of torsors $Y \to X$ with a rational point in the fibre of x. The universal torsor $\hat{X}_x$ is the projective limit of the family of torsors under finite k-group schemes having a k-rational point above x. It trivializes every object of $EF(X)$.*

The proof relies on the fact that the category $EF(X)$ is the inductive limit of finitely generated full sub-tannakian categories whose tannakian Galois groups are finite. One needs the following definition:

Definition 5.4. A tannakian category $\mathcal{T}$ is generated by a set S of objects of $\mathcal{T}$ if every object of $\mathcal{T}$ is a subquotient of the direct sum of a finite number of objects of S. More precisely, for any object E of $\mathcal{T}$, there exists a finite number of objects $F_1, \ldots, F_r$ in S and sub-objects $E_1 \subset E_2 \subset \oplus_{1 \le i \le r} F_i$, such that $E \simeq E_2/E_1$.

We will use the following general fact ([18], Theorem 1.2).

Theorem 5.5. *A k-group scheme G is finite if and only if the category $\mathrm{Rep}_k(G)$ is generated by a finite number of objects.*

Using the tannakian duality theorem, one gets the following consequence:

Corollary 5.1. *A neutral tannakian category has a finite Galois group if and only if it is generated by a finite family of objects.*

Proof of Theorem 5.4. Denote by $j : \hat{X}_x \to X$. The fact that the universal torsor trivializes the objects of $EF(X)$ is an immediate consequence of the fact that $j^* \hat{X}_x \simeq \mathrm{Isom}^{\otimes}(j^* p^* x^*, j^*)$ is trivial. Thus for any object F of $EF(X)$, $j^* F \simeq j^* p^* x^* F$ which is a trivial vector bundle.

One may apply Corollary 5.1 to the category $EF(X)$. Consider a finite set S of objects of $EF(X)$ and the smallest full sub-tensor category $\langle S\rangle$ of $EF(X)$ containing S. This is a fact that $\langle S\rangle$ is generated (in the sense of Definition 5.4) by a finite number of objects, as there is a finite number of isomorphisms classes of indecomposable objects involved in the tensor powers of objects of S ([18], Lemma 3.1).

One concludes that the full tannakian subcategories $\langle S\rangle$ of $EF(X)$ where S runs in the finite sets of objects have finite Galois groups. As a consequence, $\pi_1(X,x)$, which is the tannakian Galois group of the inductive limit of the categories $\langle S\rangle$, is the projective limit of the tannakian Galois groups $\pi_1^S(X,x)$ of the categories $\langle S\rangle$. Thus it is the projective limit of k-finite group schemes. Denote $\hat{X}_x^S = \mathrm{Isom}^{\otimes}(p^*x^*_{|\langle S\rangle}, i_{X\,|\langle S\rangle})$ the universal torsor of the tannakian category $\langle S\rangle$ based at x. It is a torsor under the finite group scheme $\pi_1^S(X,x)$, whose fiber at x is isomorphic to

$$\begin{aligned} x^*\,\mathrm{Isom}^{\otimes}(p^*x^*_{|\langle S\rangle}, i_{X\,|\langle S\rangle}) &\simeq \mathrm{Isom}^{\otimes}(x^*p^*x^*_{|\langle S\rangle}, x^*{}_{|\langle S\rangle}) \\ &\simeq \mathrm{Aut}^{\otimes}(x^*_{|\langle S\rangle}) \simeq \pi_1^S(X,x) \end{aligned}$$

and has a rational point corresponding to the neutral element of $\pi_1^S(X,x)$.

The universal property of the universal torsor stated in Proposition 5.3 (see below Paragraph 5.4) will complete the proof of Theorem 5.4. □

Remark 5.1. The fundamental group scheme and the universal torsor depends on the chosen rational point $x \in X(k)$. If $y \in X(k)$ is another rational point, $\mathrm{Isom}^{\otimes}(x^*, y^*)$ is a right torsor under $\pi_1(X,x)$ and a left torsor under $\pi_1(X,y)$. It has $\bar{k}$-rational points which induce isomorphisms $\pi_1(X,x)_{\bar{k}} \simeq \pi_1(X,y)_{\bar{k}}$ and $(\hat{X}_x)_{\bar{k}} \simeq (\hat{X}_y)_{\bar{k}}$. But in general $\hat{X}_x$ and $\hat{X}_y$ are not isomorphic. We will see that at least when $ch(k) = 0$ and X is a curve of genus at least 2, if $x \neq y$, then $\hat{X}_x$ and $\hat{X}_y$ are not isomorphic over k (Theorem 6.4).

5.4. Correspondence between fibre functors and torsors

Let G be a finite k-group scheme. We are considering in this section fiber functors $F : \mathrm{Rep}_k(G) \to X$-mod from the category of finite-dimensional representations of G to the category of coherent sheaves on X. First remark the following property.

Lemma 5.2. *The fibre functor F factors through a tensor functor $\tilde{F} : \mathrm{Rep}_k(G) \to EF(X)$, i.e., $F = i_X \circ \tilde{F}$, where i_X is the inclusion $EF(X) \to X$-mod.*

Proof. The regular representation kG satisfies the relation

$$kG \otimes_k kG \simeq dkG$$

where d is the order of the group G. So the image $F(kG)$ by the fibre functor F satisfies the relation

$$F(kG) \otimes_{O_X} F(kG) \simeq dF(kG).$$

In particular, it is a finite vector bundle. As the regular representation generates the tannakian category $\mathrm{Rep}_k(G)$, one deduces that the essential image of F lies in $EF(X)$. $\square$

Proposition 4.3 applied to the category $\mathcal{T} = \mathrm{Rep}_k(G)$ gives a correspondence between fiber functors F on X and torsors T on X under the group scheme G. The relation between the two objects is given by the following formula:

$$T \simeq \mathrm{Isom}_X^{\otimes}(p^*\mathrm{forget}_{kG}, F)$$

where forget_{kG} is the forgetful functor $\mathrm{Rep}_k(G) \to k$-mod.

More generally one has the following one to one correspondence:

Proposition 5.1. *Let G be a profinite k-group scheme, and $p : X \to \mathrm{Spec}(k)$ as before a reduced proper k-scheme such that $H^0(X, O_X) = k$. There are equivalences between the following categories:*

1. *G-torsors $f : T \to X$ with morphisms of G-torsors,*
2. *morphisms $\varphi : X \to BG$ with equivalences,*
3. *exact tensor functors $F : \mathrm{Rep}_k(G) \to EF(X)$ with tensor equivalences,*
4. *morphisms of gerbes $\tilde{F} : \mathcal{G}_X \to BG$, where $\mathcal{G}_X$ denotes the gerbe of fiber functors of the category $EF(X)$ and BG is the gerbe of G-torsors with equivalences,*
5. *in the case there exists a point $x \in X(k)$, the above correspondences restrict to equivalences between*
   (a) *G-torsors $f : T \to X$ whose fiber x^*T at x has a k-rational point, with morphisms of G-torsors,*
   (b) *morphisms $\varphi : X \to BG$ such that $\varphi(x)$ is the trivial G-torsor, with equivalences,*
   (c) *exact tensor functors $F : \mathrm{Rep}_k(G) \to EF(X)$ such that $x^* \circ F \simeq \mathrm{forget}_{kG}$, with tensor equivalences,*
   (d) *morphisms of gerbes $\tilde{F} : \mathcal{G}_X \to BG$, such that $\tilde{F}(x^*)$ is the trivial torsor, with equivalences.*
   (e) *morphisms $\alpha : \pi_1(X, x) \to G$, with conjugation by elements of $\pi_1(X, x)$.*

   Moreover these correspondences are compatible with base changes $Y \to X$.

Remark the similarity between 5(e) and the description of Galois étale covers given as a consequence of Theorem 2.3. Here finite k-group schemes replace abstract finite groups and Nori's fundamental group scheme replaces Grothendieck's étale fundamental group.

Proof. Consider first the case of a finite k-group scheme G. The equivalence between 1, 2 and 3 is an immediate consequence of Proposition 4.3, using the fact that any fiber functor $\mathrm{Rep}_k(G) \to X$-mod takes its values in $EF(X)$ and the equivalence between $\mathrm{Rep}_k(G)$ and BG-mod. The equivalence between 3 and 4 is a consequence of Corollary 4.5 applied to the gerbes $\mathcal{G}_X$ and BG.

The last part of Proposition 5.1 is a consequence of the following remark which concludes the proof of the proposition for finite groups.

Lemma 5.3. *A fiber functor* $F : \mathrm{Rep}_k(G) \to EF(X) \simeq \mathrm{Rep}_k(\pi_1(X,x))$ *satisfies the relation* $x^* \circ F \simeq \mathrm{forget}_{kG}$ *if and only if the corresponding* G*-torsor has a* k*-rational point above* x*. In this case it is equivalent to a morphism of groups* $\alpha : \pi_1(X,x) \to G$*.*

Proof of the lemma.

$$x^*T \simeq x^* \operatorname{Isom}^{\otimes}(p^*\mathrm{forget}_{kG}, F) \simeq \operatorname{Isom}^{\otimes}(x^*p^*\mathrm{forget}_{kG}, x^*F)$$

$$x^*T \simeq \operatorname{Isom}^{\otimes}(\mathrm{forget}_{kG}, \mathrm{forget}_{k\pi_1(X,x)}\tilde{x}F)$$

and thus $x^*T(k) \neq \emptyset$ if and only if the following diagram is 2-commutative

$$\begin{tikzcd} \mathrm{Rep}_k(G) \arrow[r, "\tilde{x}F"] \arrow[dr, "\mathrm{forget}_{kG}"'] & \mathrm{Rep}_k(\pi_1(X,x)) \arrow[d, "\mathrm{forget}_{k\pi_1(X,x)}"] \\ & k\text{-mod.} \end{tikzcd}$$

□

In the case of a profinite k-group scheme $G = \operatorname{proj\,lim} G_i$, the structural morphisms $\varphi_{ij} : G_j \to G_i$ induce functors $\tilde{\varphi}_{ij} : \mathrm{Rep}_k(G_i) \to \mathrm{Rep}_k(G_j)$. Objects of 3, i.e., exact tensor functors $F : \mathrm{Rep}_k(G) \to EF(X)$ are families of exact tensor functors $F_i : \mathrm{Rep}_k(G_i) \to EF(X)$ satisfying, for any i, j, $j \geq i$, $F_j \circ \tilde{\varphi}_{ij} = F_i$. As for G-torsors, they are projective limits of G_i-torsors $T_i \to X$, with structural morphisms $T_j \to T_i$ $(i \leq j)$ compatibles with the actions of the G_i's on the T_i's and with the morphisms $\varphi_{ij} : G_j \to G_i$. The correspondence between torsors and fiber functors is given at the level i by the formula

$$T_i = \operatorname{Isom}^{\otimes}(p^*\mathrm{forget}_{kG_i}, F_i)$$

and for any $j \geq i$,

$$T_i = \operatorname{Isom}^{\otimes}(p^*\mathrm{forget}_{kG_j}\tilde{\varphi}_{ij}, F_j\tilde{\varphi}_{ij}) \simeq T_j \times^{G_j} G_i$$

the last term being the contracted product of T_j by G_i along the morphism $\varphi_{ij} : G_j \to G_i$. The last isomorphism is a consequence of the following lemma whose proof is left to the reader. □

Lemma 5.4. *Let* $\Phi : \mathcal{T} \to \mathcal{T}'$ *be a tensor functor between two tannakian categories over the field* k*. Let* S *be a* k*-scheme,* F *and* G *two fibre functors over* S*. Then there is a canonical isomorphism of right torsors*

$$\operatorname{Isom}^{\otimes}(F\Phi, G\Phi) \simeq \operatorname{Isom}^{\otimes}(F, G) \times^{\operatorname{Aut}^{\otimes}(F)} \operatorname{Aut}^{\otimes}(F\Phi).$$

If one is given a morphism $\alpha : \pi_1(X,x) \to G$, we denote by

$$\widehat{X_x} \times^{\pi_1(X,x)} G$$

the contracted product for the morphism φ. This is a right G-torsor.

Proposition 5.2. *If* T *has a* k*-rational point over* x*, then* $T \simeq \widehat{X_x} \times^{\pi_1(X,x)} G$*.*

Proof. Recall that $\widehat{X_x} = \mathrm{Isom}^{\otimes}(p^\star x^*, i_X)$. Using Lemma 5.4, one has

$$(\star) \quad \widehat{X_x} \times^{\pi_1(X,x)} G \simeq \mathrm{Isom}^{\otimes}(p^\star x^* \tilde{F}, i_X \tilde{F}).$$

Using the definition of T, one gets

$$x^*T \simeq \mathrm{Isom}^{\otimes}(x^* p^\star \mathrm{forget}_{kG}, x^*\tilde{F}) = \mathrm{Isom}^{\otimes}(\mathrm{forget}_{kG}, x^*\tilde{F}).$$

The fact that T has a k-point over x means that x^*T is trivial, and then the functors forget_{kG} and $x^*\tilde{F}$ are equivalent.

Replacing in the formula $(\star)$, we get

$$\widehat{X_x} \times^{\pi_1(X,x)} G \simeq \mathrm{Isom}^{\otimes}(p^\star \mathrm{forget}_{kG}, i_X \tilde{F}) \simeq T$$

which completes the proof of the proposition. □

From Proposition 5.2 we deduce the universal property of the universal torsor $\widehat{X}_x$:

Proposition 5.3. *Let $T \to X$ be a torsor under a finite group scheme G. Suppose that the fibre of $x \in X(k)$ has a k-rational point $t \in T(k)$. Then there is a unique couple of morphisms (f, α), where $\alpha : \pi_1(X,x) \to G$ is a morphism of k-groups and $f : \widehat{X}_x \to T$ is a morphism of $\pi_1(X,x)$- and G-torsors such that $f(\hat{x}) = t$ and making the following diagram commutative:*

$$\begin{tikzcd} \widehat{X}_x \arrow[r, "f"] \arrow[dr] & T \arrow[d] \\ & X. \end{tikzcd}$$

Proof. This is just a reformulation of Proposition 5.2, once we notice that the obvious morphism

$$\widehat{X_x} \to \widehat{X_x} \times^{\pi_1(X,x)} G$$

is a morphism of $\pi_1(X,x)$- and G-torsors. □

Proposition 5.4. *With the hypothesis and notations of point* 5 *of Proposition* 5.1*, the following statements are equivalent*

1. $H^0(T, O_T) = k$
2. *α is surjective for the fpqc topology*
3. *F is fully faithful*

The proof relies on the following remark:

Lemma 5.5. *Let G be an affine group scheme and $F : \mathrm{Rep}_k(G) \to EF(X)$ a fully faithful tensor functor satisfying $F(1) = O_X$. Then for any representation V of G, $H^0(X, F(V)) \simeq V^G$.*

Proof. We have the following equalities:

$$\begin{aligned} V^G &\simeq \mathrm{Hom}(V^\vee, k) \simeq \mathrm{Hom}(F(V))^\vee, O_X) \\ &\simeq H^0(X, \underline{\mathrm{Hom}}(F(V)^\vee, O_X)) \simeq H^0(X, F(V)). \end{aligned}$$

□

Proof of the proposition. 1. Applying Lemma 5.5 to the equivalence of categories $\tilde{x}^{-1} : \pi_1(X,x)\text{-mod} \to EF(X)$, one gets that for any representation V of $\pi_1(X,x)$, $H^0(X, \tilde{x}^{-1}(V)) \simeq V^G$.

2. Let F be as in the proposition that we assume to be fully faithful and $j : T \to X$ the associated torsor. Then we have the following equalities

$$H^0(T, O_T) = H^0(X, j_\star O_T) = H^0(X, F(kG)) \simeq (kG)^G = k.$$

This proves the implication (3) $\Rightarrow$ (1).

3. Suppose that α is not faithfully flat. It factors $\alpha : \pi_1(X,x) \to H \hookrightarrow G$ where $H \hookrightarrow G$ is a closed immersion, $H \neq G$. Then $(kG)^{\pi_1(X,x)} \neq k$. Using the first point of the proof, one gets

$$H^0(T, O_T) \simeq H^0(X, j_\star O_T) \simeq H^0(X, \tilde{x}^{-1}(kG)) \simeq (kG)^G \neq k.$$

This proves (1) $\Rightarrow$ (2).

4. Finally the implication (2) $\Rightarrow$ (3) is a consequence of Proposition 4.2. $\square$

Let us finally remark that in Theorem 5.4 one can restrict the projective limit to the torsors T over X under a finite group scheme G such that the corresponding morphism $\pi_1(X,x) \to G$ is surjective for the $fpqc$ topology, or equivalently such that, $H^0(T, O_T) = k$.

Example 5.1. Consider $X = \mathrm{Spec}(k)$. A torsor $T \to \mathrm{Spec}(k)$ under a finite group G which has a k-rational point is trivial: $T \simeq G$. If one requires that $H^0(T, O_T) = k$, then $T \simeq G \simeq \mathrm{Spec}(k)$. One gets that $\pi_1(\mathrm{Spec}(k), x) = 1$.

On the other hand, the triviality of $\pi(\mathrm{Spec}(k), \star)$ is obvious, if one considers it as the tannakian Galois group of the category k-mod, of k-vector spaces of finite dimension.

Notice that the statement of Proposition 5.4 applies in particular to the universal torsor $T = \hat{X}_x$. With the notations of the proposition, it corresponds to $\alpha = Id_{\pi_1(X,x)}$ and $F = \tilde{x}^{-1}$. So one gets $H^0(\hat{X}_x, O_{\hat{X}_x}) = k$ or in other terms $(pj)_* O_{\hat{X}_x} = k$.

As the universal torsor $j : \hat{X}_x = \mathrm{Isom}^{\otimes}(p^*x^*, i_X) \to X$ is trivialized by itself, for any object E of $EF(X)$, $j^*E \simeq j^*p^*(x^*E)$ is trivial. Then the projection formula together with the fact that $(pj)_* O_{\hat{X}_x} = k$, implies that

$$H^0(\hat{X}_x, j^*E) \simeq (pj)_* j^*E \simeq (pj)_*(pj)^*(x^*E) \simeq x^*E.$$

One gets the following result:

Proposition 5.5. *The fiber functor x^* is isomorphic to the functor $H^0(\hat{X}_x, j^*(.))$ which associates to any essentially finite vector bundle E on X the vector space of global section of j^*E.*

This fact holds not only for x^* with $x \in X(k)$ but for any neutral fiber functor $\rho : EF(X) \to k$-mod.

Proposition 5.6. *Let ρ be a neutral fiber functor and $j : \hat{X}^\rho = \mathrm{Isom}^\otimes(p^*\rho, i_X) \to X$ the corresponding universal torsor. Then one recovers ρ from the universal torsor as $\rho \simeq H^0(\hat{X}^\rho, j^*(.))$.*

In the other direction let $f : T \to X$ be a torsor under a profinite k-group scheme G. Assume that

- $H^0(T, O_T) = k$
- *$f : T \to X$ trivializes all essentially finite vector bundles of X.*

There exists a unique (up to equivalence) neutral fiber functor ρ such that $f : T \to X$ is isomorphic to the universal torsor $\hat{X}^\rho \to X$.

Proof. As in the case of fiber functors coming from rational points, one has the equality $\rho(j_*O_{\hat{X}^\rho}) \simeq k\pi_1(X, \rho)$ where $k\pi_1(X, \rho)$ is the k-Hopf algebra of the fundamental group scheme based at ρ. Or equivalently, if $\tilde{\rho}$ denotes the equivalence $EF(X) \simeq \mathrm{Rep}_k(\pi_1(X, \rho))$ induced by ρ, $j_*O_{\hat{X}^\rho} \simeq \tilde{\rho}^{-1}(k\pi_1(X, \rho))$.

On the other hand, $\hat{X}^\rho \simeq \mathrm{Isom}^\otimes(p^*\mathrm{forget}_{k\pi_1(X,\rho)}, \tilde{\rho}^{-1})$, and $H^0(\hat{X}^\rho, O_{\hat{X}^\rho}) \simeq H^0(X, j_*O_{\hat{X}^\rho}) \simeq H^0(X, \tilde{\rho}^{-1}(k\pi_1(X, \rho)))$, and according to Lemma 5.5,

$$H^0(X, \tilde{\rho}^{-1}(k\pi_1(X, \rho))) \simeq (k\pi_1(X, \rho))^{\pi_1(X,\rho)} = k$$

which implies $H^0(\hat{X}^\rho, O_{\hat{X}^\rho}) = k$.

So the argument given in the proof of Proposition 5.5 for x^* holds for ρ and one gets an isomorphism $\rho \simeq H^0(\hat{X}^\rho, p^*(.))$.

Conversely suppose we are given a torsor $f : T \to X$ under a k-group scheme G such that

- $H^0(T, O_T) = k$
- $f : T \to X$ trivializes all essentially finite vector bundles of X.

Define $\rho : EF(X) \to k$-mod to be $\rho = H^0(T, f^*())$. One checks that ρ is neutral fiber functor for $EF(X)$. Moreover if $j : \hat{X}^\rho \to X$ is the universal torsor based at ρ, one has an isomorphism of functors on $EF(X) : j^* \simeq j^*p^*\rho$. Applying this to f_*O_T, one gets $j^*f_*O_T \simeq j^*p^*\rho f_*O_T$. But $\rho f_*O_T \simeq H^0(T, f^*f_*O_T) \simeq H^0(T, O_T \otimes_k kG) \simeq kG$. Using the unit element of G which defines a morphism $\epsilon : kG \to k$, one gets a morphism $j^*f_*O_T \to j^*p^*k \simeq O_{\hat{X}^\rho}$. Thus in the following cartesian diagram

$$\begin{tikzcd} T \times_X \hat{X}^\rho \arrow[r] \arrow[d] & \hat{X}^\rho \arrow[d, "j"] \\ T \arrow[r, "f"] & X \end{tikzcd}$$

the first horizontal map has a section, of equivalently, there is an X-morphism $h : \hat{X}^\rho \to T$. Then there exists a unique morphism of groups $\pi_1(X, \rho) \to G$ such that h is a morphism of torsors [19] (Lemma 1). On the other hand, as $f : T \to X$ trivializes every object of $EF(X)$, it trivializes $j_*O_{\hat{X}^\rho}$, which means that the left vertical map of the above diagram has a section. This gives a X-morphism $T \to \hat{X}^\rho$, and thus $h : \hat{X}^\rho \to T$ is an isomorphism of torsors. $\square$

5.5. Base change

Let $\mathcal{T}$ be a tannakian category over the field k and L a finite extension of k. One can define a new category $\mathcal{T}_L$ in the following manner: the objects of $\mathcal{T}_L$ are couples (X, α) where X is an object of $\mathcal{T}$ and $\alpha : L \to \mathrm{End}_{\mathcal{T}}(X)$ is a morphism of k-algebras (with $k \subset \mathrm{End}_{\mathcal{T}}(X)$ by $a \to a1_X$). Morphisms between two objects $f : (X, \alpha_X) \to (Y, \alpha_Y)$ are morphisms $f : X \to Y$ in $\mathcal{T}$ compatible with the action of L via α_X and α_Y.

The tensor product of two objects in $\mathcal{T}_L$ is defined as follows: let (X, α_X) and (Y, α_Y) be two objects. The tensor product $(X, \alpha_X) \otimes (Y, \alpha_Y)$ in the new category is the biggest quotient in $\mathcal{T}$ of $X \otimes Y$ where $1 \otimes \alpha_Y(a) = \alpha_X(a) \otimes 1$ for all $a \in L$.

Moreover $\mathcal{T}_L$ is endowed with a k-linear tensor functor $t : \mathcal{T} \to \mathcal{T}_L$, inducing for any objects X, Y of $\mathcal{T}$ an isomorphism $\mathrm{Hom}_{\mathcal{T}}(X, T) \otimes_k L \simeq \mathrm{Hom}_{\mathcal{T}_L}(t(X), t(Y))$ ([25], Th. 1.3.18).

Proposition 5.7. ([6], [25], Th. 3.1.3) *$\mathcal{T}_L$ is a tannakian category.*

Example 5.2. If $\mathcal{T} = k$-mod, then $(k\text{-mod})_L \simeq L$-mod.

Example 5.3. Let G be a k-group scheme and $\mathcal{T} = \mathrm{Rep}_k(G)$. Then $(\mathrm{Rep}_k(G))_L \simeq \mathrm{Rep}_L(G \times_k L)$.

Let $A = kG$ be the Hopf algebra of G. An object of $(\mathrm{Rep}_k(G))_L$ is a finitely generated k-vector space V with a co-action $\delta : V \to A \otimes_k V$ together with an action of L on V compatible with the co-action, This boils down to a L-vector space V with a co-action $\delta : V \to A \otimes_k V$ which is L-linear (L acting on $A \otimes_k V$ through V). But $A \otimes_k V \simeq (A \otimes_k L) \otimes_L V$ canonically and δ can be reinterpreted as a L-co-action of $A \otimes_k L$ on V, or as a representation of $G \times_k L$ on V viewed as L-vector space.

Theorem 5.6. ([6] Prop. 3.11, [25] Prop. 3.1.2) *Let $\mathcal{T}$ be a tannakian category over a field k, and consider a field extension L of k. For every L-scheme S', the functor $() \circ t$*

$$\{\text{fibre functors on } \mathcal{T}_L \text{ over } S'\} \simeq \{\text{fibre functors on } \mathcal{T} \text{ over } S'\}$$

is an equivalence of categories.

Let us interpret this extension of scalars in the case of the category of essentially finite vector bundles. Let $X \to \mathrm{Spec}(k)$ be a locally noetherian reduced proper scheme satisfying the condition $H^0(X, O_X) = k$. Suppose that there exists a rational point $x \in X(k)$. Let L be a finite extension of k. Denote by f the morphism $X_L \to X$. We assume X_L to be reduced. The following interpretation of $EF(X)_L$ can be extracted from the proof of Proposition 3.1 of [15].

Lemma 5.6. *The following categories are equivalent:*

1. $EF(X)_L$
2. $\mathrm{Rep}_L(\pi_1(X, x)_L)$
3. *The full subcategory $EF(X_L)'$ of $EF(X_L)$ of objects F such that there exists an object F_1 of $EF(X)$ such that F is a subobject of f^*F_1.*

Proof of the lemma. The equivalence between the two first categories comes from the equivalence between $EF(X)$ and $\mathrm{Rep}_k(\pi_1(X,x))$ induced by the fiber functor $x^* : EF(X)_L \simeq \mathrm{Rep}_k(\pi_1(X,x))_L \simeq \mathrm{Rep}_L(\pi_1(X,x) \times_k L)$.

Let W be an object of $\mathrm{Rep}_L(\pi_1(X,x) \times_k L)$. This representation factors through a finite quotient $\alpha : \pi_1(X,x) \to G$. One knows that to this morphism α corresponds a morphism $\varphi : X \to BG$ and a G-torsor on X.

Consider the following cartesian diagram:

$$\begin{tikzcd} X_L \arrow[r, "f"] \arrow[d, "\varphi_L"] & X \arrow[d, "\varphi"] \\ BG_L \arrow[r, "g"'] & BG. \end{tikzcd}$$

To W we associate $F = \varphi_L^* W$ which is an object of $EF(X_L)$. One can easily check that F does not depend on the group G. Suppose indeed that α factors as $\alpha = \beta \circ \alpha'$ where $\alpha' : \pi_1(X,x) \to G'$ and $\beta : G' \to G$. One deduces morphisms

$$X_L \xrightarrow{\varphi'_L} BG'_L \xrightarrow{\tilde{\beta}} BG_L$$

and $\varphi_L = \tilde{\beta} \circ \varphi'_L$. And thus ${\varphi'_L}^* \tilde{\beta}^* W \simeq \varphi_L^* W$.

Moreover W can be embedded in the sum of a finite number of copies of the regular representation $W \subset (LG_L)^{\oplus d} \simeq g^*(kG)^{\oplus d}$. Thus $F = \varphi_L^* W \subset \varphi_L^* g^*(kG)^{\oplus d} \simeq f^* \varphi^*(kG)^{\oplus d}$. Then $F_1 = \varphi^*(kG)^{\oplus d}$ is an object of $EF(X)$ and F is a subobject of $f^* F_1$. We proved that F is an object of $EF(X_L)'$.

In the other direction, if F is an object of $EF(X_L)'$, it corresponds to some representation W of $\pi_1(X_L,x)$. Let F_1 be an object of $EF(X)$ such that $F \subset f^* F_1$. It corresponds to a representation V of $\pi_1(X,x)$. Then $V \times_k L$ is a representation of $\pi_1(X,x)_L$ and can be considered also as a representation of $\pi_1(X_L,x)$ through the morphism $\pi_1(X_L,x) \to \pi_1(X,x)_L$. By hypothesis $W \subset V \otimes_k L$. Thus the representation W factors through a group G_L where G is a finite quotient of $\pi_1(X,x)$. It means that W is an object of $\mathrm{Rep}_k(\pi_1(X,x)_L)$ and $F = \varphi_L^* W$ an object of $EF(X)_L$. □

Proposition 5.8. *Let L be a finite separable extension of k. For any X as in the statement of Lemma* 5.6*, X_L is reduced, and there is an equivalence of tannakian categories*

$$EF(X)_L \simeq EF(X_L)$$

and an isomorphism of group schemes over L

$$\pi_1(X_L,x) \simeq \pi_1(X,x) \times_k L.$$

Moreover there is a unique isomorphism of pointed torsors compatible with the preceding isomorphism of groups schemes:

$$\hat{X}_L^x \simeq (\hat{X}^x)_L.$$

Proof. As X is reduced and $X_L \to X$ is étale, X_L is reduced [21]. We have to show that $EF(X_L)' = EF(X_L)$. It suffices to show that generators of the category $EF(X_L)$ are objects of $EF(X_L)'$. We know that p_*O_T generates $EF(X_L)$ when $p : T \to X_L$ runs in the family of pointed torsors under finite group schemes satisfying the condition $H^0(T, O_T) = L$. We are going to show that any such torsor is dominated by a finite torsor of the form $p'_L : f^*T' \to X_L$, where $p' : T' \to X$ is a finite torsor pointed above x satisfying $H^0(T', O_{T'}) = k$: precisely there exists a faithfully flat morphism $q : T'_L \to T$ such that $p'_L = p \circ q$. It follows that $O_T \subset q_*O_{T'_L}$ is a subobject, and thus $p_*O_T \subset p_*q_*O_{T'_L} = p'_{L*}O_{T'_L} = f^*p_*O_T$. We conclude by Lemma 5.6 that p_*O_T is an object of $EF(X_L)'$.

As the universal torsor $\hat{X}^x_L$ is the projective limit of torsor $p : T \to X_L$ of the type considered above, the following fact will conclude the proof of the proposition: there is a morphism of torsors $(\hat{X}^x)_L \to \hat{X}^x_L$. We follow the proof of Proposition 5 of [18], assuming that L is a Galois extension of k of group $\Gamma = \mathrm{Gal}(L/k)$. For any $\sigma \in \Gamma$, there exists a unique morphism of torsor $f_\sigma : {}^\sigma\hat{X}^x_L \to \hat{X}^x_L$ sending ${}^\sigma\hat{x}_L$ to $\hat{x}_L$ by the universal property of the universal torsor (Proposition 5.3). The morphisms f_σ satisfy clearly Weil cocycle condition and by descent give rise to a torsor under a k-pro-finite group scheme $p : T \to X$ pointed above x and such that $p_L : T_L \to X_L$ is isomorphic to $\hat{X}^x_L \to X_L$. The condition that $H^0(\hat{X}^x_L, O_{\hat{X}^x_L}) = L$ implies that $H^0(T, O_T) = k$ and by the universal property of the universal torsor again, there is a unique morphism of pointed torsors $\hat{X}^x \to T$. Extending the scalars to L one gets finally a morphism of pointed torsors $(\hat{X}^x)_L \to \hat{X}^x_L$. It is clear that this morphism is the inverse of the natural morphism $\hat{X}^x_L \to (\hat{X}^x)_L$ and that it is an isomorphism. This concludes the proof in the case of a finite Galois extension L of k. In the general case one introduces the Galois closure K of L over k, and compare the universal torsors over k, L and K. □

Remark 5.2. From the equivalence $EF(X_L) \simeq EF(X)_L$ one deduces that for any fiber functor F of $EF(X)$ over k, the same isomorphism holds

$$\pi_1(X_L, F_L) \simeq \pi_1(X, F) \times_k L$$

$$\hat{X}^F_L \simeq (\hat{X}^F)_L.$$

Indeed according to Theorem 5.6 the fiber functor F_L can be considered as a fiber functor on the category $EF(X)_L \simeq EF(X_L)$. And $\pi_1(X_L, F_L) = \mathrm{Aut}^\otimes(F_L)$. But $\mathrm{Aut}^\otimes(F_L) \simeq \mathrm{Aut}^\otimes(F) \times_k L$, where F_L in the left-hand side is considered as a fiber functor in $EF(X)_L$.

In [18] Nori conjectured that the isomorphism of Proposition 5.8 holds for an arbitrary field extension L of k. However some counterexamples were given later, first by V.B. Mehta and S. Subramanian in [15] (where X is some singular curve) and then by C. Pauly [20] (where X is some smooth projective curve over an algebraically closed field of characteristic 2).

6. Characteristic 0 case

In this paragraph the base field k is supposed to be of characteristic 0.

Proposition 6.1. *Let X be a scheme over a characteristic 0 field k, any torsor over X under a finite group scheme G is étale over X.*

This is a consequence of the following fact [28]:

Theorem 6.1. *Every finite k-group over a characteristic 0 field k is étale.*

The aim of this section is to compare the fundamental group scheme of X introduced by Nori and the Grothendieck's étale fundamental group. The geometric fundamental group and the algebraic fundamental groups fit in the classical short exact sequence:

$$1 \to \pi_1^{et}(X \times_k \bar{k}, \bar{x}) \to \pi_1^{et}(X, \bar{x}) \to \mathrm{Gal}(\bar{k}/k) \to 1$$

where $\bar{x}$ is the geometric point attached to x and to the choice of an algebraic closure $\bar{k}$ of k. The rational point $x \in X(k)$ defines a section s_x of this exact sequence, and thus a continuous action of $\mathrm{Gal}(\bar{k}/k)$ on the geometric fundamental group $\pi_1^{et}(X \times_k \bar{k}, \bar{x})$. This defines a k-pro-algebraic group scheme; as we will see this is the Nori's fundamental group scheme.

We first remark the following fact:

Proposition 6.2. *Let X be a proper and reduced scheme over a characteristic 0 field. Then every object of $EF(X)$ is finite (cf. Definition 5.1).*

Proof. By the tannakian duality theorem it is sufficient to show that the objects of $\mathrm{Rep}_k(G)$ are finite, where G denotes a finite k group scheme. The representations of an étale finite group scheme are direct sums of irreducible (or indecomposable) representations, and there is a finite number of isomorphic classes of irreducible representations of G. Then for any representation V of G, there is only a finite number of indecomposable representations involved in the power $V^{\otimes n}$, which is enough for V to be finite ([18], Lemma 3.1). □

As any algebraic extension of a characteristic 0 field is separable, from Proposition 5.8 one gets the following isomorphism:

Theorem 6.2. *Let X be a reduced and proper geometrically connected k-scheme, where k is a characteristic 0 field. Let $\bar{k}$ be an algebraic closure of k. Then*

$$\pi_1(X, x) \times_k \bar{k} \simeq \pi_1(X_{\bar{k}}, x).$$

Corollary 6.1. *Let X be a reduced and proper scheme over a characteristic 0 field k, and $x \in X(k)$. Then*

$$\pi_1(X, x) \times_k \bar{k} \simeq \pi_1^{et}(X \times_k \bar{k}, \bar{x})$$

where the second member denotes Grothendieck étale fundamental group viewed as constant pro-algebraic group. The universal torsor $\hat{X}_{\bar{k}}^{\bar{x}}$ is isomorphic to the pro-universal object pointed at $\bar{x}$ in the Galois category of étale covering of $X_{\bar{k}}$.

Moreover the tannakian category of vector bundles on X finite in the sense of Definition 5.1 *is the category of vector bundles on X trivialized by $\hat{X}^{\bar{x}}_{\bar{k}}$ or equivalently by some Galois étale cover $Y \to X$.*

Proof. Torsors over $X_{\bar{k}}$ under a $\bar{k}$-finite group scheme are just finite Galois covers of $X_{\bar{k}}$. Moreover such torsors have always $\bar{k}$-rational points above x. One then uses Theorem 6.2. □

6.1. The groupoid $\mathbf{Spec}(\bar{k}) \times_{\mathbf{Spec}(k)} \mathbf{Spec}(\bar{k})$

Consider the identity morphism

$$id : G = \mathrm{Spec}(\bar{k}) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(\bar{k}) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$$

It defines a k-groupoid acting on $\mathrm{Spec}(\bar{k})$. For any couple of k-morphisms $\alpha, \beta : S \to \mathrm{Spec}(\bar{k})$ of a k-scheme S to $\mathrm{Spec}(\bar{k})$, there is a unique morphism from α to β: there exists a unique $\sigma \in \mathrm{Gal}(\bar{k}/k)$ such that $\beta = \tilde{\sigma} \circ \alpha$, where $\tilde{\sigma} : \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(\bar{k})$ is the k-morphism induced by σ.

It follows that the $\bar{k}$-points of the groupoid $\mathrm{Spec}(\bar{k}) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$ are in bijection with $\mathrm{Gal}(\bar{k}/k)$, the composition of morphisms in the groupoid corresponding to the product in the group $\mathrm{Gal}(\bar{k}/k)$.

6.2. The short exact sequence

Let $\bar{x} \in X(\bar{k})$ and $\bar{x}^* : EF(X) \to \bar{k}$-mod be the corresponding tannakian fibre functor over $\mathrm{Spec}(\bar{k})$. We will denote by $\bar{x}^\star : \mathrm{Rev}(X_{\bar{k}}) \to Sets$ the Galois fiber functor associated to $\bar{x}$ on the Galois category $\mathrm{Rev}(X_{\bar{k}})$ of étale covers of $X_{\bar{k}}$. The functors $\bar{x}^*$ and $\bar{x}^\star$ fit together in the following sense: for any étale cover $h : Y \to X_{\bar{k}}$, $\bar{x}^\star(Y \to X_{\bar{k}})$ is the set of geometric points of $\mathrm{Spec}(\bar{x}^*(f_* O_Y))$.

To the fiber functor $\bar{x}^*$ is associated a k-groupoid

$$(s,t) : \pi_1(X_{\bar{k}}, \bar{x}^*) = \mathrm{Aut}^{\otimes}_{\bar{k}}(\bar{x}^*) \to \mathrm{Spec}(\bar{k}) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k}) = G$$

acting on $\mathrm{Spec}(\bar{k})$ (cf. Definition 4.6). Define

$$pr_1 \circ (s,t) : \pi_1(X, \bar{x}^*)_s = \pi_1(X, \bar{x}^*) \to \mathrm{Spec}(\bar{k})$$

where

$$pr_1 : G_s = \mathrm{Spec}(\bar{k}) \times_k \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(\bar{k})$$

is the first projection. One gets

$$\pi_1(X, \bar{x}^*)_s \to G_s \to \mathrm{Spec}(\bar{k}).$$

For $(\alpha, \beta) : \mathrm{Spec}(\bar{k}) \to \mathrm{Spec}(\bar{k}) \times_{\mathrm{Spec}(k)} \mathrm{Spec}(\bar{k})$, by definition

$$(\alpha, \beta)^* \pi_1(X, \bar{x}^*) = \mathrm{Isom}^{\otimes}(\alpha^* \bar{x}^*, \beta^* \bar{x}^*).$$

So if $\beta \in G_s(\bar{k}) = \mathrm{Gal}(\bar{k}/k)$, the fibre of β in $\pi_1(X, \bar{x}^*)_s(\bar{k}) \to G_s(\bar{k})$ is

$$\mathrm{Isom}^{\otimes}(\bar{x}^*, \beta^\star \bar{x}^*).$$

Denote $\Gamma = \pi_1(X, \bar{x}^*)_s(\bar{k}) = \bigcup_{\beta \in \mathrm{Gal}(\bar{k}/k)} \mathrm{Isom}^{\otimes}(\bar{x}^*, \beta^\star \bar{x}^*)$. One can equip this set with the following group law: if (φ, σ) and (ψ, τ) are elements of Γ, where $\varphi : \bar{x}^* \to \sigma^\star \bar{x}^*$ and $\psi : \bar{x}^* \to \tau^\star \bar{x}^*$ are isomorphisms, then define the product

$$(\varphi, \sigma) * (\psi, \tau) = ((\tau^\star \varphi) \circ \psi, \sigma\tau).$$

And it is clear that the map $(\varphi, \sigma) \to \sigma$, $\Gamma \to \mathrm{Gal}(\bar{k}/k)$ is a morphism of groups. The kernel of this morphism is $\mathrm{Isom}^{\otimes}(\bar{x}^*, \bar{x}^*)$.

One gets the following exact sequence of groups:

$$1 \to \mathrm{Isom}^{\otimes}_{\bar{k}}(\bar{x}^*, \bar{x}^*) \to \pi_1(X, \bar{x}^*)_s(\bar{k}) \to \mathrm{Gal}(\bar{k}/k) \to 1. \qquad (\star)$$

Theorem 6.3. *The gerbe over* $\mathrm{Spec}(k)_{et}$ *of fiber functors of the tannakian category* $F(X)$ *is equivalent to the gerbe of sections of the short exact sequence* $(\star)$.

Moreover if $\bar{x}$ *come from a rational point* $x \in X(k)$*, the short exact sequence* $(\star)$ *can be rewritten*

$$1 \to \pi_1(X, x)(\bar{k}) \to \pi_1(X, \bar{x}^*)_s(\bar{k}) \to \mathrm{Gal}(\bar{k}/k) \to 1. \qquad (\star)$$

The fiber functor x^* *corresponds to a section* s_x *of* $(\star)$ *and Nori's fundamental group scheme* $\pi_1(X, x)$ *is isomorphic to the* k*-group scheme defined by the action of* $\mathrm{Gal}(\bar{k}/k)$ *over* $\pi_1(X, x)(\bar{k})$ *by conjugation through the section* s_x.

Proof. Sections of this exact sequence have the following interpretation: let $\sigma \to (\varphi_\sigma, \sigma)$ be a map $\mathrm{Gal}(\bar{k}/k) \to \pi(X, \bar{x}^*)_s(\bar{k})$. It is a morphism of groups if and only if $(\varphi_\sigma)_{\sigma \in \mathrm{Gal}(\bar{k}/k)}$ is a descent data from $\bar{k}$ to k for $\bar{x}^*$. As any descent data is effective for quasi-coherent modules and then for fibre functors, the section induces a fibre functor Φ such that $\Phi \times_k \bar{k} \simeq \bar{x}^*$. As two fibre functors over $\bar{k}$ are always equivalent, a section of the exact sequence gives rise to a fibre functor over k. We get in this way a correspondence between sections of the exact sequence and fiber functors defined over k. The same argument holds on any finite extension L of k. Sections $s : \mathrm{Gal}(\bar{k}/L) \to \pi_1(X, \bar{x}^*)$ correspond to fiber functors defined over L.

Let Φ_1 and Φ_2 be fiber functors defined over k, corresponding to sections s_1 and s_2 (or equivalently descent data from $\bar{k}$ to k for $\bar{x}^*$), isomorphisms over any finite extension L of k between Φ_1 and Φ_2 are automorphisms of $\bar{x}^*$ which are compatible with the descent data defining Φ_1 and Φ_2. In other words, they are elements γ of $\mathrm{Isom}^{\otimes}_{\bar{k}}(\bar{x}^*, \bar{x}^*)$ verifying

$$\forall \sigma \in \mathrm{Gal}(\bar{k}/L) \quad \gamma \star s_1(\sigma) = s_2(\sigma) \star \gamma.$$

One can define a fibered category on the étale site of $\mathrm{Spec}(k)$ whose objects over some finite extension L of k are sections $\mathrm{Gal}(\bar{k}/L) \to \pi_1^{et}(X, \bar{x})$ of the exact sequence and morphism over L between two sections s_1 and s_2 defined over L are elements γ of the geometric étale fundamental group $\pi_1^{et}(X_{\bar{k}}, \bar{x})$ satisfying the preceding relation.

If $\bar{x}$ comes from a rational point $x \in X(k)$ the fiber functor x^* corresponds to a section s_x of $(\star)$. Moreover for any $\gamma \in \mathrm{Isom}^{\otimes}_{\bar{k}}(\bar{x}^*, \bar{x}^*)$, ${}^\sigma\gamma = s(\sigma) \star \gamma \star s(\sigma)^{-1}$. $\square$

Proposition 6.3. *If one identifies* $\pi_1(X \times_k \bar{k}, \bar{x}^*)$ *to* $\pi_1^{et}(X \times_k \bar{k}, \bar{x})$ *by the isomorphism of Corollary* 6.1*, the exact sequence*

$$1 \to \pi_1(X \times_k \bar{k}, \bar{x}^*) \to \pi(X, \bar{x}^*)_s(\bar{k}) \to \mathrm{Gal}(\bar{k}/k) \to 1$$

identifies with Grothendieck exact sequence

$$1 \to \pi_1^{et}(X \times_k \bar{k}, \bar{x}) \to \pi_1^{et}(X, \bar{x}) \to \mathrm{Gal}(\bar{k}/k) \to 1.$$

Proof. For all $\sigma \in \mathrm{Gal}(\bar{k}/k)$ and for any étale covering $h : Y \to X$, we have a cartesian diagram

$$\begin{tikzcd}
Y_{\bar{k}} = {}^{\sigma}Y_{\bar{k}} \arrow[r, "\beta_\sigma"] \arrow[d, "{}^{\sigma}h_{\bar{k}}"] & Y_{\bar{k}} \arrow[d, "h_{\bar{k}}"] \\
X_{\bar{k}} \arrow[r, "\alpha_\sigma"] \arrow[d] & X_{\bar{k}} \arrow[d] \\
\mathrm{Spec}(\bar{k}) \arrow[r, "\tilde{\sigma}"] & \mathrm{Spec}(\bar{k})
\end{tikzcd}$$

which defines ${}^{\sigma}Y_{\bar{k}}$ and ${}^{\sigma}h_{\bar{k}}$.

The restrictions of β_σ to the fibers of $\bar{x}$ and ${}^{\sigma}\bar{x}$ induce maps between finite sets

$$\beta_\sigma : ({}^{\sigma}\bar{x})^\star({}^{\sigma}Y) \to \bar{x}^\star(Y)$$

They define a natural transformation that we still denote α_σ from ${}^{\sigma}\bar{x}^\star$ to $\bar{x}^\star$. Let $\gamma \in \pi_1(X, \bar{x}^*)_s(\bar{k})$; it is an isomorphism $\gamma : \bar{x}^* \simeq {}^{\sigma}\bar{x}^*$; let us associate to γ, $\tilde{\gamma} : \bar{x}^\star \Rightarrow {}^{\sigma}\bar{x}^\star$ and define $\Phi(\gamma) = \alpha_\sigma \circ \tilde{\gamma} \in \pi_1^{et}(X, \bar{x})$[2]. The following commutative diagram proves that Φ is a group homomorphism:

$$\begin{tikzcd}
({}^{\sigma\tau}\bar{x})^\star({}^{\sigma\tau}Y) \arrow[r, "{}^{\sigma}\beta_\tau"] & ({}^{\sigma}\bar{x})^\star({}^{\sigma}Y) \arrow[r, "\beta_\sigma"] & (\bar{x})^\star(Y) \\
& ({}^{\sigma}\bar{x})^\star({}^{\sigma}Y) \arrow[r, "\beta_\sigma"] \arrow[ul, "{}^{\sigma}\tilde{\delta}"] \arrow[u, "{}^{\sigma}\Phi(\delta)"] & (\bar{x})^\star(Y) \arrow[u, "\Phi(\delta)"] \\
& & (\bar{x})^\star(Y). \arrow[ul, "\tilde{\gamma}"] \arrow[u, "\Phi(\gamma)"]
\end{tikzcd}$$

To verify that Φ is an isomorphism, it suffices to check that the diagram

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X, \bar{x}^*)(\bar{k}) \arrow[r] \arrow[d, "\Phi_{|\pi_1(X,\bar{x}^*)(\bar{k})}"] & \pi_1(X, \bar{x}^*)_s(\bar{k}) \arrow[r] \arrow[d, "\Phi"] & \mathrm{Gal}(\bar{k}/k) \arrow[r] \arrow[d, "="] & 1 \\
1 \arrow[r] & \pi_1^{\acute{e}t}(X_{\bar{k}}, \bar{x}) \arrow[r] & \pi_1^{\acute{e}t}(X, \bar{x}) \arrow[r] & \mathrm{Gal}(\bar{k}/k) \arrow[r] & 1
\end{tikzcd}$$

[2]The category of étale finite covering of X can be identified to a subcategory of $EF(X)$ by the functor which sends a finite étale covering $f : Y \to X$ to $f_*\mathcal{O}_Y$. We are identifying the restriction to this subcategory of $\bar{x}^*$ with $\bar{x}^\star$

is commutative and that the first vertical map is an isomorphism. It is a consequence of Corollary 6.1.

To check that the diagram is commutative we only have to check that the right square is commutative. The morphism $\pi_1^{ét}(X,\bar{x}) \to \mathrm{Gal}(\bar{k}/k)$ is associated in the Galois theory with the functor which sends any finite étale k-algebra $k \subset K$ to the purely arithmetic covering $X \times_{\mathrm{Spec}(k)} \mathrm{Spec}(K) \to X$.

Let $k \subset K$ be a finite étale k-algebra. The structural morphism $\bar{x}^\star(X_K) \to \mathrm{Spec}(\bar{k})$ can be identified canonically to $\mathrm{Spec}(K \otimes_k \bar{k}) \simeq \mathrm{Spec}(\bar{k}^{S_K}) \to \mathrm{Spec}(\bar{k})$, where S_K is the set of k-embeddings of K in $\bar{k}$, corresponding to the diagonal morphism $\bar{k} \hookrightarrow \bar{k}^{S_K}$. In particular it does not depend on the $\bar{k}$-point $\bar{x}$.

Let γ be in $\mathrm{Isom}^{\otimes}(\bar{x}^*, {}^\sigma\bar{x}^*) \subset \pi_1(X,\bar{x}^*)_s(\bar{k})$ where $\sigma \in \mathrm{Gal}(\bar{k}/k)$. When we restrict γ to the full subcategory $\mathcal{T}$ of $EF(X)$ whose objects are $\mathcal{O}_{X_K}$, where $k \hookrightarrow K$ runs among finite étale k-algebras (or more generally finite k-vector spaces), we get a tensor automorphism of the trivial fibre functor extended to $\bar{k}$ from the category $EF(\mathrm{Spec}(k))$. It is easy to check that the Nori fundamental group of $\mathrm{Spec}(k)$ is trivial, and thus, the restriction of γ to $\mathcal{T}$ is trivial.

On the other hand, when we restrict the natural transformation α_σ to objects of the form $X_K \to X$, where K is a finite étale k-algebra, σ induces $1_K \otimes \sigma : K \otimes_k \bar{k} \to K \otimes_k \bar{k}$, and modulo the isomorphism $K \otimes_k \bar{k} \simeq \bar{k}^{S_K}$, the isomorphism $\bar{k}^{S_K} \to \bar{k}^{S_K}$ given by the following formula:

$$(\lambda_\varphi)_{\varphi \in S_K} \to (\sigma(\lambda_{\sigma^{-1}\varphi}))_{\varphi \in S_K}. \tag{$\star\star$}$$

Finally, the restriction of $\Phi(\gamma) = \alpha_\sigma \circ \tilde{\gamma}$ to objects of the form $X_K \to X$ is given by the formula ($\star\star$), which corresponds on the set S_K of $\bar{k}$ points of $\bar{k}^{S_K}$ to the map

$$\varphi \to \sigma \circ \varphi.$$

We have checked that the image of $\Phi(\gamma) \in \pi_1(X,\bar{x})$ in $\mathrm{Gal}(\bar{k}/k)$ is $\sigma \in \mathrm{Gal}(\bar{k}/k)$ as expected. □

One can summarize the results of Theorem 6.3 and Proposition 6.3 in the following statement

Corollary 6.2. *The gerbe of fibre functors of the tannakian category $F(X)$ is equivalent to the gerbe of sections of the Grothendieck exact sequence.*

6.3. Sections of the Grothendieck short exact sequence

In a letter to Faltings [12], Grothendieck conjectured that, if X is a smooth projective geometrically connected curve of genus at least 2 over a finitely generated field extension k of $\mathbf{Q}$, all sections over k of the exact sequence come from rational points. This can be reformulated using the above equivalence in terms of fiber functors of the tannakian category $EF(X)$: every neutral fiber functor should be equivalent to x^* for some rational point $x \in X(k)$. This is the point of view adopted in [8].

The conjecture is open. But in the same letter Grothendieck mentioned an injectivity property for sections which can be rephrased in these terms:

Theorem 6.4. *Let x, y be rational points on X, then $x^* \simeq y^*$ if and only if $x = y$.*

The following proof is essentially borrowed from [8].

Proof. Recall that if $\hat{X}^x = \mathrm{Isom}^{\otimes}(p^*x^*, i_X)$ denotes the universal torsor based at x, then $y^*\hat{X}^x \simeq \mathrm{Isom}^{\otimes}(x^*, y^*)$. Thus the existence of a rational point in $y^*\hat{X}^x(k)$ is equivalent to $x^* \simeq y^*$.

Using the rational point $x \in X(k)$ we embed X in its jacobian $X \hookrightarrow \mathrm{Jac}(X) = A$ such that x goes to 0. It is easy to see that it suffices to show the statement of injectivity for A and 0 in place of X and x.

From the Lang-Serre theorem, one knows that the universal torsor of A at 0 is the projective limit

$$\hat{A}_0 = \lim(A \xrightarrow{[n]} A).$$

And the theorem is the consequence of the fact that there is no infinitely divisible rational point on A except 0. □

Remark 6.1. According to the remark at the end of Paragraph 5.4, there is a one to one correspondence between neutral fiber functor and torsors $f : T \to X$ such that $H^0(T, O_T) = k$ and f^*E is trivial for any object E of $EF(X)$. In the characteristic 0 case, these are regular models $T \to X$ over k of the universal pro-object $\hat{X}_{\bar{k}} \to X_{\bar{k}}$. On the other hand, we have seen that a neutral fiber functor ρ on $EF(X)$ is isomorphic to x^* for some $x \in X(k)$ if and only if $\hat{X}^{\rho}(k) \neq \emptyset$. So the section conjecture is equivalent to the following statement: any regular k-model of the universal pro-object $\hat{X}_{\bar{k}} \to X_{\bar{k}}$ has a k-rational point.

7. Examples

7.1. Case of the projective line

Theorem 7.1. *Let k be a perfect field. The fundamental group scheme of the projective line $\pi_1(\mathbf{P^1_k}, x)$ is trivial.*

Proof. One knows that $\pi_1(\mathbf{P^1_k}, \mathbf{x}) \times_{\mathbf{k}} \bar{\mathbf{k}} \simeq \pi_{\mathbf{1}}(\mathbf{P^1_{\bar{k}}}, \bar{\mathbf{x}})$. So one is reduced to the case where k is algebraically closed.

On the other hand, one knows that the objects of $EF(\mathbf{P^1_k})$ are semi-stable of degree 0. By the Grothendieck theorem, every vector bundle F on $\mathbf{P^1_{\bar{k}}}$ is isomorphic to $\oplus_{i \in I} O_{P^1_{\bar{k}}}(i)$, where I is a finite subset of $\mathbf{Z}$. Suppose there is $i \in I$ such that $i > 0$. Then $G = \oplus_{i \in I, i > 0} O_{P^1_{\bar{k}}}(i)$ is a sub-bundle of F of degree strictly positive, which is impossible if F is semi-stable of degree 0. The same proof with the dual of F shows that there is no $i \in I$ with $i < 0$. And finally $F \simeq \oplus_{i \in I} O_{P^1_{\bar{k}}}$ is trivial. □

Corollary 7.1. *Let k be a perfect field. Any torsor on $\mathbf{P^1_k}$ under a finite k-group scheme is trivial.*

7.2. Case of an abelian variety

Lemma 7.1. *Soit $f : Y \to X$ a finite flat morphism of schemes over a field k. Then if F is a vector bundle on Y, f_*F is a vector bundle on X and $\chi(f_*(F)) = \chi(F)$.*

Proof. The morphism f being finite and flat, it is clear that the direct image by f of a locally free 0_Y-module is a locally free O_X-module. The fact that f is affine implies that f_* is an exact functor from the category of quasi-coherent sheaves on Y to the category of quasi-coherent sheaves on X (use [13], Prop. 8.1 of Ch. III and Th. 3.5 of Ch. III). This implies that for any quasi coherent sheaf F on Y, and any $i \geq 0$, $H^i(Y, F) \simeq H^i(X, f_*F)$ ([13], Ex. 8.2, p. 252, Ch. III). □

Corollary 7.2. *Let $f : Y \to X$ be a finite flat morphism of degree n between smooth geometrically connected projective curves over k, then, for any vector bundle F on Y*

$$rk(F)(1 - g_Y) + \deg(F) = rk(F)n(1 - g_X) + \deg(f_\star(F)).$$

Proof. This uses Riemann-Roch's formula and the fact that $rk(f_*F) = nrk(F)$. □

Applying this formula to $F = O_Y$ and the fact that $f_\star(O_Y)$ is finite, and then semi-stable of degree 0, one gets the following result:

Corollary 7.3. *Let $f : Y \to X$ be a torsor under a finite flat k-group G of order n, where X and Y are smooth geometrically connected projective curves over k. Then*

$$1 - g_Y = n(1 - g_X).$$

Corollary 7.4. *Let $f : Y \to X$ be a torsor under a finite flat k-group G of order n, where X is of genus 1 and Y is a projective curve over k. Then Y is of genus 1. Moreover, suppose that X has a rational point x and Y has a rational point y over x. Then if X and Y can be endowed with the structure of elliptic curves where x and y are the neutral elements of the group laws, and f is an isogeny.*

Proof. The first assertion is an immediate consequence of the formula $1 - g_Y = n(1-g_X)$. If X and Y are endowed with rational points x and y as in the statement of the corollary, they get the structure of elliptic curves, and f is a surjective morphism. As $f(0_Y) = 0_X$, f is a thus a morphism for the group law ([23], Th. 4.8, Ch. III). □

As any isogeny is dominated by an isogeny of the form "multiplication by n", where n is an integer, one gets the following:

Corollary 7.5. *Let X be an elliptic curve defined over a field k. Then the universal torsor $\hat{X}_0$ based at the origin 0 of X is the projective limit of the morphisms "multiplication by n" $[n] : X \to X$ and the fundamental group scheme $\pi_1(X, 0)$ is the projective limit of the finite group schemes $X[n]$.*

This result on the fundamental group scheme of an elliptic curve is a particular case of a more general theorem on the fundamental group scheme of an abelian variety proved by Nori [19]:

Theorem 7.2. *Let X be an abelian variety defined over a field k. Then the universal torsor $\hat{X}_0$ based at the origin 0 of X is the projective limit of the morphisms "multiplication by n" $[n] : X \to X$ and the fundamental group scheme $\pi_1(X, 0)$ is the projective limit of the finite group schemes $X[n]$.*

8. Appendix: "twisting" by a torsor

The aim of this section is to explain how G-torsors are tools for twisting objects endowed with an action of G. Let S be a scheme and $\mathcal{C}$ a stack over the category of S-schemes. We are also given a faithfully flat S-group G and a G-torsor $\xi : T \to X$ over some S-scheme X.

Consider the category $\mathcal{C}_G(X)$ of objects V of $\mathcal{C}(X)$ endowed with a morphism of sheaves $\varphi : G_X \to \mathrm{Aut}_X(V)$. A morphism from (V, φ) to (V', φ') in the category $\mathcal{C}_G(X)$ is a morphism $f : V \to V'$ in $\mathcal{C}(X)$ compatible with the data φ, φ'.

Theorem 8.1.

1. *Let $\xi : P \to X$ be a G-torsor on some S-scheme X. It induces a functor*

$$\Phi = \xi \times^G (-) : \mathcal{C}_G(X) \to \mathcal{C}(X)$$

and for any object (V, φ) of $\mathcal{C}_G(X)$ an isomorphism of sheaves

$$\mathrm{Isom}_{\mathcal{C}(X)}(V, \Phi V) \to \xi \times^{G_X} \underline{\mathrm{Aut}}(V)$$

where $\xi \times^{G_X} \underline{\mathrm{Aut}}(V)$ is the contracted product of ξ with $\underline{\mathrm{Aut}}(V))$ with respect to $\varphi : G_X \to \underline{\mathrm{Aut}}(V)$.

2. *In the opposite direction if we are given two objects V and V' of $\mathcal{C}(X)$ which are locally isomorphic for the $fpqc$-topology, then $\xi = \underline{\mathrm{Isom}}(V, V')$ is a torsor under $\underline{\mathrm{Aut}}(V)$. Moreover the twisted $\xi \times^G V$ of V by the torsor ξ is canonically isomorphic to V'.*
3. *If we are given two S-stacks $\mathcal{C}_1$ and $\mathcal{C}_2$ and a morphism of stacks $F : \mathcal{C}_1 \to \mathcal{C}_2$, then for any object V of $\mathcal{C}_{1,G}(X)$, there is a canonical isomorphism*

$$F(\xi \times^G V) \simeq \xi \times^G F(V).$$

Proof. 1. Let $u_i : U_i \to X$, $i \in I$ be a $fpqc$-covering of X trivializing the torsor ξ. One gets a cocycle $g_{ij} \in G(U_{ij})$ and its image $\varphi(g_{ij}) = \bar{g}_{ij} \in \mathrm{Aut}(\xi_{|U_{ij}})$. These $\bar{g}_{ij}$ induce descent data for the family of objects $u_i^\star V$ which are effective in the stack $\mathcal{C}$. There exists a unique object $\Phi(V) = \xi \times^G V$ over X with isomorphisms

$\theta_i : u_i^\star V \simeq u_i^\star \Phi V$ making the following diagrams commutative:

$$\begin{tikzcd} u_{ij}^\star V \arrow[r, "\theta_{i|U_{ij}}"] \arrow[d, "\bar{g}_{ij}"'] & u_i^\star \Phi(V)_{|U_{ij}} \arrow[d] \\ u_{ij}^\star V \arrow[r, "\theta_{j|U_{ij}}"] & u_j^\star \Phi(V)_{|U_{ij}} \end{tikzcd} \tag{1}$$

where second vertical map is identity.

One checks that the object $\Phi(V)$ does not depend on the trivializing covering neither on the representative g_{ij}.

Let $\bar{h}_{ij}$ be another cocycle with values in $\underline{\mathrm{Aut}}(\xi)$ defining another object $\Phi'(V)$. A morphism $\lambda : \Phi'(V) \to \Phi(V)$ is equivalent to the data of morphisms $\lambda_i \in \mathrm{Hom}(u_i^\star V, u_i^\star V)$ making the following diagrams commutative:

$$\begin{tikzcd} u_{ij}^\star V \arrow[r, "\lambda_{i|U_{ij}}"] \arrow[d, "\bar{h}_{ij}"] & u_{ij}^\star V \arrow[d, "\bar{g}_{ij}"] \\ u_{ij}^\star V \arrow[r, "\lambda_{j|U_{ij}}"] & u_{ij}^\star V. \end{tikzcd}$$

If one takes in particular h_{ij} the trivial cocycle, $\Phi(V) = V$ and the commutative diagrams resume to

$$\lambda_{i|U_{ij}} \circ \lambda_{j|U_{ij}}^{-1} = \bar{g}_{ij}.$$

So the family (λ_i) is a section of the torsor $\xi \times^{G_X} \underline{\mathrm{Aut}}(V)$ corresponding to the cocycle $\bar{g}_{ij} = \varphi(g_{ij})$. This shows a one to one correspondence between sections of the torsor $\mathrm{Isom}(V, \xi \times^G V)$ and sections of $\xi \times_{G_X} \underline{\mathrm{Aut}}(\xi)$.

2. In the other direction the first assertion is clear. Let U_i, $i \in I$ be a covering of X such that there exist isomorphisms $\lambda_i : u_i^\star V \to u_i^\star V'$. The cocycle associated to the torsor $\underline{\mathrm{Isom}}(V, V')$ and this covering is $\bar{g}_{ij} = \lambda_{j|U_{ij}}^{-1} \circ \lambda_{i|U_{ij}}$. Thus the following diagrams are commutative

$$\begin{tikzcd} u_{ij}^\star V \arrow[r, "\lambda_{i|U_{ij}}"] \arrow[d, "\bar{g}_{ij}"] & u_{ij}^\star V \arrow[d, "Id"] \\ u_{ij}^\star V \arrow[r, "\lambda_{j|U_{ij}}"] & u_{ij}^\star V \end{tikzcd}$$

which proves that V' is obtained from V by descent data $\bar{g}_{ij}$; in other words $V' = \xi \times^G V$.

3. Let (V, φ) be an object of $\mathcal{C}_{1,G}(X)$. Then $(F(V), F \circ \varphi)$ is an object of $\mathcal{C}_{2,G}(X)$. The twisted object $\Phi(V) = \xi \times^G V$ is given by diagrams (1). Its image by the functor F is given by the images of diagrams (1) by F. Taking in account

the fact that F commutes with base changes one gets commutative diagrams

$$\begin{array}{ccc} u_{ij}^{\star}F(V) & \xrightarrow{F(\theta_i)_{|U_{ij}}} & u_i^{\star}F(\Phi(V))_{|U_{ij}} \\ {\scriptstyle F(\bar{g}_{ij})}\downarrow & & \downarrow \\ u_{ij}^{\star}F(V) & \xrightarrow{F(\theta_j)_{|U_{ij}}} & u_j^{\star}F(\Phi(V))_{|U_{ij}} \end{array} \tag{2}$$

which means that $F(\Phi(V)) \simeq \Phi(F(V))$. □

One may apply this construction with $\mathcal{C}_1 = BG$ and $\mathcal{C}_2$ the category of quasi-coherent sheaves. Let F be an object of BG-mod, i.e., a morphism of the stack $\mathcal{C}_1$ to the stack $\mathcal{C}_2$. The trivial torsor G_d is an object of $\mathcal{C}_{1,G}(S)$ and we can twist it by a G-torsor ξ. It is clear from the above construction that one gets $\xi \times^G G_d \simeq \xi$. Then

$$F(\xi) \simeq F(\xi \times^G G_d) \simeq \xi \times^G F(G_d)$$

the last isomorphism being a consequence of the point (3) of Theorem 8.1. This proves formula (2) in the proof of Lemma 4.1

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Extension of Galois Groups by Solvable Groups, and Application to Fundamental Groups of Curves

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Abstract. The issue of extending a given Galois group is conveniently expressed in terms of embedding problems. If the kernel is an abelian group, a natural method, due to Serre, reduces the problem to the computation of an étale cohomology group, that can in turn be carried out thanks to Grothendieck-Ogg-Shafarevich formula. After introducing these tools, we give two applications to fundamental groups of curves.

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1. Informal introduction

In what follows, we will be mainly concerned by the description of the structure of the (étale) fundamental groups of algebraic curves. To have a glimpse of what the main issues are, let us fix k be an algebraically closed field of characteristic 0. It is then well known that:

$$\pi_1^{et}(\mathbb{P}^1\backslash\{0,1,\infty\},\overline{x}) \simeq \widehat{F_2} \tag{1.1}$$

where F_2 is a free group on 2 generators and $\widehat{\cdot}$ stands for profinite completion. The proof, however, uses in an essential way analytic techniques. It is now an old but still open question to find a *purely algebraic proof* of the above isomorphism. This issue seems to be first mentioned in Grothendieck's masterpiece [1], where the author also explained that the only thing that was proven algebraically in the 1960's was the isomorphism between the abelianizations of the groups:

$$\pi_1^{et}(\mathbb{P}^1\backslash\{0,1,\infty\},\overline{x})^{ab} \simeq \widehat{F_2}^{ab}. \tag{1.2}$$

The proof relies on class field theory, or to put it more simply, on the description of the generalized Jacobian of the curve.

Since then, not much progress has been done. In a recent joint work with Michel Emsalem [5], we could extend the scope of algebraic methods to give a proof of the isomorphism of the largest solvable quotients of the groups:

$$\pi_1^{et}(\mathbb{P}^1\backslash\{0,1,\infty\},\overline{x})^{\mathrm{solv}} \simeq \widehat{F_2}^{\mathrm{solv}}. \tag{1.3}$$

These quotients are unfortunately very small: one can indeed use the classification of finite groups to show that any finite simple group can be generated by two generators, hence is a quotient of $\widehat{F_2}$, but such a group is of course not a quotient of $\widehat{F_2}^{\mathrm{solv}}$, except if it is abelian. Thus our result is very far from giving an algebraic proof of (1.1), and moreover the isomorphism (1.3) is the best we can get from our method.

Strangely enough, our work stems from Serre's proof of Abhyankar's conjecture for solvable covers of the affine line in positive characteristic [14]. Let thus now k be an algebraically closed field of characteristic $p > 0$. Abhyankar's conjecture states that, for a finite group G:

$$\exists\, \pi_1^{et}(\mathbb{A}^1,\overline{x}) \twoheadrightarrow G \iff G \text{ is quasi-}p \tag{1.4}$$

a group being, by definition, quasi-p when it is generated by its p-Sylow-subgroups.

After a brief review of classical results on the étale fundamental groups of curves (Section 2), we will explain Serre's device that reduces the issue of building covers with solvable Galois groups to the computation of an étale cohomology group (Section 3). In characteristic 0, Ogg-Shafarevich's formula finally solves the problem, leading in Section 4 to the algebraic proof of the obvious generalization (1.3) for an arbitrary affine curve. In characteristic p, the full Grothendieck-Ogg-Shafarevich is needed, which is explained, without a proof, in Section 5. We finally go back to the origin of the subject by sketching Serre's celebrated proof of (1.4) for solvable groups.

2. Fundamental groups of curves over an algebraically closed field

2.1. Étale fundamental group

Let us start with a quick reminder of the étale fundamental group. Let X be a connected scheme, endowed with a geometric point $\overline{x} : \operatorname{spec}\Omega \to X$. The étale fundamental group $\pi_1^{et}(X,\overline{x})$ is defined as the automorphism group of the functor $\overline{x}^* : \operatorname{Cov} X \to \operatorname{Sets}$ that sends a finite étale cover $Y \to X$ to its fiber $Y(\overline{x})$. One can show (see [1]) that this group is profinite (that is, this is a topological group isomorphic to an inverse limit of finite discrete groups) and that the functor above factors through an equivalence $\overline{x}^* : \operatorname{Cov} X \to \pi_1^{et}(X,\overline{x}) - \operatorname{Sets}$. In particular for a finite group G:

$$\exists \pi_1^{et}(X,\overline{x}) \twoheadrightarrow G \iff \exists Y \to X \text{ finite connected étale cover} \,/\, \operatorname{Gal}(Y/X) \simeq G.$$

2.2. Comparison theorems

We suppose in this section that $X \to \operatorname{spec} \mathbb{C}$ is a connected scheme, locally of finite type. Let X^{an} the associated complex analytic space. Then it is known (see [1], XII) that the functor

$$\operatorname{Cov} X \to \operatorname{Cov} X^{an}$$

that sends a finite cover $Y \to X$ to $Y^{an} \to X^{an}$ identifies the finite étale covers of X with those of X^{an}. An obvious consequence is that

$$\widehat{\pi_1(X^{an}, \overline{x})} \simeq \pi_1^{et}(X, \overline{x})$$

where in the left-hand side π_1 stands for the usual topological fundamental group and $\widehat{\cdot}$ for the profinite completion of a group G:

$$\widehat{G} = \varprojlim_{\#G/I<\infty} G/I.$$

2.3. Invariance under algebraically closed base change

Let now k be an arbitrary algebraically closed field, and X a proper connected k-scheme. If k'/k is an algebraically closed extension and $\overline{x'} : \operatorname{spec} \Omega \to X_{k'}$ is a geometric point, define $\overline{x} : \operatorname{spec} \Omega \to X$ as its image in X. Then there is a canonical isomorphism (see [1], X):

$$\pi_1^{et}(X_{k'}, \overline{x'}) \simeq \pi_1^{et}(X, \overline{x}).$$

2.4. Curves in characteristic zero

From these general results, one can deduce the structure of the étale fundamental group of a smooth projective curve X of genus g defined over an algebraically closed field k of characteristic 0. The principle is that such a X is in fact defined over a subfield $k_0 \subset k$ of finite transcendence degree over the prime field $\mathbb{Q}$. This enables to embed k_0 into $\mathbb{C}$, and applying the results of §2.3 (twice) and §2.2, one gets that $\pi_1^{et}(X, \overline{x})$ is isomorphic to the profinite completion of the topological fundamental group of a Riemann surface of genus g.

More generally, similar techniques apply to work out the structure of the étale fundamental group of any smooth curve over an algebraically closed field k of characteristic 0. With the above notations, let $U \subset X$ be a non-empty open subset, and $r = \#(X \backslash U)$ the number of "holes" (possibly 0). One can show along the above lines that

$$\pi_1^{et}(U, x) \simeq \widehat{\Gamma_{g,r}}$$

where $\Gamma_{g,r}$ is the quotient of the free group on $2g+r$ generators $a_1,\ldots,a_g,b_1,\ldots,b_g$, $c_1,\ldots,c_r$ by the unique relation $[a_1, b_1] \cdots [a_g, b_g] = c_1 \cdots c_r$. In particular, $\pi_1^{et}(U, x)$ is free (as a profinite group) on $2g + r - 1$ generators as soon as $r > 0$ (that is, when U is affine).

2.5. Positive characteristic phenomenons

We follow the notations of the previous section, but we now work over an algebraically closed field k of characteristic $p > 0$. For $g \geq 2$, there is not a single example of a curve X of genus g where the structure of the étale fundamental group $\pi_1^{et}(X, \overline{x})$ is fully understood! So we must somehow simplify the problem, and for this purpose we introduce, for a profinite group G, two quotients:

$$G^{p'} = \varprojlim_{\substack{I \lhd G \text{ open} \\ [G:I] \text{ prime to } p}} \frac{G}{I}$$

and

$$G^{p} = \varprojlim_{\substack{I \lhd G \text{ open} \\ [G:I] \text{ a power of } p}} \frac{G}{I}.$$

2.5.1. p' part. Thanks to *specialisation theory*, one can show:

$$\pi_1^{et}(U, x)^{p'} \simeq \widehat{\Gamma_{g,r}}^{p'}.$$

This isomorphism was one of the early successes of Grothendieck's theory of the étale fundamental group (see [1]). So as far as p'-quotients are concerned, nothing new occurs in comparison with characteristic 0. The only known proof uses comparison theorems.

2.5.2. p part (complete curves). But for p-quotients the situation is completely different. They are no longer controlled by the genus but by the *Hasse-Witt invariant*

$$h = \dim_{\mathbb{F}_p} H^1(X, \mathbb{F}_p)$$

that is, the first étale cohomology group with coefficients in the constant sheaf $\mathbb{F}_p$. One can show than $0 \leq h \leq g$, and thanks to cohomological arguments, Shafarevich proved the following:

Theorem 2.1 (Shafarevich). *The group $\pi_1^{et}(X, x)^p$ is a free pro-p group on h-generators, that is*

$$\pi_1^{et}(X, x)^p \simeq \widehat{F_h}^p.$$

Remark 2.2.

1. Shafarevich's original proof was quite intricate and was heavily simplified with the rise of étale cohomology (see [6]). In contrast to the previous result, this is an algebraic theorem.
2. In particular, if one considers the abelianizations of the above groups, one gets, with obvious notations, for a fixed prime l:

$$\pi_1^{et}(X, x)^{ab,l} \simeq \begin{cases} \mathbb{Z}_l^{\oplus 2g} & \text{for } l \neq p \\ \mathbb{Z}_p^{\oplus h} & \text{for } l = p. \end{cases}$$

   This illustrates the general trend that (for complete curves) there are less covers in positive characteristic.

2.5.3. Mixed covers (affine curves). We now consider only affine curves, that is, the number r of holes is greater than 1. Because of *wild ramification* strange things occur:

- The affine line $\mathbb{A}^1$ is not simply connected[1].
- Even worse, the profinite group $\pi_1^{et}(U,x)$ is not topologically of finite type.

However, the set of finite quotients of the étale fundamental group is known[2]. To state this, for a finite group G, we denote by $p(G)$ the group generated by its p-Sylow subgroups, and n_G the minimal number of generators of G. Then the celebrated Abhyankar conjecture states:

Theorem 2.3 (Raynaud [10], Harbater [8]).

$$\exists \pi_1^{et}(U,\overline{x}) \twoheadrightarrow G \iff n_{G/p(G)} \leq 2g + r - 1.$$

The proof, unfortunately, uses a transcendental argument at some point. But a first crucial step, performed by Serre, was to prove the theorem when $X = \mathbb{A}^1$, the affine line, and G is solvable, and this was done by algebraic means (see [14], and Section 6).

3. Embedding problems

3.1. Definition

An embedding problem is a diagram in the category of profinite groups:

$$\begin{tikzcd} & & & \pi \arrow[d, two heads, "a"] & \\ 1 \arrow[r] & A \arrow[r] & G \arrow[r, "q"] & H \arrow[r] & 1 \end{tikzcd}$$

where the vertical arrow is an epimorphism and the horizontal sequence is exact. It is said to have a *weak solution* if there exists a continuous homomorphism $\beta : \pi \to G$ lifting α, i.e., $q \circ \beta = \alpha$. There is a *strong solution* if one can choose moreover β to be an epimorphism.

Clearly, weak solutions are in one to one correspondence with the sections of the exact sequence:

$$1 \to A \to G \times_H \pi \to \pi \to 1\ .$$

3.2. Embedding problems with irreducible kernels

Let l be a prime number. We assume that G is finite and A is a l-elementary abelian group irreducible as $\mathbb{F}_l[H]$-module. Then a weak solution is strong if and only if it does not come from a section of the exact sequence:

$$1 \to A \to G \to H \to 1\ .$$

One can use this fact to give a cohomological criterion of existence of a strong solution of the embedding problem.

[1] As the existence of Artin-Schreier covers shows.

[2] This set does not determine the group up to isomorphism, see also Proposition 4.2.

We distinguish between the two following situations:

3.2.1. Case of a non-split exact sequence. We denote (abusively) $\mathrm{cl}(G)$ the class of the extension $1 \to A \to G \to H \to 1$ in $H^2(H, A)$. Then the embedding problem has a strong solution if and only if the image of $\mathrm{cl}(G)$ by $H^2(H, A) \to H^2(\pi, A)$ is the trivial class.

3.2.2. Case of a split exact sequence. If the exact sequence we started from splits, and $\mathcal{S}$ denotes the set of its sections, one has the equality: $|A^H| \cdot |\mathcal{S}| = |H^1(H, A)| \cdot |A|$. Similarly if $\tilde{\mathcal{S}}$ stands for the set (possibly infinite) of sections of the exact sequence $1 \to A \to G \times_H \pi \to \pi \to 1$, then $|A^H| \cdot |\tilde{\mathcal{S}}| = |H^1(\pi, A)| \cdot |A|$ Note that $H^1(H, A) \hookrightarrow H^1(\pi, A)$. We deduce from these facts that the embedding problem has a strong solution in this case if and only if:

$$\dim_{\mathbb{F}_l} H^1(\pi, A) > \dim_{\mathbb{F}_l} H^1(H, A) \ .$$

3.3. Étale sheaves

We will be interested in such embedding problems mainly when $\pi = \pi_1^{et}(X, \overline{x})$ is the étale fundamental group of a smooth, connected algebraic curve over an algebraically closed field. In this case, the data of the epimorphism $\alpha : \pi_1^{et}(X, \overline{x}) \to H$, together with the action $\rho : H \to \mathrm{Aut}(A)$ given by conjugation, define a locally constant sheaf of $\mathbb{F}_l$-vector spaces $\underline{A}$ on the étale site X_{et}, by the formula $\underline{A} = (\pi_*(A_Y))^H$, where $\pi : Y \to X$ is the cover associated to α, and $A_Y = \mathrm{Hom}_Y(\cdot, A \times Y)$ is the constant sheaf with stalk A on Y. We will also denote this locally constant sheaf by $\pi_*^H(A_Y)$ in the sequel.

This is a well-known fact from descent theory that this process defines, when α and ρ vary, an equivalence between continuous representations of $\pi_1^{et}(X, \overline{x})$ with values in $\mathbb{F}_l$-vector spaces and locally constant sheaves of $\mathbb{F}_l$-vector spaces on the étale site X_{et}. In the opposite direction, one simply associates to such a sheaf its stalk $F_{\overline{x}}$ at the chosen geometric point, with the natural action.

3.4. Comparison of cohomologies

The reason to switch to étale sheaves is that we have both a better intuition and a better grasp of their cohomology than the one of the corresponding representations. To compare them, remember that to an H-Galois cover $\pi : Y \to X$ is associated the Hochschild-Serre spectral sequence:

$$E_2^{p,q} = H^p(H, H^q(Y, \pi^* F)) \Longrightarrow H^{p+q}(X, F) = E^{p+q}.$$

This spectral sequence is cohomological (that is $E_2^{p,q} = 0$ for $p < 0$ or $q < 0$) hence gives rise to a five-term short exact sequence, that in the case of $F = \underline{A}$ amounts to

$$0 \to H^1(H, A) \to H^1(X, \underline{A}) \to H^1(Y, A_Y)^H \to H^2(H, A) \to H^2(X, \underline{A}).$$

Going to the inductive limit over all $\alpha : \pi_1^{et}(X, \overline{x}) \to H$, we get the following facts:

- $H^1(\pi_1^{et}(X, \overline{x}), A) \simeq H^1(X, \underline{A})$
- $H^2(\pi_1^{et}(X, \overline{x}), A) \hookrightarrow H^2(X, \underline{A})$.

3.5. l-cohomological dimension of a curve

We recall a general definition:

Definition 3.1. Let X be a scheme, and l be a prime number.

1. an abelian sheaf F on X_{et} is l-torsion if the natural morphism
$$\varinjlim_{n\to\infty} {}_{l^n}F \to F$$
where ${}_{l^n}F = \ker(F \overset{\times l^n}{\to} F)$, is an isomorphism.
2. The *l-cohomological dimension* of X is the greatest integer $n = \mathrm{cd}_l(X)$ (possibly ∞) such that there exists a l-torsion sheaf F with $H^n(X, F) \neq 0$.

The cohomology of étale torsion sheaves is controlled by the following classical result.

Theorem 3.2 (Artin [2]). *Let X be a complete smooth algebraic curve over a separably closed field k of characteristic p, and l be a prime number distinct from p.*

1. $\mathrm{cd}_l\, X = 2$
2. *if $U \subsetneq X$ is a non empty affine open subset then* $\mathrm{cd}_l\, U = 1$.

Sketch of the proof. 1. We have to show that $H^n(X, F) = 0$ for $n > 2$ and F a l-torsion sheaf. It is enough to show this when F is constructible (for curves, this means locally constant on a dense open subset, with finite stalks, see also §5.3). Indeed, the cancellation is stable by extension, and any l-torsion sheaf can be filtered by constructible sheaves. Then, since a constructible sheaf is locally constant on a stratification, one can in turn reduce to the case where $F = j_! F'$ for $j : U \to X$ an open immersion, and F' is locally constant. Here $j_!$ denotes the "extension by 0" operation, described on the stalks by:
$$(j_! F)_{\overline{x}} = \begin{cases} F_{\overline{x}} & \text{for } x \in U \\ 0 & \text{for } x \notin U. \end{cases}$$
Using a trick called "la méthode de la trace", one reduces the problem again to the case where $F = j_!\,(\mathbb{Z}/l)_U$. The idea is that it is enough to control the cancellation of the cohomology after a pullback to a finite étale cover. If we denote by $i : X' = X \backslash U \to X$ the closed immersion, with the reduced structure, the exact sequence
$$0 \to j_! \left(\frac{\mathbb{Z}}{l}\right)_U \to \left(\frac{\mathbb{Z}}{l}\right)_X \to i_* \left(\frac{\mathbb{Z}}{l}\right)_{X'} \to 0$$
shows that one can suppose that $F = (\mathbb{Z}/l)_X$. Since $l \neq p$, there is a non canonical isomorphism $(\mathbb{Z}/l)_X \simeq \mu_l$. One can then use Kummer's theory, and Tsen's theorem, that asserts that $H^n(X, \mathbb{G}_m)(l) = 0$ for $n \geq 2$ (where $\cdot(l)$ stands for the l-primary part), to work out the following:
$$H^n(X, \mu_l) = \begin{cases} 0 & \text{for } n > 2 \\ \frac{\mathrm{Pic}\, X}{l\, \mathrm{Pic}\, X} & \text{for } n = 2 \end{cases}$$
which concludes the proof of the first case.

2. For the same reason, it is enough to show that $H^n(U,\mu_l)=0$ for $n\geq 2$. But if $A=\underline{\mathrm{Pic}}^0_{X/k}$, then $A(k)\twoheadrightarrow \mathrm{Pic}(U)$ (because U is affine), and $A(k)$ is l-divisible (because $A\overset{\times l}{\to}A$ is étale). Hence $H^2(U,\mu_l)=\frac{\mathrm{Pic}\,U}{l\,\mathrm{Pic}\,U}=0$. □

4. Largest pro-solvable p'-quotient of the fundamental group of an affine curve

The aim of this section is to prove the following theorem.

4.1. Statement

We fix some notations:

- X a smooth projective curve over an algebraically closed field k of characteristic $p\geq 0$,
- g the genus of X,
- $U=X\setminus\{a_1,\dots,a_r\}$, with $r\geq 1$ (so that U is affine),
- for a profinite group G, let $G^{\mathrm{solv},p'}$ be the inverse limit of its finite solvable quotients of order prime to p,
- $\widehat{F_N}$ a free group on N generators.

Theorem 4.1 (B.-Emsalem [5] for the algebraic proof). *If $\overline{x}$ is a geometric point of U then:*

$$\pi_1^{et}(U,\overline{x})^{\mathrm{solv},p'}\simeq \widehat{F_{2g+r-1}}^{\mathrm{solv},p'}.$$

4.2. The $\mathcal{P}_G$ property

For a finite group G, we denote by n_G the minimal number of generators of G. Let $\mathcal{P}_G$ be the property

$$\boxed{n_G\leq 2g+r-1 \iff \exists\, \pi_1^{et}(U,\overline{x})\twoheadrightarrow G.}$$

Theorem 4.1 implies that $\mathcal{P}_G$ is true for G solvable of order prime to p. But the following well-known proposition shows that the converse is also true.

Proposition 4.2 (see for instance [7]). *For π a profinite group, define* $\mathrm{Im}(\pi)=\{G/H, H\lhd G, H$ *open* $\}$. *If π and π' are two profinite groups such that* $\mathrm{Im}(\pi)=\mathrm{Im}(\pi')$ *and π is topologically of finite type, then $\pi\simeq\pi'$.*

Sketch of a proof. The main tool is the following fact: if $(E_i)_{i\in I}$ is a projective system of non empty finite sets, then $\varprojlim_{i\in I}E_i\neq\emptyset$. □

To now prove that the property $\mathcal{P}_G$ holds for G solvable of order prime to p, we will show the slightly stronger statement:

Proposition 4.3. *Fix an exact sequence of finite groups:* $1\to A\to G\to H\to 1$. *If A is solvable, $\#G$ is prime to p, and $\mathcal{P}_H$ holds, then $\mathcal{P}_G$ holds.*

Moreover, it is easy to see that it is enough to show the proposition when A is abelian, l-elementary (for a prime $l \neq p$), and irreducible as a $\mathbb{F}_l[H]$-module.

The hypothesis is that $\mathcal{P}_H$ is true. If both assertions in $\mathcal{P}_H$ are false then the same holds for $\mathcal{P}_G$, hence $\mathcal{P}_G$ is true. One can thus suppose that both assertions in $\mathcal{P}_H$ are true. In particular, one can fix an epimorphism $\pi_1^{et}(U, \overline{x}) \twoheadrightarrow H$. Let $\pi : V \to U$ the corresponding H-Galois cover. One can now apply the general technique explained in §3, and this leads to the following discussion.

4.2.1. Case of a non split exact sequence. Let us suppose that $\mathrm{cl}(G)$ is not the trivial class in $H^2(H, A)$.

Then on the one hand $H^2(\pi_1^{et}(U, \overline{x}), A) = 0$ since U is affine, according to Theorem 3.2 and §3.4. The argument in §3.2.1 shows that the fixed epimorphism $\pi_1^{et}(U, \overline{x}) \twoheadrightarrow H$ always lifts to G.

On the other hand, the fact that the exact sequence $1 \to A \to G \to H \to 1$ does not split, and the fact that A is irreducible, enable to show easily that $n_G = n_H$. Hence $n_G \leq 2g + r - 1$ holds.

So both assertions in $\mathcal{P}_G$ are true, and $\mathcal{P}_G$ holds.

4.2.2. Case of a split exact sequence. Let now suppose that $\mathrm{cl}(G) = 0$ in $H^2(H, A)$. Then the arguments in §3.2.2 and in §3.4 show that the fixed epimorphism $\pi_1^{et}(U, \overline{x}) \twoheadrightarrow H$ lifts to G if and only if

$$\dim_{\mathbb{F}_l} H^1(U, \underline{A}) > \dim_{\mathbb{F}_l} H^1(H, A) \ .$$

Using Ogg-Shafarevich formula to compute the first term in the next section, we will show that this last condition is equivalent to $n_G \leq 2g + r - 1$. This will conclude the proof. Indeed then $n_G \leq 2g + r - 1 \implies \exists\, \pi_1^{et}(U, \overline{x}) \twoheadrightarrow G$ is clear. The other way round, if we assume $\exists\ \pi_1^{et}(U, \overline{x}) \twoheadrightarrow G$, then we have lifted the composite $\pi_1^{et}(U, \overline{x}) \twoheadrightarrow G \twoheadrightarrow H$ (which does not need to coincide with the one we started with), hence $n_G \leq 2g + r - 1$.

Remark 4.4. The proof shows in fact that if $n_G \leq 2g + r - 1$ then *every* embedding problem has a strong solution. In this sense the issue is much simpler in the present situation than in Serre's original context (see §6).

4.3. Ogg-Shafarevich formula

We recall that X is a smooth projective curve over an algebraically closed field k of characteristic $p \geq 0$, g denotes the genus of X, and $U = X \backslash \{a_1, \ldots, a_r\}$ with $r \geq 1$, is an affine open subset. Ogg-Shafarevich formula enables to compute the Euler-Poincaré characteristic $\chi(X, F) = \sum_{i=0}^{2} (-1)^i \dim_{\mathbb{F}_l} H^i(X, F)$ of a constructible sheaf F (see §5.3 for more details on this notion). Using the exact sequence of relative cohomology, it translates into the following affine version.

Theorem 4.5 (Ogg-Shavarevich, see [9]). *Let F be a constructible sheaf of $\mathbb{F}_l$-vector spaces on X that is tamely ramified at infinity and unramified on U. Then*

$$\chi(U, F_{|X}) = \chi(U, \mathbb{F}_l) \dim_{\mathbb{F}_l} F_{\overline{\nu}}$$

where $\overline{\nu}$ is the generic point of U.

This formula enables to conclude the proof of Proposition 4.3 (and thus of Theorem 4.1). To explain this, we fix an epimorphism $\pi_1^{et}(U,\overline{x}) \twoheadrightarrow H$. Let $\pi : V \to U$ be the associated H-Galois cover. By a slight abuse, we denote also by $\pi : Y \to X$ its normalisation in X. Let A be an irreducible $\mathbb{F}_l[H]$-module, and $G = A \rtimes H$.We can now apply Theorem 4.5 to the constructible sheaf $\pi_*^H(A_Y)$ on X. Note that this sheaf does not need to be locally constant, but its restriction $\pi_*^H(A_Y)_{|U} = \underline{A}$ is. Using the standard fact $\chi(U,\mathbb{F}_l) = 2 - 2g - r$, we get that

$$\dim_{\mathbb{F}_l} H^1(U,\underline{A}) = (2g + r - 2)\dim_{\mathbb{F}_l} A + \dim_{\mathbb{F}_l} A^H \ .$$

So the equivalence of $\dim_{\mathbb{F}_l} H^1(U,\underline{A}) > \dim_{\mathbb{F}_l} H^1(H,A)$ with $n_G \leq 2g + r - 1$ results from the following easily shown group-theoretic Lemma, applied with $N = 2g + r - 1$:

Lemma 4.6. *Let l be a prime, N an integer. Let moreover A be an l-elementary abelian group that is irreducible for the action of a group H whose minimal number of generators n_H is less than N. Denote by G the semi-direct product $G = A \rtimes H$. Then:*

$$\dim_{\mathbb{F}_l} H^1(H,A) < (N-1)\dim_{\mathbb{F}_l} A + \dim_{\mathbb{F}_l} A^H \iff n_G \leq N \ .$$

4.4. Remark on groups whose order is divisible by p

In the proof of Proposition 4.3, the hypothesis that $\#G$ is prime to the characteristic p of k is only used to ensure that the constructible sheaf $F = \pi_*^H(A_Y)$ on X is tamely ramified. We can in fact weaken this hypothesis and allow $\#H$ to be divisible by p, if we impose instead this condition on F.

Proposition 4.7. *Fix an epimorphism $\pi_1(U,\overline{x}) \twoheadrightarrow H$ where H is finite group of any order, and let $\pi : V \to U$ be the corresponding Galois H-cover. Suppose that A is an l-elementary abelian group that is irreducible for the action of H, and consider the embedding problem:*

$$\begin{tikzcd}
 & & & \pi_1(U,\overline{x}) \arrow[d, two heads] & \\
1 \arrow[r] & A \arrow[r] & G \arrow[r] & H \arrow[r] & 1.
\end{tikzcd}$$

If the corresponding sheaf on $F = \pi_*^H(A_Y)$ on X is tamely ramified, and $n_G \leq 2g + r - 1$, the embedding problem has a strong solution.

Remark 4.8.

1. If $\pi : Y \to X$ is tamely ramified, then so is F.
2. Moreover by specialisation theory and analytical methods, one can show that

$$\widehat{F_{2g+r-1}} \twoheadrightarrow \pi_1(U,\underline{x})^{\text{tame}}$$

(see [1]). So in other words, the condition $n_G \leq 2g + r - 1$ on a finite group G is necessary to be realised as a Galois group of a tame cover of X. The proposition above says that, for some very special groups G, this condition is

sufficient. Since the epimorphism is not an isomorphism (there are less tame covers than in characteristic zero), it is not always sufficient. It is interesting to note, however, that in this situation algebraic and analytic techniques are complementary, rather than opposed.

5. Grothendieck-Ogg-Shafarevich formula

There are two reasons why we now need a refined version of Ogg-Shafarevich formula, due to Grothendieck, that takes into account the wild ramification of constructible sheaves. The first reason is that the former, tame version of the formula, was originally proved by transcendental methods, using precisely the theorem describing the structure of the largest prime to p-quotient of the fundamental group of a curve. The second reason is this refined formula is the crux of Serre's approach of Abhyankar's conjecture.

5.1. Artin and Swan characters

5.1.1. Definition. Let

- R be a complete discrete valuation ring,
- $k = R/\mathfrak{m}$ its residue field,
- π a uniforming parameter,
- $K = \operatorname{frac} R$,
- L/K a finite Galois extension with group G,
- v_L the (normalized) valuation of L.

We suppose that k algebraically closed of characteristic p. For g in G, $g \neq 1$, put $i_G(g) = v_L(g\pi - \pi)$.

Definition 5.1. The *Artin character* $a_G : G \to \mathbb{Z}$ is defined by

$$g \mapsto \begin{cases} -i_G(g) & \text{if } g \neq 1 \\ \sum_{g \neq 1} i_G(g) & \text{if } g = 1. \end{cases}$$

Remark 5.2.

1. $a_G(1) = v_L(\mathcal{D}_{L/K})$ is the valuation of the different.
2. Define the *higher ramification groups* by

$$g \in G_i \iff i_G(g) \geq i + 1 \text{ or } g = 1.$$

These groups obviously form a decreasing sequence of normal groups starting from $G_0 = G$; one can moreover show that $G_i = \{1\}$ for $i \gg 0$, that G_i is a p-group for $i \geq 1$, and that G_0/G_1 is cyclic of order prime to p. An alternative description of a_G is then given by the easily proved formula:

$$a_G = \sum_{i=0}^{\infty} \frac{\#G_i}{\#G} \operatorname{Ind}_{G_i}^{G}(u_{G_i})$$

where u_{G_i} is the character of the augmentation representation: $u_{G_i} = r_{G_i} - 1$, where r_{G_i} stands for the character of the regular representation. In particular $a_G = 0$ if and only if $G = 1$.

Definition 5.3. The *Swan character* $sw_G : G \to \mathbb{Z}$ is defined by

$$sw_G = a_G - u_G.$$

Remark 5.4.

$$sw_G = \sum_{i=1}^{\infty} \frac{\#G_i}{\#G} \operatorname{Ind}_{G_i}^{G}(u_{G_i})$$

and $sw_G = 0$ if and only if $G_1 = 1$ (that is, exactly when L/K is tamely ramified).

5.1.2. Artin and Swan representations. The functions a_G and sw_G are central, that is, constant over conjugacy classes. Moreover, it was already known to Weil (in 1948, see [15]) that they come from complex representations, more precisely that for any complex character $\chi : G \to \mathbb{C}$, the scalar product $\langle a_G, \chi \rangle$ is a nonnegative integer. But a lot more can be said:

Theorem 5.5 (Serre [12]). *Let l a prime distinct from p.*

1. *Artin and Swan characters can be realized over $\mathbb{Q}_l$.*
2. *There exists a* projective *$\mathbb{Z}_l[G]$-module Sw_G so that $\mathbb{Q}_l \otimes_{\mathbb{Z}_l} Sw_G$ has sw_G for character.*

Remark 5.6.

1. Sw_G is unique up to isomorphism.
2. The augmentation character u_G is defined (over any field) as the character of the augmentation representation $U_G = \ker(\mathrm{tr} : \mathbb{Q}_l[G] \to \mathbb{Q}_l)$, so a_G is the character of the representation (called the Artin representation) $A_G = Sw_G \oplus U_G$.

5.2. Weil's formula

Let us now recall Weil's original motivation to introduce these representations.

Let $\pi : Y \to X$ be a Galois cover of smooth projective curves over an algebraically closed field k, with Galois group G. We denote by g_Y and g_X the genus of the curves. By functoriality G acts on

$$H^i(Y, \mathbb{Q}_l) \simeq \begin{cases} \mathbb{Q}_l & i = 0, 2 \\ \mathbb{Q}_l^{\oplus 2g_Y} & i = 1 \\ 0 & i > 2. \end{cases}$$

Weil's formula will compute the characters of these representations.

Let $y \in |Y|_0$ be a closed point, $x = \pi(y)$. We can apply what we have just seen in §5.1 to $R = \widehat{\mathcal{O}_{X,x}}$ and $L = \operatorname{frac} \widehat{\mathcal{O}_{Y,y}}$. The Galois group is the decomposition group and is denoted by G_y. We will write A_y for the Artin representation, this is a finite type $\mathbb{Q}_l[G_y]$-module.

Now let $x \in |X|_0$ be a closed point, and put $A_x = \mathrm{Ind}_{G_y}^G A_y$ for any lifting $y \mapsto x$. This is independent of the choice of the lifting.

Let $R_{\mathbb{Q}_l}(G)$ be the subgroup of the character group $R_{\overline{\mathbb{Q}}_l}(G)$ generated by characters of G over $\mathbb{Q}_l$ (or equivalently, the Grothendieck group of the category of finite type $\mathbb{Q}_l[G]$-modules). For such a module V, denote by $[V]$ its class in $R_{\mathbb{Q}_l}(G)$.

Theorem 5.7 (Weil's formula, see [13]).

$$\sum_{i=0}^{2}(-1)^i[H^i(X,\mathbb{Q}_l)] = (2-2g_x)[\mathbb{Q}_l[G]] - \sum_{x\in|X|_0}[A_x].$$

Remark 5.8.

1. This can be seen as an equivariant version of Hurwitz formula.
2. The proof uses a Lefschetz formula in étale cohomology, see [9].

5.3. Constructible sheaves

Since Grothendieck-Ogg-Shafarevich formula deals with constructible sheaves, we give a more precise definition of these, valid on any scheme.

Definition 5.9.

1. A sheaf of abelian groups on X_{et} is *locally constant* if there exists an étale covering $(X_i \to X)_{i\in I}$ and abelian groups $(G_i)_{i\in I}$ such that
$$F_{|X_i} \simeq \mathrm{Hom}_{X_i}(\cdot, X_i \times G_i).$$
2. F is *locally constant finite* if the G_i's are finite.

Remark 5.10.

1. Let G a finite étale commutative étale group scheme over X. Then the sheaf $\mathrm{Hom}_X(\cdot, X\times G)$ represented by G is locally constant. Besides, descent theory asserts that this functor gives an equivalence of categories from the category of finite étale commutative étale group schemes over X to the category of locally constant finite abelian sheaves on X_{et}.
2. Locally constant finite sheaves are not stable under direct images. For if $\mathrm{supp}\, F = \overline{\{x \in X/F_{\overline{x}} \neq 0\}}$ and $i : X' \to X$ is a closed immersion, then for any étale sheaf F' on X'_{et}, by definition of the stalks $\mathrm{supp}\, i_*F' \subset X'$. But if F is locally constant finite and non-zero, and X is irreducible, then $\mathrm{supp}\, F = X$.

Definition 5.11. A sheaf of abelian groups on X_{et} is *constructible* if for every irreducible closed subscheme X' of X, there exists a non-empty open subset $U \subset X'$ such that $F_{|U}$ is locally constant finite.

Remark 5.12. One can show:

1. constructible sheaves form an abelian category,
2. if $f : X' \to X$ is a proper (and finitely presented) morphism and F is a constructible abelian sheaf on X'_{et}, so is R^qf_*F on X_{et} for all $q \geq 0$.

5.4. Wild conductor

We return for a while to the local setting. Let R be a discrete valuation ring, with fraction field K, and perfect residue field k. We are interested in describing constructible sheaves on the étale site of $\operatorname{spec} R$. The decomposition theorem in étale cohomology takes the following simple form. Denote the closed point by $x : \operatorname{spec} k \to \operatorname{spec} R$ and the generic point by $\nu : \operatorname{spec} K \to \operatorname{spec} R$. Let $\overline{K}$ be a separable closure of K, $\overline{v}$ an extension from v to $\overline{K}$, and $I(\overline{v})$ the corresponding inertia group. An étale sheaf F on $\operatorname{spec} R$ gives rise to

$$\begin{cases} F_{\overline{\nu}} & \text{a } G_K\text{-module} \\ F_{\overline{x}} & \text{a } G_k\text{-module} \\ F_{\overline{x}} \to (F_{\overline{\nu}})^{I(\overline{v})} & \text{a } G_k\text{-equivariant morphism.} \end{cases}$$

The sheaf F is constructible is these modules are finite. Moreover one can then recover F from this data.

Suppose from now on that R is complete, and that k is algebraically closed of characteristic $p \geq 0$. Let F be a constructible sheaf of $\mathbb{F}_l$-modules, with $l \neq p$, and let L/K be Galois extension with group G trivializing $F_{\overline{\nu}}$.

Definition 5.13. The (exponent of the) wild conductor of F is

$$\alpha(F) = \dim_{\mathbb{F}_l} \operatorname{Hom}_G(Sw_G, F_{\overline{\nu}}).$$

Remark 5.14.

1. $\alpha(F) = \sum_{i=1}^{\infty} \frac{\#G_i}{\#G} \dim_{\mathbb{F}_l} \frac{F_{\overline{\nu}}}{F_{\overline{\nu}}^{G_i}}$, in particular $\alpha(F) = 0$ if and only if G_1 acts trivially on F (one says that F is tamely ramified),
2. $\alpha(F)$ is additive in (short exact sequences) in F (because Sw_G is projective),
3. $\alpha(F)$ is independent of the choice of L/K.

5.5. Conductor

We now return to the global situation. Let X be a smooth algebraic curve over an algebraically closed field k of characteristic p, and let F be a constructible sheaf of $\mathbb{F}_l$-modules, with $l \neq p$. Fix $\pi : Y \to X$ a Galois étale cover such that $\pi^* F$ is generically constant. Denote the generic point by $\nu : \operatorname{spec} K \to X$ and fix a closed point $x : \operatorname{spec} k \to X$.

Applying what we have seen in §5.4 to $R = \widehat{\mathcal{O}_{X,x}}$, and to the restriction of F to $\operatorname{spec} R$, we get a local wild conductor $\alpha_x(F)$.

Definition 5.15. The (exponent of the) conductor of F at x is

$$\epsilon_x(F) = \alpha_x(F) + \dim_{\mathbb{F}_l} F_{\overline{\nu}} - \dim_{\mathbb{F}_l} F_{\overline{x}}.$$

Remark 5.16. $\epsilon_x(F)$ is additive in (short exact sequences) in F.

Lemma 5.17. *Let $\nu : \operatorname{spec} K \to X$ be the generic point of X and suppose that the natural morphism $F \to \nu_* \nu^* F$ is an isomorphism. Then for any lifting $y \mapsto x$:*

1. $\dim_{\mathbb{F}_l} F_{\overline{x}} = \dim_{\mathbb{F}_l} (F_{\overline{\nu}})^{G_y}$,
2. $\epsilon_x(F) = \sum_{i=0}^{\infty} \frac{\#G_{y,i}}{\#G_y} \dim_{\mathbb{F}_l} \frac{F_{\overline{\nu}}}{F_{\overline{\nu}}^{G_{y,i}}}$.

5.6. Euler-Poincaré formula

We keep the notations of previous paragraph. As usual, for a constructible sheaf F, $\chi(X,F) = \sum_{i=0}^{2}(-1)^i \dim_{\mathbb{F}_l} H^i(X,F)$, and $\chi(X) = \chi(X,\mathbb{F}_l) = 2 - 2g_X$.

Theorem 5.18 (Grothendieck-Ogg-Shafarevich, see [11]).

$$\chi(X,F) = \chi(X)\dim_{\mathbb{F}_l} F_{\overline{\nu}} - \sum_{x\in|X|_0} \epsilon_x(F)\ .$$

References to a proof. Apart from Raynaud's report on Grothendieck's proof [11], one may want to refer to similar proofs in [3] and [9], or to the more recent and completely different proof in [4]. □

Corollary 5.19. *Let $U \subsetneq X$ be a nonempty (affine) open subset such that F is unramified on U. Then:*

$$\chi(U,F_{|U}) = \chi(U)\dim_{\mathbb{F}_l} F_{\overline{\nu}} - \sum_{x\in|U|_0} \alpha_x(F)\ .$$

The corollary is clear from the sequence of relative cohomology of the pair (X,U) and from the fact that $\dim_{\mathbb{F}_l} F_x = \sum_i (-1)^i \dim_{\mathbb{F}_l} H^i_x(X,F)$.

Remark 5.20. Note that if r is the number of points of $X \setminus U$, and g is the genus of X, then $\chi(U) = 2 - 2g - r$.

6. Serre's proof of solvable Abhyankar's conjecture for the affine line

6.1. Statement

Let p a prime number. Remember that, for a finite group G, we denote by $p(G)$ the subgroup generated by the p-Sylow subgroups of G. We will call $\mathcal{P}_G$ the following property:

$$\boxed{G = p(G) \iff \exists\, \pi_1^{et}(\mathbb{A}^1,\overline{x}) \twoheadrightarrow G.}$$

Abhyankar's conjecture for the affine line states that $\mathcal{P}_G$ is true for any finite group G. Serre proved this conjecture for G solvable at the beginning of the 1990s. He in fact showed the following stronger statement.

Theorem 6.1 (Serre, [14]). *Fix an exact sequence of finite groups: $1 \to A \to G \to H \to 1$. If A is solvable, and $\mathcal{P}_H$ holds, then $\mathcal{P}_G$ holds.*

In the property $\mathcal{P}_G$, the direct sense is the difficult one, so we will mainly concentrate on this.

6.2. Sketch of a proof

6.2.1. Reduction steps. By standard *dévissages*, we can reduce to the case where A is abelian, l-elementary (l any prime, possibly p) and irreducible for the action of H.

6.2.2. A local system. By hypothesis, the property $\mathcal{P}_H$ is true. The case where both assertions of $\mathcal{P}_H$ are false is easy, as before. So we can assume that $H = p(H)$, and that we are given a $\phi : \pi_1^{et}(\mathbb{A}^1, \overline{x}) \twoheadrightarrow H$, and try to extend it to G. Let $\pi : V \to U = \mathbb{A}^1$ the étale H-cover corresponding to ϕ. The data of ϕ, together with the action of H on A by conjugation, defines a local system $\underline{A}_\phi$ of $\mathbb{F}_l$-vector spaces on $\mathbb{A}^1_{et}$ by the usual formula: $\underline{A}_\phi = \pi_*^H(A_V)$. The reason why we put emphasis on ϕ will appear later.

6.2.3. Case of a non split exact sequence. Let us assume the exact sequence $1 \to A \to G \to H \to 1$ does not split. According to Theorem 3.2, we have that $\mathrm{cd}_l\, \mathbb{A}^1 = 1$ (this is actually also true for $l = p$, albeit with a different proof). Thus according to §3.4, $H^2(\pi_1^{et}(\mathbb{A}^1, \overline{x}), A) = 0$. By the reasoning explained in §3.2.1, we get that ϕ always lifts to G. Moreover it is easy to check that $G = p(G)$, so $\mathcal{P}_G$ is true.

6.2.4. Case of a split exact sequence. We now suppose that the exact sequence $1 \to A \to G \to H \to 1$ splits. We will only deal with the case $l \neq p$ and show that if $G = p(G)$, then $\exists\ \pi_1^{et}(\mathbb{A}^1, \overline{x}) \twoheadrightarrow G$ (although ϕ does not necessary lift to G). According to the conclusions of §3.2.2, ϕ lifts to G if and only if

$$\dim_{\mathbb{F}_l} H^1(H, A) < \dim_{\mathbb{F}_l} H^1(\mathbb{A}^1, \underline{A}_\phi)\ ,$$

(note that according to §3.4, $H^1(\pi_1^{et}(\mathbb{A}^1, \overline{x}), A) = H^1(\mathbb{A}^1, \underline{A}_\phi)$). Applying Grothendieck-Ogg-Shafarevich formula to compute the last term, we get:

Lemma 6.2. $$\dim_{\mathbb{F}_l} H^1(\mathbb{A}^1, \underline{A}_\phi) = \alpha_\infty(\underline{A}_\phi) - \dim_{\mathbb{F}_l} A.$$

Proof. Grothendieck-Ogg-Shafarevich formula gives

$$\chi(\mathbb{A}^1, \underline{A}_\phi) = \chi(\mathbb{A}^1) \dim_{\mathbb{F}_l} A - \alpha_\infty(\underline{A}_\phi)\ .$$

But $\chi(\mathbb{A}^1) = 2-2g-r = 1$, and $\chi(\mathbb{A}^1, \underline{A}_\phi) = \dim_{\mathbb{F}_l} H^0(\mathbb{A}^1, \underline{A}_\phi) - \dim_{\mathbb{F}_l} H^1(\mathbb{A}^1, \underline{A}_\phi)$. Now $H^0(\mathbb{A}^1, \underline{A}_\phi) = A^H$, and this last group must be trivial. Indeed else by irreducibility $A^H = A$, and $G \simeq A \times H$, and this contradicts the fact that $G = p(G)$, since we assume $l \neq p$. □

So to sum-up, we have

$$\dim_{\mathbb{F}_l} H^1(H, A) \leq \alpha_\infty(\underline{A}_\phi) - \dim_{\mathbb{F}_l} A\ ,$$

and ϕ lifts to G if and only if the inequality is strict. However, it may happen that the inequality above is an equality. This occurs for instance for Artin-Schreier covers. Suppose that we are in this situation. It is then necessary to increase the ramification by the following trick. Fix an integer $m \geq 1$, not divisible by p. Denote by $V_m \to \mathbb{A}^1$ the base change of the original étale H-cover $V \to \mathbb{A}^1$ by the Kummer morphism $\mathbb{A}^1 \to \mathbb{A}^1$ defined by $T \mapsto T^m$. Because H is quasi-p, and p does not divide m, the covers $V \to \mathbb{A}^1$ and $\mathbb{A}^1 \to \mathbb{A}^1$ are linearly disjoint, so V_m is irreducible, and $V_m \to \mathbb{A}^1$ defines in turn an epimorphism $\phi_m : \pi_1^{et}(\mathbb{A}^1, \overline{x}) \twoheadrightarrow H$. The next easy Lemma shows that $\alpha_\infty(\underline{A}_{\phi_m}) = m\alpha_\infty(\underline{A}_\phi)$, so ϕ_m lifts to G.

Lemma 6.3. *Let $f : X' \to X$ a finite separable morphism, where X and X' are smooth curves over an algebraically closed field of characteristic p. Let F be a constructible sheaf of $\mathbb{F}_l$-vector spaces on X_{et}, with $l \neq p$, and $x' \in X'$ a closed point. Then*

$$\alpha_{x'}(f^*F) = (\deg f)_{x'} \alpha_{f(x')}(F) \ .$$

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On the "Galois Closure" for Finite Morphisms

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Abstract. We give necessary and sufficient conditions for a finite flat morphism of schemes of characteristic $p > 0$ to be dominated by a torsor under a finite group scheme. We show that schemes satisfying this property constitute the category of covers for the fundamental group scheme.

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Introduction

The fundamental construction in Galois theory is that any separable field extension can be embedded in a Galois extension. Grothendieck [7] has generalized Galois theory to schemes (and potentially to even more abstract situations: Galois categories). Again, the basic step is, starting from a finite étale morphism $\pi : X \to S$, to construct a finite group G, a subgroup $H \leq G$ and a diagram

$$\begin{tikzcd} Y \arrow[r, "h"] \arrow[dr, "g"'] & X \arrow[d, "\pi"] \\ & S \end{tikzcd} \tag{1}$$

where g and h are finite étale Galois covers of groups G and H respectively. Recall that a finite étale morphism $X \to S$ is a Galois cover if a finite group G acts on X without fixed points and S is identified with the quotient of X by this action (cf. [10], § 7). This is equivalent to saying that X is a principal homogenous space (or torsor) over S under G, i.e., that the map $G \times X \to X \times_S X$ given by $(\gamma, x) \mapsto (\gamma x, x)$ is an isomorphism.

In characteristic $p > 0$ or in an arithmetic context it is often necessary to consider not only actions by abstract groups but infinitesimal actions as well. For instance an isogeny between abelian varieties may have an inseparable component (or degenerate to one). One is then led to consider torsors under finite flat group schemes (cf. [10], § 12).

In this note, we start with a finite flat morphism $\pi : X \to S$ of schemes of characteristic $p > 0$ and we try to find a "Galois closure" as in diagram (1), where g and h are torsors under group schemes G and $H \leq G$ defined over the prime field $\mathbb{F}_p$.

First of all, not any finite flat morphism π will do: indeed, if a "Galois closure" Y as above can be found at all, X will be a twisted form (in the flat topology) of the homogeneous scheme G/H, so π will have to be a local complete intersection morphism. It turns out that the right class of morphism, namely the *differentially homogeneous* morphisms, has been studied thoroughly by Sancho de Salas [13], who has developed a differential calculus extending Grothendieck's for smooth and étale morphisms. As for smoothness and étaleness, this is a local notion.

Our first result (Theorem 2.3) is that any finite differentially homogeneous morphism $\pi : X \to S$ of schemes in characteristic $p > 0$ fits in a diagram as in (1) above, where g and h are torsors under group schemes G and $H \leq G$ defined over the prime field $\mathbb{F}_p$.

As we shall explain shortly, Grothendieck's construction of the Galois closure for finite étale morphisms does not apply when one drops the étaleness assumption. We thus have to give a direct construction of a universal torsor dominating π: in many cases, it will much larger than the actual "Galois closure".

Let us describe our construction in the case of fields: a separable extension $L = K[x]/f(x)$ of degree n can be seen as a twist of K^n. The automorphism group of the geometric fibre of $K \subseteq L$ (i.e., the set of roots of f in an algebraic closure of K) is the symmetric group $\mathfrak{S}_n$, so L defines a Galois cohomology class in $H^1(K, \mathfrak{S}_n)$, represented by a Galois K-algebra A such that $A \otimes_K L \simeq A^n$. Any étale K-algebra B such that $B \otimes_K L \simeq B^n$ receives a map from A, and in particular the Galois closure of L/K is a direct summand of A. Moreover $L \subseteq A$ consists of elements fixed by the stabilizer $\mathfrak{S}_{n-1}$ of a given root of f.

Unfortunately, the group schemes acting on our universal torsor are not finite in general; for instance they are not in the case of the Frobenius morphism $\pi : \mathbb{P}^1 \to \mathbb{P}^1$. The reason is that, in contrast with the étale case, the automorphism group scheme of a fibre of π is not finite.

Our main result, Theorem 2.11, gives necessary and sufficient conditions for the existence of a finite Galois closure Y as in (1). In contrast with the étale case, these conditions are of a global nature, as can be expected from the counterexample above.

Except when one can reduce to the case of field extensions (e.g., when all schemes involved are normal), Grothendieck's construction of the Galois closure of an étale morphism is indirect and relies on his theory of the fundamental group [7], V § 4. Let us now briefly review it, disregarding base points for simplicity. Grothendieck first proves that the category of finite étale covers of a given scheme is filtered: this relies on the fact that fibred products of étale morphisms are again étale. In fact, existence of finite fibred products is the first axiom that any Galois category should satisfy. This fails dismally for arbitrary finite flat morphisms.

Grothendieck's second step is formal: being filtered, the category of étale covers has a projective limit, which is the universal cover. He then turns his attention to connected covers, as any cover breaks down as a disjoint union of connected ones. Since any endomorphism of a connected cover is an automorphism, he defines the Galois objects as the simple, connected covers. Tautologically, these form a filtering subsystem, thus any cover is dominated by a smallest Galois cover, which is the Galois closure. Obviously, this process cannot be replicated with flat covers: a trivial torsor under any infinitesimal group scheme is connected.

The arithmetic fundamental group $\pi_1(S)$ is the projective limit of all the Galois groups over S, i.e., the automorphism groups of the Galois covers of S. If S is given over a base scheme B, later in his seminar (X 2.5), Grothendieck suggested to look for a profinite B-group scheme classifying torsors over S under finite flat B-group schemes. This *fundamental group scheme* $\boldsymbol{\pi}(S/B)$ should be the projective limit of all finite group schemes occurring as structure groups of torsors over S. In terms of Galois theory as outlined above, this approach forgets the general category of covers to focus solely on Galois objects.

This program has been pursued by Nori [11] (over a base field) and Gasbarri [4] (over a Dedekind base). Much progress has been made recently on the fundamental group scheme. This is especially true in the case of proper reduced schemes over a field, where again Nori [11] gave a Tannakian interpretation of the fundamental group scheme in terms of vector bundles, whence a connection with motivic fundamental groups.

The basic existence criterion for the fundamental group scheme is that the category of torsors should admit finite fibred products: a formal argument due to Nori shows then that the category of torsors is filtered and the universal cover is just the limit of this category. As is to be expected from the above-mentioned pathologies, the existence of fibred products can only be proven under quite restrictive assumptions on S and B. In Theorem 4.5, as a consequence of our main result, we improve slightly on previously known existence results for the fundamental group scheme.

The conceptual significance of the Galois closure problem is that it pinpoints the essential property of covers for abstract fundamental groups: for the flat topology, it allows us to trace Grothendieck's steps backwards, from Galois objects to covers. "Covers" should indeed be taken to mean morphisms that can be dominated by a finite torsor. A formal argument (Theorem 4.13) shows that the fundamental group scheme exists if and only if the category of "covers" admits fibred products, and that the universal cover is indeed the initial object among covers. The merit of Theorem 2.11 is to show these speculations to be non-vacuous. In fact, it allows us to determine completely the category of covers for flat schemes over a perfect field in positive characteristic. What is sorely missing is a similar characterization of "covers" for arithmetic schemes.

Let us now review in more detail the structure of the paper. Until the last section, we work in characteristic $p > 0$.

In § 1, after reviewing Sancho de Salas' work [13] on differentially homogeneous morphisms, we focus on the subcategory of finite differentially homogeneous morphisms. We show that a finite morphism is differentially homogeneous if and only if it is a twisted form in the flat topology of a finite $\mathbb{F}_p$-scheme, completely determined by the differential structure of the morphism.

In § 2, we first prove that any finite differentially homogeneous morphism can be dominated by a torsor under a flat, but not necessarily finite, $\mathbb{F}_p$-group scheme. We next prove our main result, Theorem 2.11, giving necessary and sufficient conditions for a finite morphism to admit a finite Galois closure. A morphism with this property is called F-constant.

M. Antei and M. Emsalem have introduced in [1] another class of finite flat morphisms (called essentially finite), admitting a Galois closure. Their construction is based on Nori's tannakian approach to the fundamental group scheme: it is thus restricted to reduced schemes proper over a field, but provides a description of the Galois group. In § 3, we show that, whenever they may be compared, essentially finite and F-constant morphisms are equivalent (Theorem 3.5).

Finally, in § 4 we give applications to the fundamental group scheme. We first give an existence result (Theorem 4.5): let S be a flat scheme over a Dedekind base which has a fundamental group scheme, then if $X \to S$ is a finite flat map with étale or F-constant generic fibre, X has a fundamental group scheme too. If moreover X itself can be dominated by a finite torsor, then its fundamental group scheme injects into that of S (Theorem 4.9). The remainder of the section is devoted to speculations on Galois theory for the flat topology.

I am indebted to Pedro Sancho de Salas for pointing out a mistake in an earlier version of this paper, providing Example 1.7 below. It is a pleasure to thank Noriyuki Suwa for many interesting conversations and useful comments.

1. Differentially homogeneous morphisms

Notations and conventions: After Example 1.2 below and until § 4 all schemes are assumed to be noetherian of characteristic $p > 0$. We fix a separated scheme of finite type S.

If Z is a scheme of characteristic p, denote $F_Z : Z \to Z$ the absolute Frobenius. If U is a Z-scheme, $U^{(i/Z)}$ denotes the pullback of U by the ith iterate of F_Z and $F_{U/Z} : U \to U^{(1/Z)}$ the relative Frobenius, a morphism of Z-schemes. We shall simplify and write $U^{(i)}$ for $U^{(i/\mathbb{F}_p)}$.

If G is an $\mathbb{F}_p$-group scheme, we denote by ${}_{F^i}G \trianglelefteq G$ the kernel of $F^i_{G/\mathbb{F}_p} : G \to G^{(i)}$.

Definition 1.1. An S-scheme X of finite type is *differentially homogeneous*[1] if it is flat and for all $r \geq 0$ the $\mathcal{O}_X$-module $\mathcal{O}_X \otimes_{\mathcal{O}_S} \mathcal{O}_X/\mathcal{I}^{r+1}$ is coherent and locally free, where $\mathcal{I}$ is the sheaf of ideals defined by the diagonal map $X \to X \times_S X$.

[1] Or *normally flat along the diagonal* in the EGA lingo: [5] IV.6.10.1.

A morphism $\pi : X \to S$ is said to be differentially homogeneous at $x \in X$ if $\operatorname{Spec} \mathcal{O}_{X,x}$ is differentially homogeneous over $\operatorname{Spec} \mathcal{O}_{S,\pi(x)}$. From the definition (and the behaviour of the differential sheaves) it follows immediately that this property is local on the source and stable under base change and faithfully flat descent. For any $\pi : X \to S$, the set of points $x \in X$ such that π is differentially homogeneous at x is open.

Example 1.2. Smooth morphisms are differentially homogeneous. Twisted forms in the flat topology of differentially homogeneous schemes are differentially homogeneous. If k is a field and S is a k-scheme, torsors over S under an algebraic k-group scheme are differentially homogeneous.

Differentially homogeneous morphisms have been investigated by Sancho de Salas [13]. In characteristic zero, a morphism is differentially homogeneous if and only if it is smooth. In characteristic $p > 0$, differentially homogeneous schemes can be characterized in terms of pth powers. For any $n \geq 0$, let X_{p^n} be the scheme with the same underlying topological space as X and whose structure sheaf is $\mathcal{O}_S[\mathcal{O}_X^{p^n}]$, the $\mathcal{O}_S$-subalgebra of $\mathcal{O}_X$ generated by p^nth powers of sections of $\mathcal{O}_X$.

Proposition 1.3 (Sancho de Salas [13]). *Let S be a connected scheme and $\pi : X \to S$ a flat morphism of finite type.*

1) *X is differentially homogeneous if and only if $\Omega^1_{X_{p^r}/S}$ is a flat $\mathcal{O}_{X_{p^r}}$-module for any $r \geq 0$* ([13], Proposition 2.4).
2) *X is differentially homogeneous if and only if for every $x \in X$ there are affine neighborhoods $V = \operatorname{Spec} B$ of x and $U = \operatorname{Spec} A$ of $\pi(x)$ such that $\pi(V) \subseteq U$, and there exists a chain $B_0 \subset B_1 \subset \cdots \subset B_n = B$, where B_0 is a smooth A-algebra and $B_{i+1} = B_i[x_i]/(x_i^{p^{e_i}} - b_i)$ for some $b_i \in A[B_i^{p^{e_i}}]$* ([13], *Theorem* 3.4).
3) *If X is differentially homogeneous over S then X is finite and differentially homogeneous over X_{p^n} for all n and X_{p^n} is smooth over S for $n \gg 0$* ([13], Corollary 2.5 and Theorem 2.6).

Remark 1.4. The condition $b_i \in A[B_i^{p^{e_i}}]$ in Proposition 1.3.2 has the unpleasant consequence that if Y is differentially homogeneous over a scheme X that is differentially homogeneous (even smooth) over S then Y may not be differentially homogeneous over S. For instance, the affine curve Y given by $y^p = x^{p+1}$ is differentially homogeneous over $\mathbb{A}^1 = \operatorname{Spec} \mathbb{F}_p[x]$, but $\Omega^1_{Y/\mathbb{F}_p}$ is not flat at the origin, so Y is not differentially homogeneous over $\mathbb{F}_p$.

Definition 1.5. We will use the acronym *qfdh* (respectively *fdh*) to indicate a quasi-finite (resp. finite) differentially homogeneous morphism $X \to S$.

Example 1.6. A flat S-group scheme of finite height (i.e., $G = \ker F^h_{G/S}$ for some $h \geq 0$) is qfdh. Indeed its fibres are fdh and $G_{p^i} = \ker F^{h-i}_{G^{(i/S)}/S}$, hence $G \to G_{p^i}$ is faithfully flat. We can apply [13], proposition 2.8: X is differentially homogeneous

if and only if its fibres are differentially homogeneous and $X \to X_{p^i}$ is faithfully flat for all $i > 0$.

Example 1.7 (Sancho de Salas). Unfortunately, qfdh morphisms are not composable: let $A = \mathbb{F}_p[x](x^p)$, $B = A[u]/(u^p)$ and $C = B[v](v^p - xu)$. Then $X = \operatorname{Spec} B$ is fdh over $S = \operatorname{Spec} A$ and $Y = \operatorname{Spec} C$ is fdh over X, by the criterion 1.3.2, but Y is not differentially homogeneous over S since $\Omega^1_{C/A} = Cdu \oplus Cdv/(xdu)$ is not a flat C-module.

Remark 1.8. J.-M. Fontaine (unpublished) defined quiet morphisms as the smallest class of syntomic morphisms closed under composition and containing étale maps and morphisms of the type $\operatorname{Spec} A[x]/(x^p - a) \to \operatorname{Spec} A$. All such morphisms are qfdh and, by [13], Proposition 1.7, a differentially homogeneous morphism is a complete intersection morphism. Therefore, qfdh morphisms are the building blocks of Fontaine's quiet topology.

In the following, we will show that any scheme X qfdh over a connected scheme S of characteristic p is a twisted form in the flat topology of a "constant" scheme defined over the prime field $\mathbb{F}_p$. The first step is to attach to $X \to S$ such "typical fibre". The starting point is the following remark.

Lemma 1.9. *If $X \to S$ is a qfdh morphism of connected schemes,* $\operatorname{rk} \Omega^1_{X/S} \geq \operatorname{rk} \Omega^1_{X_p/S}$.

Proof. We may assume that $S = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ are local. Let $dz_1, \ldots, dz_r$ be a basis of $\Omega^1_{B/A}$ and define a map $\varphi : C = A[Z_1, \ldots, Z_r] \to B$ by $Z_i \mapsto z_i$. Since $d\varphi : B \otimes_C \Omega^1_{C/A} \to \Omega^1_{B/A}$ is an isomorphism, φ induces an isomorphism at the level of tangent spaces and is therefore surjective. φ maps the subalgebra $A[C^p] = A[Z_1^p, \ldots, Z_r^p]$ to the subalgebra $A[B^p]$. Let $\bar{f} \in A[B^p]$ and $f \in C$ such that $\varphi(f) = \bar{f}$. Since $d\bar{f} = 0$ in $\Omega^1_{B/A} = \Omega^1_{B/A[B^p]}$ and $d\varphi$ is an isomorphism, $df = 0$ hence $f \in A[Z_1^p, \ldots, Z_r^p]$. Therefore $\varphi : A[C^p] \to A[B^p]$ is again surjective and so $\Omega^1_{A[B^p]/A}$ is generated by the $d\varphi(Z_i^p) = d(z_i^p)$ and has thus rank $\leq r$. $\square$

Definition 1.10. Let X be a qfdh, connected S-scheme and consider the factorization

$$X \to X_p \cdots \to X_{p^i} \cdots \to S.$$

We shall say that an integer $\nu \geq 1$ is a *break* if $\operatorname{rk} \Omega^1_{X_{p^\nu}/S} \lneq \operatorname{rk} \Omega^1_{X_{p^{\nu-1}}/S}$.

Definition 1.11. Let X be a qfdh, connected S-scheme and $r = \operatorname{rk} \Omega^1_{X/S}$. To $X \to S$ we associate the following data:

1. The r-tuple $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_r)$ of breaks, each one repeated $\operatorname{rk} \Omega^1_{X_{p^{\nu-1}}/S} - \operatorname{rk} \Omega^1_{X_{p^\nu}/S}$ times, arranged in increasing order.
2. The scheme $\Sigma^{\boldsymbol{\nu}} = \operatorname{Spec} \mathbb{F}_p[t_1, \ldots, t_r]/(t_1^{p^{\nu_1}}, \ldots, t_r^{p^{\nu_r}})$.
3. If $X \to S$ is finite, the degree $d = \deg(X_{p^{\nu_r}}/S)$ of the étale subcover.

Proposition 1.12. *A finite scheme X over a connected scheme S is fdh if and only if, locally for the flat topology on S, it is isomorphic to $\coprod_{i=1}^{d} \Sigma_S^{\boldsymbol{\nu}}$.*

Proof. The if part is clear. We may assume that $S = \operatorname{Spec} A$ is local. Replacing A by its strict henselization, we may assume that $X = \coprod_{i=1}^{d} \operatorname{Spec} B$ with B/A fdh and radicial. Let thus $d = 1$. We may also assume that $X \to S$ has a section: indeed, by [13], Corollary 3.5, there is a section over the pullback by a qfdh A-algebra A'. The kernel $J = \ker[B \to A]$ of this section is a nilpotent ideal, since X and S have the same topological space. By [13], Theorem 1.6, there is a faithfully flat base change $A \to A''$ such that $B'' = A'' \otimes_A B \cong A''[t_1, \ldots, t_r]/(t_1^{p^{e_1}}, \ldots, t_r^{p^{e_r}})$, for suitable integers $e_1 \leq \cdots \leq e_r$. Computing the breaks of $\Omega^1_{B''/A''} = B'' \otimes_B \Omega^1_{B/A}$, one checks immediately that $(e_1, \ldots, e_r) = (\nu_1, \ldots, \nu_r)$. $\square$

2. Galois closures

Definition 2.1. Let $X \to S$ be a finite flat morphism. We shall say that a torsor T/S under a group scheme G *dominates* X if $T \to S$ factors through a flat morphism $T \to X$ which is a torsor under a suitable subgroup $H \subseteq G$.

$$\begin{tikzcd} T \arrow[r, "H"] \arrow[dr, "G"'] & X \arrow[d] \\ & S. \end{tikzcd}$$

In the previous section we have established that an fdh scheme $X \to S$ is a twisted form of a disjoint sum of "constant" schemes $\Sigma^{\boldsymbol{\nu}}$. In order to construct a torsor T dominating X, we should investigate the automorphisms of $\Sigma^{\boldsymbol{\nu}}$ as a sheaf for the flat topology. The idea is to mimic the following process: the symmetric group $\mathfrak{S}_n$ is the automorphism group of the set $\Sigma = \{1, \ldots, n\}$. Evaluation at $1 \in \Sigma$ yields a surjective map $\mathfrak{S}_n \to \Sigma$ identifying the latter as the homogeneous space $\mathfrak{S}_n/\mathfrak{S}_{n-1}$.

By [2] II § 1, 2.7 (see also the proof of the following lemma), the sheaf of automorphisms of $\Sigma^{\boldsymbol{\nu}}$ is representable by an affine group scheme $\underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}})$ of finite type over $\mathbb{F}_p$. Let $o \in \Sigma^{\boldsymbol{\nu}}(\mathbb{F}_p)$ be the origin. We denote by $\underline{\mathrm{Aut}}_{\,o}(\Sigma^{\boldsymbol{\nu}})$ its stabilizer and by $q : \underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}}) \to \Sigma^{\boldsymbol{\nu}}$ the canonical morphism defined, for any $\mathbb{F}_p$-algebra A, by mapping an automorphism g of $\Sigma_A^{\boldsymbol{\nu}}$ to $g(o) \in \Sigma^{\boldsymbol{\nu}}(A)$.

The following lemma gathers the information we will need about $\underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}})$ and some of its subgroups. It is probably well known, but we include it for lack of references.

Lemma 2.2. *The morphism $q : \underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}}) \to \Sigma^{\boldsymbol{\nu}}$ is faithfully flat. For any integer $n \geq \nu_r$ it induces an isomorphism ${}_{F^n}\underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}})/{}_{F^n}\underline{\mathrm{Aut}}_{\,o}(\Sigma^{\boldsymbol{\nu}}) \cong \Sigma^{\boldsymbol{\nu}}$.*

Proof. Let $N = \{[0, p^{\nu_1} - 1] \times \cdots \times [0, p^{\nu_r} - 1]\} \cap \mathbb{N}^r$ and let $N_i = \{J \in N \,|\, p^{\nu_i} J \in N\}$.

The t^J, with $J \in N$ form a basis of the $\mathbb{F}_p$-vector space

$$\mathbb{F}_p[t_1, \dots, t_r]/(t_1^{p^{\nu_1}}, \dots, t_r^{p^{\nu_r}}).$$

The functor on $\mathbb{F}_p$-algebras $A \mapsto \mathrm{Hom}_{A-sch}(\Sigma_A^{\boldsymbol{\nu}}, \mathbb{A}_A^r)$ is represented by $\mathbb{A}^{r|N|} = \mathrm{Spec}\, R[x_{i,J}]$, a morphism $\Sigma_A^{\boldsymbol{\nu}} \to \mathbb{A}_A^r$ being defined by a map

$$\begin{array}{rcl} A[t_1, \dots, t_r] & \longrightarrow & A \otimes \mathbb{F}_p[t_1, \dots, t_r]/(t_1^{p^{\nu_1}}, \dots, t_r^{p^{\nu_r}}) \\ t_i & \longmapsto & \sum_J x_{i,J} \otimes t^J. \end{array} \tag{2}$$

This map factors through $\Sigma_A^{\boldsymbol{\nu}}$ if and only if

$$\left(\sum_J x_{i,J} \otimes t_1^{j_1} \dots t_r^{j_r} \right)^{p^{\nu_i}} = \sum_J x_{i,J}^{p^{\nu_i}} \otimes t_1^{p^{\nu_i} j_1} \dots t_r^{p^{\nu_i} j_r} = 0$$

for $i = 1, \dots, r$. Hence the sheaf of monoids $A \mapsto End_{A-sch}(\Sigma_A^{\boldsymbol{\nu}})$ is represented by $\underline{\mathrm{End}}\,(\Sigma^{\boldsymbol{\nu}}) = \mathrm{Spec}\, \mathbb{F}_p[x_{i,J}]/(x_{i,J}^{p^{\nu_i}} \mid i = 1, \dots, r; J \in N_i)$.

From (2) we infer that the action $\underline{\mathrm{End}}\,(\Sigma^{\boldsymbol{\nu}}) \times \Sigma^{\boldsymbol{\nu}} \to \Sigma^{\boldsymbol{\nu}}$ (described on A-valued points by $(g, x) \mapsto g(x)$) is given by

$$\begin{array}{rcl} \mathbb{F}_p[t_1, \dots, t_r]/(t_1^{p^{\nu_1}}, \dots, t_r^{p^{\nu_r}}) & \longrightarrow & \mathbb{F}_p[x_{i,J}]/(x_{i,J}^{p^{\nu_i}} \mid J \in N_i) \\ & & \otimes\, \mathbb{F}_p[t_1, \dots, t_r]/(t_1^{p^{\nu_1}}, \dots, t_r^{p^{\nu_r}}) \\ t_i & \longmapsto & \displaystyle\sum_J x_{i,J} \otimes t^J \end{array}$$

and therefore $q : \underline{\mathrm{End}}\,(\Sigma^{\boldsymbol{\nu}}) \to \Sigma^{\boldsymbol{\nu}}$, given by

$$\begin{array}{rcl} \mathbb{F}_p[t_1, \dots, t_r]/(t_1^{p^{\nu_1}}, \dots, t_r^{p^{\nu_r}}) & \longrightarrow & \mathbb{F}_p[x_{i,J}]/(x_{i,J}^{p^{\nu_i}} \mid J \in N_i) \\ t_i & \longmapsto & x_{i,0} \end{array} \tag{3}$$

is faithfully flat (since $0 \in N_i\ \forall i$) and so is the restriction to the open subscheme $\underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}}) \subset \underline{\mathrm{End}}\,(\Sigma^{\boldsymbol{\nu}})$.

For any $n \geq 0$ the endomorphisms whose pull-back by the nth iterate of Frobenius is the identity form a submonoid ${}_{F^n}\underline{\mathrm{End}}\,(\Sigma^{\boldsymbol{\nu}}) \subseteq \underline{\mathrm{End}}\,(\Sigma^{\boldsymbol{\nu}})$. If $n \geq \nu_r$, from (2), we deduce that

$${}_{F^n}\underline{\mathrm{End}}\,(\Sigma^{\boldsymbol{\nu}}) = \mathrm{Spec}\, \mathbb{F}_p[x_{i,J}]/(x_{i,J}^{p^{\nu_i}} \mid J \in N_i;\ x_{i,J}^{p^n} \mid J \notin N_i)$$

and from (3) that the induced map $q_n : {}_{F^n}\underline{\mathrm{End}}\,(\Sigma^{\boldsymbol{\nu}}) \to \Sigma^{\boldsymbol{\nu}}$ is faithfully flat for all $n \geq \nu_r$. Therefore, so is the restriction to the open subscheme ${}_{F^n}\underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}})$. Let us consider the diagram:

$$\begin{tikzcd} {}_{F^n}\underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}}) \arrow[r] \arrow[dr, "q_n"'] & {}_{F^n}\underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}})/{}_{F^n}\underline{\mathrm{Aut}}_o(\Sigma^{\boldsymbol{\nu}}) \arrow[d, "\iota_n"] \\ & \Sigma^{\boldsymbol{\nu}}. \end{tikzcd}$$

By [2] III § 3 5.2, the quotient ${}_{F^n}\underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}})/{}_{F^n}\underline{\mathrm{Aut}}_o(\Sigma^{\boldsymbol{\nu}})$ is representable and the canonical map ι_n is an immersion. By [2] III § 3 2.5, the horizontal projection is

faithfully flat. Hence ι_n is flat and is thus an open immersion. Since $\Sigma^{\boldsymbol{\nu}}$ is local, ι_n is an isomorphism. $\square$

Theorem 2.3. *Let S be a connected scheme, $X \to S$ an fdh morphism. There exists a torsor T/S in the fppf topology under an affine $\mathbb{F}_p$-group scheme of finite type, dominating X and such that $T \times_S X \cong \coprod_{i=1}^d \Sigma_T^{\boldsymbol{\nu}}$.*

Proof. As an automorphism of a scheme induces an automorphism of the set of connected components, $\underline{\mathrm{Aut}}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$ is a (split) extension of the symmetric group $\mathfrak{S}_d$ by $\prod_{i=1}^d \underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}})$. The fppf sheaf $\underline{\mathrm{Isom}}_S(\coprod_{i=1}^d \Sigma_S^{\boldsymbol{\nu}}, X)$ is an $\underline{\mathrm{Aut}}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$-torsor over S and is thus representable (e.g., [9], III, 4.3) by a scheme T.

Let o_1 be the origin of the first connected component of $\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}$ and $\underline{\mathrm{Aut}}_{o_1}(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$ its stabilizer (an extension of $\mathfrak{S}_{d-1}$ by $\underline{\mathrm{Aut}}_o(\Sigma^{\boldsymbol{\nu}}) \times \prod_{i=2}^d \underline{\mathrm{Aut}}(\Sigma^{\boldsymbol{\nu}})$). If U is any S-scheme, to any $\varphi_U : \coprod_{i=1}^d \Sigma_U^{\boldsymbol{\nu}} \to X_U$ we can associate $\varphi_U(o_1) \in X(U)$. These data define an $\underline{\mathrm{Aut}}_{o_1}(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$-equivariant morphism

$$f : T = \underline{\mathrm{Isom}}_S(\coprod_{i=1}^d \Sigma_S^{\boldsymbol{\nu}}, X) \to X.$$

Around any closed point of X, locally for the flat topology, f is isomorphic to the "evaluation at o_1" map $q : \underline{\mathrm{Aut}}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}) \to \coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}$ followed by the projection onto the first factor. Hence f is faithfully flat by Lemma 2.2. Finally, one checks immediately that the diagram

$$\begin{array}{ccc} T \times \underline{\mathrm{Aut}}_{o_1}(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}) & \longrightarrow & T \times_X T \\ \downarrow & & \downarrow \\ T \times \underline{\mathrm{Aut}}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}) & \longrightarrow & T \times_S T \end{array}$$

where the horizontal maps are given by $(\varphi_U, g_U) \mapsto (\varphi_U, \varphi_U \circ g_U)$, is cartesian. Since T is an $\underline{\mathrm{Aut}}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$-torsor, the bottom map is an isomorphism, hence so is the top map. $\square$

Remark 2.4. The datum of an isomorphism $X \times_S X \cong \Sigma_X^{\boldsymbol{\nu}}$ as X-schemes is equivalent to a section $X \to T = \underline{\mathrm{Isom}}_S(\Sigma_S^{\boldsymbol{\nu}}, X)$ of $f : T \to X$; in such a situation, T is a trivial torsor over X. This is the case in particular when X is itself a torsor over S.

Being a torsor under an algebraic group scheme, T is differentially homogeneous but never finite: as seen in the proof of Lemma 2.2, the reduced connected component of $\underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}})$ is positive-dimensional. The remainder of this section is devoted to the following question: is it possible to find a torsor Y/S under a *finite* group scheme dominating X? In other words, when does T admit a reduction of the structure group to a finite subgroup?

Proposition 2.5. *Locally on S for the Zariski topology, an fdh morphism $X \to S$ is dominated by a torsor under a finite $\mathbb{F}_p$-group scheme.*

Proof. If $S = \operatorname{Spec} A$ is local then $X = \operatorname{Spec} B$ admits a chain $B_0 \subset B_1 \subset \cdots \subset B_n = B$ as in Proposition 1.3. Since $B_{i+1} = B_i[x_i]/(x_i^{p^{e_i}} - b_i)$ can be seen as an $\boldsymbol{\alpha}_{p^{e_i}}$-torsor over B_i, replacing B_0 by its finite étale Galois closure over A, we get a factorization of X as a tower of finite torsors. By [3], Theorem 2, X is dominated by a torsor under a finite $\mathbb{F}_p$-group scheme. $\square$

Another explanation for the fact that locally on the base an fdh morphism can be dominated by a finite torsor will be provided in Proposition 3.8 in the next section. In general however, it is not possible to dominate an fdh morphism by a finite torsor, as shown in Example 2.7 below. The example and the subsequent results are based on the following remark.

Remark 2.6. Let Σ be a finite $\mathbb{F}_p$-scheme, $G = \underline{\mathrm{Aut}}\,(\Sigma)$ and let $X \to S$ be a twisted form of Σ_S. The Frobenius morphism $F^n_{G/\mathbb{F}_p} : G \to G^{(n)}$ induces an exact sequence in flat cohomology

$$\check{H}^1(S,\ {}_{F^n}G) \longrightarrow \check{H}^1(S, G) \longrightarrow \check{H}^1(S, G^{(n)}).$$

The second map sends the class of $T = \underline{\mathrm{Isom}}_S(\Sigma_S, X)$ to that of $\underline{\mathrm{Isom}}_S(\Sigma_S, X^{(n/S)})$. Hence $X^{(n/S)}$ is isomorphic to Σ_S if and only if T is induced from a torsor Y under the finite subgroup ${}_{F^n}G$. The canonical map $Y \to Y \times G \to Y \wedge^{{}_{F^n}G} G \cong T$ gives a point in $T(Y) = \mathrm{Isom}_Y(\Sigma_Y, X_Y)$, hence X becomes isomorphic to Σ over Y.

Example 2.7. Let k be a perfect field, $X = S = \mathbb{P}^1_k$ and $\pi : X \to S$ be the relative (k-linear) Frobenius. X is a twisted form of $\Sigma^1_S = S \times \operatorname{Spec} \mathbb{F}_p[t]/t^p$. Suppose that X trivializes over a torsor under a finite subgroup $H \leq G = \underline{\mathrm{Aut}}\,(\Sigma^1)$. As there are no étale covers of $\mathbb{P}^1$, there is no loss in generality in assuming H connected and thus $H \leq {}_{F^n}G$ for a suitable integer n. In other words, X would become isomorphic to Σ^1_S over the nth iterate $F^n_S : S \to S$ of the absolute Frobenius. In particular the pullback $p_2^*\Omega^1_{X/S} = p_2^*\Omega^1_X$ would have to be constant over $S \times_{S, F^n_S} X$ and so would then be the pullback $F_X^{n*}\Omega^1_X$. This is absurd, since $\Omega^1_X = \mathcal{O}(-2)$ and $F_X^{n*}\Omega^1_X = \mathcal{O}(-2p^n)$ is never constant.

Definition 2.8. Let $X \to X^{\acute{e}t} \to S$ be an fdh morphism, factored into a radicial and an étale morphism. We will say that X is *F-constant* over S if the pull-back of X over a suitable iterate of the absolute Frobenius $F_S : S \to S$ becomes isomorphic to $\Sigma^{\boldsymbol{\nu}}_{X^{\acute{e}t}}$.

Remark 2.9. Notice that since $X^{\acute{e}t} \to S$ is étale, the diagram

$$\begin{array}{ccc} X^{\acute{e}t} & \xrightarrow{F_{X^{\acute{e}t}}} & X^{\acute{e}t} \\ \downarrow & & \downarrow \\ S & \xrightarrow{F_S} & S \end{array}$$

is cartesian, so X is F-constant over S if and only if it is F-constant over $X^{\acute{e}t}$.

Remark 2.10. F-constance can be checked after finite étale base change: X is F-constant over S if and only if, for any finite étale base $S' \to S$ the scheme $X' = S' \times_S X$ is F-constant over S'. By the above remark, we may assume $S = X^{ét}$. Composing a section over S' with the projection yields a finite S-morphism $\sigma : S' \to X' \to X$. Since S'/S is étale while X/S is radicial, one checks immediately that the image of σ is isomorphic to S, thus providing a section to π.

Theorem 2.11. *Let S be a connected scheme and X a finite S-scheme. The following conditions are equivalent:*

1. *X is F-constant;*
2. *there are finite $\mathbb{F}_p$-group schemes $H \leq G$ and an X-scheme Y which is a G-torsor over S and an H-torsor over X;*
3. *there exists a torsor Y/S under a finite $\mathbb{F}_p$-group scheme such that $Y \times_S X$ is a finite disjoint union of copies of $\Sigma_Y^{\boldsymbol{\nu}}$.*

Proof. By [3], Theorem 2, X is dominated by a torsor under a finite $\mathbb{F}_p$-group scheme.
1) $\Rightarrow$ 2) By Thm. 2.3, X becomes isomorphic to $\coprod_{i=1}^d \Sigma_T^{\boldsymbol{\nu}}$ over the $\underline{\mathrm{Aut}}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$-torsor $T = \underline{\mathrm{Isom}}_S(\coprod_{i=1}^d \Sigma_S^{\boldsymbol{\nu}}, X)$. Since $\underline{\mathrm{Aut}}\,(\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}})$ is an extension of the étale group $\mathfrak{S}_d$ by the connected component $\prod_{i=1}^d \underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}})$, we can factor $T \to S$ through an étale $\mathfrak{S}_d$-cover $Z \to S$, which we can interpret as a disjoint union of $[\mathrm{Gal}(X^{ét}/S) : \mathfrak{S}_d]$ copies of the Galois closure of the maximal étale subcover $X^{ét} \to S$. We have to show that the connected torsor $T \to Z$ is induced by a finite subgroup of the structure group $\prod_{i=1}^d \underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}})$ so, replacing S by Z and X by a connected component of $Z \times_{X^{ét}} X$ we may assume that X is radicial over S.
Since X is F-constant, $X^{(p^n/S)} \cong \Sigma_S^{\boldsymbol{\nu}}$ for $n \gg 0$. Hence, by Remark 2.6, there is an ${}_{F^n}\underline{\mathrm{Aut}}\,(\Sigma^{\boldsymbol{\nu}})$-torsor Y such that $X \times_S Y = \Sigma_Y^{\boldsymbol{\nu}}$. Taking $n \geq \nu_r$, so that Lemma 2.2 applies, the same argument as in Theorem 2.3 shows that Y is an ${}_{F^n}\underline{\mathrm{Aut}}\,_o(\Sigma^{\boldsymbol{\nu}})$-torsor over X.

2) $\Rightarrow$ 3) Denoting by $\mu : Y \times G \to Y$ the action and by m the multiplication in G, we have a commutative diagram

$$\begin{array}{ccc} Y \times G \times H & \xrightarrow{id_Y \times m} & Y \times G \\ {\scriptstyle id_Y \times \mu \times id_H}\downarrow & & \downarrow{\scriptstyle id_Y \times \mu} \\ Y \times_S Y \times H & \xrightarrow{id_Y \times \mu} & Y \times_S Y \end{array}$$

whose vertical arrows are isomorphisms because Y is a G-torsor over S. Hence the quotient $Y \times (G/H)$ by the top action is isomorphic, as an Y-scheme, to the quotient $Y \times_S X$ by the bottom one. Therefore X becomes isomorphic over Y to G/H and the latter, by [2], III § 3, 6.1, is a scheme of type $\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}$.

3) $\Rightarrow$ 1) Being a twisted form of $\coprod_{i=1}^d \Sigma^{\boldsymbol{\nu}}$ in the flat topology, X certainly is differentially homogeneous, and we can factor it as $X \to X^{ét} \to S$ as the composition of a radicial and an étale morphism. According to Remark 2.9, to check that X is

F-constant we may assume that $X^{ét} = S$. Since G is an extension of an étale group $G^{ét}$ by a connected one G^0, we can also factor the cover $Y \to Z \to S$, where the first is G^0-torsor and the second a Galois étale cover. By [10] II, § 7, Proposition 2, there is an equivalence of categories between coherent sheaves on S and coherent $G^{ét}$-sheaves on Z. Since the absolute Frobenius commutes with automorphisms, $X \times_S Z$ is F-constant over Z if and only if X is F-constant over S.
We may therefore assume that Y/S is a torsor under G^0. The latter is a finite connected group scheme, hence has finite Frobenius height $\leq h$. Therefore Y is an fdh S-scheme with $Y_{p^h} = S$ and we have a factorization of F_S^h as $S \to Y \to S$. From the isomorphism $Y \times_S X \cong \Sigma_Y^{\boldsymbol{\nu}}$ we then deduce that $S \times_{F_S^h} X \cong \Sigma_S^{\boldsymbol{\nu}}$. □

Corollary 2.12. *Let k be a field of characteristic $p > 0$, S a connected k-scheme and X a finite S-scheme. Then in conditions* 2 *and* 3 *in Theorem* 2.11 *we may replace $\mathbb{F}_p$-group schemes by k-group schemes.*

Proof. This is just a little dévissage. It suffices to prove 3) $\Rightarrow$ 1). Let thus G be a finite k-group scheme and Y/S a G-torsor such that X trivializes over Y. We may replace S by $Y^{ét}$, the maximal étale subcover of $Y \to S$ and X by $Y^{ét} \times_S X$. The group G is then replaced by its connected component, whose Hopf algebra we denote by R. If $r = \dim_k R$, we have an embedding $G \subseteq {}_{F^n}GL(R) = {}_{F^n}GL_r \times_{\mathbb{F}_p} k$, for a suitable integer n. Let Y' be the ${}_{F^n}GL_r$-torsor over S induced by this embedding. Since $Y \times_S X = \Sigma_Y^{\boldsymbol{\nu}}$, a fortiori $Y' \times_S X = \Sigma_{Y'}^{\boldsymbol{\nu}}$. We can now conclude by Theorem 2.11. □

3. Essentially finite morphisms

In this section, k is a perfect field of characteristic $p > 0$. When S is a connected and reduced scheme, proper over k, Antei and Emsalem [1] have introduced another class of finite flat morphisms $X \to S$ that can be dominated by a finite torsor. Their construction is based on the tannakian approach to Nori's fundamental group scheme ([11], Chapter I).

Definition 3.1 (Nori [11]). Let S be a connected, reduced, proper k-scheme.

1) A vector bundle $\mathcal{V}$ on S is *finite* if there exist polynomials $f(t) \neq g(t)$ in $\mathbb{N}[t]$ such that $f(\mathcal{V}) = g(\mathcal{V})$.
2) Let $SS(S)$ be the category of semistable vector bundles on S. The category $EF(S)$ of *essentially finite* vector bundles on S is the full subcategory of $SS(S)$ whose objects are sub-quotients of finite bundles. In other words, a vector bundle $\mathcal{E}$ is essentially finite if there exists a finite bundle $\mathcal{V}$ and sub-bundles $\mathcal{V}'' \subset \mathcal{V}' \subseteq \mathcal{V}$ such that $\mathcal{E} \simeq \mathcal{V}'/\mathcal{V}''$.

Of course, Definition 3.1.2 relies on the fact that every finite vector bundle is semistable ([11], Corollary I.3.1).

If S has a rational point $s \in S(k)$, the fibre functor $\mathcal{E} \mapsto \mathcal{E}_s$ from $EF(S)$ to k-vector spaces makes $EF(S)$ into a neutral tannakian category ([11], § I.3). It is

thus equivalent to the category of representations of an affine group scheme of finite type $\boldsymbol{\pi}(S/k;s)$, the *fundamental group scheme* of S. The crucial result is then:

Proposition 3.2 (Nori [11], I.3.10). *If $\mathcal{E}$ is any essentially finite vector bundle, the representation $\boldsymbol{\pi}(S/k;s) \to GL(\mathcal{E}_s)$ factors through a finite quotient of $\boldsymbol{\pi}(S/k;s)$.*

It follows from this that $\boldsymbol{\pi}(S/k;s)$ is a profinite group scheme.

Definition 3.3 (Antei-Emsalem [1]). Let S be a connected, reduced, proper k-scheme. A finite flat morphism $\pi : X \to S$ is *essentially finite* if the vector bundle $\pi_*\mathcal{O}_X$ is essentially finite.

Proposition 3.4 (Antei-Emsalem [1], 3.2). *Let S be a connected, reduced, proper k-scheme with a rational point $s \in S(k)$. Let $\pi : X \to S$ be an essentially finite morphism. Assume that $H^0(S, \pi_*\mathcal{O}_X) = k$ and that there exists a point $x \in X(k)$ above s. Then X is dominated by a torsor under a finite k-group scheme.*

As a matter of fact, the main result of [1] is much more precise: it describes the actual "Galois group" of X/S as the quotient of $\boldsymbol{\pi}(S/k;s)$ determined by $\pi_*\mathcal{O}_X$, as in Proposition 3.2.

Theorem 3.5. *Let S be a connected, reduced, proper k-scheme, $\pi : X \to S$ a finite flat morphism.*

1) *If X is F-constant, then π is essentially finite.*
2) *If π is essentially finite and $H^0(X, \mathcal{O}_X)$ is an étale k-algebra, then X is F-constant over S.*

Proof. 1) If X is F-constant, by Theorem 2.11 there is a torsor Y/S under a finite flat group scheme such that the pullback to Y of $\pi_*\mathcal{O}_X$ becomes constant as a sheaf of $\mathcal{O}_Y$-algebras and therefore as an $\mathcal{O}_Y$-module. Hence $\pi_*\mathcal{O}_X$ is essentially finite by [11], Proposition I.3.8.

2) Replacing k by a finite extension and X by a connected component, we may assume that the hypotheses of Proposition 3.4 are satisfied. Then X is dominated by a torsor $Y \to S$ under a finite k-group scheme G. Since Y is a torsor over X under a subgroup $H \subseteq G$ we have that $Y \times_S X \cong (G/H)_Y$. Then X is F-constant by Corollary 2.12. □

Remark 3.6. The condition on $H^0(X, \mathcal{O}_X)$ in Proposition 3.4 ensures not only that X is connected but also reduced (in the sense of covers, cf. [11] Definition II.3). Specifically, it guarantees that the action of the Galois group G on the fibre X_s is transitive ([1] Lemma 3.18). As a consequence $X_s \cong G/G_x$, where G_x is the stabilizer at x. In particular it implies that X is fdh. Hence this global condition in Antei-Emsalem's construction translates into a local one in ours.

We would like now to address the apparent inconsistency between the F-constance condition, requiring that a pullback of $\pi_*\mathcal{O}_X$ trivializes as a sheaf of algebras, and essential finiteness, requiring only a trivialization as a sheaf of modules. This becomes even more glaring if we recall the following fact, whose proof inspired Remark 2.6 above.

Proposition 3.7 (Mehta-Subramanian [8], § 2). *A vector bundle $\mathcal{E}$ on a k-scheme S trivializes over a torsor under a finite local k-group scheme if and only if $(F_S^n)^*\,\mathcal{E}$ is the trivial bundle for some integer $n > 0$ (such a bundle is called F-*finite*).*

Let $\pi : X \to S$ be an essentially finite morphism and let $f : Y \to S$ be a torsor under a finite group scheme trivializing the vector bundle $\pi_*\mathcal{O}_X$. We can factor the finite cover $Y \to S' \to S$ into a radicial torsor followed by an étale one. Then $\pi' : X' = S' \times_S X \to S'$ is essentially finite and the bundle $\pi'_*\mathcal{O}_{X'}$ trivializes over a torsor under a finite local group scheme, namely $Y \to S'$ (we could call such a morphism F-*finite*). Summarizing:

- $\pi : X \to S$ is essentially finite $\Longleftrightarrow$ $\exists$ an integer $n > 0$ and a finite étale cover $S' \to S$ such that $(F_{S'}^n)^*\,\pi'_*\mathcal{O}_{X'}$ is a free $\mathcal{O}_{S'}$-module.
- $\pi : X \to S$ is F-constant $\Longleftrightarrow$ $\exists$ an integer $n > 0$ and a finite étale cover $S' \to S$ such that $(F_{S'}^n)^*\,\pi'_*\mathcal{O}_{X'} \cong \mathcal{O}_{S'}[t_1, \ldots, t_r]/(t_1^{p^{\nu_1}}, \ldots, t_r^{p^{\nu_r}})$.

Yet, according to Theorem 3.5, on a proper reduced scheme, the weaker first condition is equivalent to the second. To clarify this point we shall see that on an arbitrary scheme of characteristic p, the F-constance of a morphism is equivalent to the trivialization of a suitable subquotient of the direct image of the structure sheaf. Therefore, in cases where it is possible to apply the tannakian formalism, the two notions coincide. We will only treat the simplest situation, the general case being conceptually similar but notationally messy.

Let $\pi : X \to S$ be an fdh morphism such that $X_p = S$. Then the relative Frobenius $F_{X/S}$

$$\begin{tikzcd} X \arrow[r, "F_{X/S}"] \arrow[dr, "\pi"'] & X^{(1/S)} \arrow[r] \arrow[d, "\pi^{(1)}"] & X \arrow[d, "\pi"] \\ & S \arrow[r, "F_S"] & S \end{tikzcd}$$

factors through a section $\varepsilon : S \to X^{(1/S)}$ of $\pi^{(1)}$. Let $\omega_{\pi^{(1)}} = \varepsilon^*\Omega^1_{X^{(1/S)}/S}$.

Proposition 3.8. *Let S be a scheme of characteristic $p > 0$ and $\pi : X \to S$ an fdh morphism such that $X_p = S$. Then π is F-constant if and only if $\omega_{\pi^{(1)}}$ is a free $\mathcal{O}_S$-module.*

Proof. If π is F-constant, $\Omega^1_{X^{(1/S)}/S}$ is free and so does $\omega_{\pi^{(p)}}$. Conversely, let $\mathcal{I} \subset \pi_*^{(1)}\mathcal{O}_{X^{(1/S)}}$ be the ideal defined by the closed embedding ε. We have a canonical surjection from the conormal bundle of ε to $\omega_{\pi^{(1)}}$:

$$\mathcal{I}/\mathcal{I}^2 \longrightarrow \omega_{\pi^{(1)}} \longrightarrow 0. \tag{4}$$

If $\omega_{\pi^{(1)}}$ is free, any lifting to $\mathcal{I}$ of a basis of $\omega_{\pi^{(1)}}$ defines a surjection of algebras

$$\vartheta : \mathcal{O}_S[t_1, \ldots, t_r] = \mathrm{Sym}\,(\omega_{\pi^{(1)}}) \longrightarrow \pi_*^{(1)}\mathcal{O}_{X^{(1/S)}}. \tag{5}$$

Any section $z \in \mathcal{I}$ satisfies $z^p = 0$. Therefore ϑ factors through a surjection:

$$\mathcal{O}_S[t_1, \ldots, t_r]/(t_1^p, \ldots, t_r^p) \longrightarrow \pi_*^{(1)} \mathcal{O}_{X^{(p/S)}}.$$

Since π is fdh, this is a nontrivial map between twists and is thus an isomorphism. $\square$

Example 3.9. In the situation of Example 2.7, we have $\omega_{\pi^{(1)}} = \mathcal{O}(-2)$. This shows again that the k-linear Frobenius $\pi : \mathbb{P}^1_k \to \mathbb{P}^1_k$ is not F-constant.

If S is reduced and proper over a perfect field, from surjections (4) and (5) above we see that $\omega_{\pi^{(1)}}$ generates the same tannakian subcategory of $EF(S)$ as $F_S^* \pi_* \mathcal{O}_X = \pi_*^{(1)} \mathcal{O}_{X^{(1/S)}}$. Therefore, if the latter is the trivial bundle, so is $\omega_{\pi^{(1)}}$ and thus $\pi : X \to S$ if F-constant.

4. Fundamental group schemes

Notations and conventions: Let B be a fixed base scheme. In this section all schemes are assumed to be B-schemes of finite type. We fix a separated flat B-scheme S with a marked rational point $s \in S(B)$.

Definition 4.1 (Nori [11]). Let $\mathfrak{C}(S/B; s)$ be the category whose objects are triples (X, G, x) consisting of a finite flat B-group scheme G, a G-torsor $f : X \to S$ and a rational point $x \in X(B)$ such that $f(x) = s$. A morphism $(X', G', x') \to (X, G, x)$ in $\mathfrak{C}(S/B; s)$ is the datum of an S-morphism $\alpha : X' \to X$ such that $\alpha(x') = x$ and a B-group scheme homomorphism $\beta : G' \to G$ making the following diagram, where the horizontal arrows are the group actions, commute:

$$\begin{array}{ccc} G' \times X' & \xrightarrow{\mu'} & X' \\ {\scriptstyle \beta \times \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G \times X & \xrightarrow{\mu} & X. \end{array}$$

Definition 4.2 (Nori [11]). A scheme S *has a fundamental group scheme* $\boldsymbol{\pi}(S/B; s)$ if the category $\mathrm{Pro}(\mathfrak{C}(S/B; s))$ has an initial object $(\tilde{S}, \boldsymbol{\pi}(S/B; s), \tilde{s})$.

Nori [11], Proposition II.9 (resp. Gasbarri [4], § 2) have shown that if S is reduced and B is the spectrum of a field (resp. a Dedekind scheme) then S has a fundamental group scheme. If S is reduced and proper over a perfect field, its fundamental group scheme in the sense of Definition 4.2 is identical to the tannakian group considered in § 3. If B is a Dedekind scheme, S has a fundamental group scheme and X/S is a torsor under a finite flat group scheme, then X admits a fundamental group scheme ([3], Theorem 3).

All of the above results are proved using the following criterion:

Proposition 4.3 (Nori [11], Proposition II.1, Gasbarri [4], 2.1). *A flat B-scheme S has a fundamental group scheme if and only if $\mathfrak{C}(S/B;s)$ admits finite fibered products, i.e., for any $(Y,G,y) \in \mathfrak{C}(S/B;s)$ and any pair of morphisms $\alpha_i : (Y_i,G_i,y_i) \to (Y,G,y)$ in $\mathfrak{C}(S/B;s)$, the triple $(Y_1 \times_Y Y_2, G_1 \times_G G_2, (y_1,y_2))$ belongs to $\mathfrak{C}(S/B;s)$.*

Remark 4.4 (Nori [11], Lemma II.1). For any given torsor $(Y,G,y) \in \mathfrak{C}(S/B;s)$ and any pair of morphisms $\alpha_i : (Y_i,G_i,y_i) \to (Y,G,y)$ in $\mathfrak{C}(S/B;s)$, the triple $Y_1 \times_Y Y_2$ is a $G_1 \times_G G_2$-torsor over a closed subscheme of S containing s. So it is a torsor over S if and only if it is faithfully flat over S.

Theorem 4.5. *Let B be a Dedekind scheme and η its generic point. Let (S,b) a flat pointed B-scheme which has a fundamental group scheme. Let $\pi : X \to S$ be a finite flat B-morphism, equipped with a point $x \in X(B)$ such that $\pi(x) = s$. If the generic fibre $\pi_\eta : X_\eta \to S_\eta$ is étale or F-constant, then also (X,x) has a fundamental group scheme.*

Proof. We will apply the criterion above. Let thus (Y_i,G_i,y_i), for $i = 0,1,2$, be three torsors in $\mathfrak{C}(X,x)$ and $\alpha_i : (Y_i,H_i,y_i) \to (Y_0,H_0,y_0)$, for $i = 1,2$, be two morphisms in $\mathfrak{C}(X,x)$. We have to show that the triple $(Y_1 \times_{Y_0} Y_2, H_1 \times_{H_0} H_2, (y_1,y_2))$ belongs to $\mathfrak{C}(X,x)$.

In light of Remark 4.4, it suffices to prove this when B is the spectrum of a field. Indeed, since X is the closure of its generic fibre X_η, by [5] IV.2.8.5, the case of a general Dedekind scheme follows by taking the scheme theoretic closure of the objects defined over η: the proof of [4], Proposition 2.1 goes through verbatim.

Let thus B be the spectrum of a field. By Grothendieck's Galois theory [7], chap. V (in characteristic 0) or by Theorem 2.11 (in positive characteristic) we can dominate X by a finite torsor:

$$\begin{tikzcd} X' \arrow[r, "f"] \arrow[dr] & X \arrow[d, "\pi"] \\ & S. \end{tikzcd}$$

Pullback via f provides us with the H_i-torsors $Y_i' = X' \times_X Y_i$. Since X'/S is a finite torsor, by [3] Theorem 3, it has a fundamental group scheme. Hence $Y_1' \times_{Y_0'} Y_2'$ is an $H_1 \times_{H_0} H_2$-torsor over X'. In particular, it is faithfully flat over X'. Also f is faithfully flat: by descent we get that $Y_1 \times_{Y_0} Y_2$ is faithfully flat over X, and we conclude by Remark 4.4. □

Having established that X has a fundamental group scheme, by functoriality from $\pi : (X,x) \to (S,s)$ we obtain a group homomorphism $\boldsymbol{\pi}(X/B,x) \to \boldsymbol{\pi}(S/B,s)$. If X is a torsor over S, this is an embedding of $\boldsymbol{\pi}(X/B,x)$ as a closed normal subgroup of $\boldsymbol{\pi}(S/B,s)$ ([3], Theorem 4). More generally, we show below that it is an injection if π admits a Galois closure. In order not to have to spell

out this condition every time, we introduce the following definition, which should not be taken too seriously.

Definition 4.6. A morphism $\pi : (X, x) \to (S, s)$ of pointed B-schemes will be called *submissive* if it is finite, flat and it can be dominated, in the sense of Definition 2.1, by a torsor under a finite flat B-group scheme with a marked B-point lying over x.

Proposition 4.7. *Let $\pi : (X, x) \to (S, s)$ be a finite flat morphism of pointed B-schemes. Then π is submissive in the following cases:*

1. *π is étale;*
2. *π is F-constant and B is the spectrum of a perfect field.*

Proof. The domination property is guaranteed for an étale cover by Grothendieck's Galois theory and by Theorem 2.11 for an F-constant morphism (even for imperfect fields). The issue is to deal with base points. Let X'/S be a torsor under a finite flat group scheme G dominating X and denote G' the group of X'/X. It may happen that X' has no integral points over x, but only acquires one over a finite étale (since B is perfect) extension $\widetilde{B}$ of B. In this case, denoting $\widetilde{T}$ the base change of a B-scheme T, we may replace G and G' by the Weil restrictions $\Re_{\widetilde{B}/B}(\widetilde{G})$ and $\Re_{\widetilde{B}/B}(\widetilde{G'})$ and X' by $\Re_{\widetilde{S}/S}(\widetilde{X'}) = \Re_{\widetilde{X}/X}(\widetilde{X'})$. □

Remark 4.8. The perfectness assumption is needed in the proof because Weil restriction only behaves nicely with respect to étale morphisms. The reason to invoke Weil restriction, instead of descent theory, is the nasty behaviour of fundamental group schemes under base change. If $\widetilde{B}/B$ is a faithfully flat extension, functoriality yields a morphism $\boldsymbol{\pi}(\widetilde{S}/\widetilde{B}) \to \boldsymbol{\pi}(S/B) \times_B \widetilde{B}$, but this is by no means an isomorphism: see [8], § 3 for a counterexample with S an integral projective curve and B and $\widetilde{B}$ algebraically closed fields. A counterexample with S a smooth curve has been given by Pauly in [12].

Theorem 4.9. *Let B be a Dedekind scheme, (S, b) and (X, x) flat pointed B-schemes admitting a fundamental group scheme. Let $\pi : X \to S$ be a submissive B-morphism with $\pi(x) = s$. Then π induces a closed immersion $\boldsymbol{\pi}(X/B, x) \to \boldsymbol{\pi}(S/B, s)$ of fundamental group schemes.*

Proof. Let X'/S be a marked torsor under a finite flat group scheme G dominating X and denote G' the group of X'/X.

Any quotient H of $\boldsymbol{\pi}(X/B, x)$ corresponds to a marked H-torsor (Y, y) over (X, x). Let $Y' = X' \times_X Y$. By [3], Theorem 2 (if $\dim B = 1$ one has to repeat the scheme-theoretic closure argument above) we can find a finite flat B-group scheme $\Phi = \Phi(G, H)$ and a scheme Z' which is a Φ-torsor over X' dominating Y'. Moreover, Φ is equipped with an action of G and Z' is a $\Phi \rtimes G$-torsor over S. It follows from

this that Z' is a $\Phi \rtimes G'$-torsor over X.

$$\begin{tikzcd}
Z' \arrow[d] \arrow[dr] & \\
Y' \arrow[r] \arrow[d, "H"'] & Y \arrow[d, "H"] \\
X' \arrow[r, "G'"] \arrow[dr, "G"'] & X \arrow[d] \\
& S.
\end{tikzcd}$$

In other words, any quotient H of $\boldsymbol{\pi}(X/B, x)$ fits in a diagram:

$$\begin{tikzcd}
\boldsymbol{\pi}(X/B, x) \arrow[r] \arrow[d] & \boldsymbol{\pi}(S/B, s) \arrow[d] \\
\Phi \arrow[r] \arrow[d] & \Phi \rtimes G' \\
H. &
\end{tikzcd}$$

Since $\boldsymbol{\pi}(X/B, x)$ is the projective limit of such H's and the bottom horizontal arrow is a closed immersion, the top one is a monomorphism, and it is a closed immersion by [6] IV.8.10.5. $\square$

The previous theorem suggests that submissive morphisms play, for the fundamental group scheme, the role that covers have for the étale fundamental group. The remainder of this section is devoted to making this hunch more precise.

Definition 4.10. Let (S, s) be a pointed B-scheme. Let $\mathfrak{Sub}(S/B; s)$ be the category whose objects are pairs (X, x) consisting of a submissive B-scheme $\pi : X \to S$ and a point $x \in X(B)$ such that $\pi(x) = s$. A morphism $(X', x') \to (X, x)$ is a morphism of pointed (S, s)-schemes.

The forgetful functor $(X, G, x) \mapsto (X, x)$ embeds $\mathfrak{C}(S/B; s)$ into $\mathfrak{Sub}(S/B; s)$ [though not as a full subcategory: if B is a perfect field k of characteristic $p > 0$, A is a k-algebra and $a \in A^\times$, then $X = \operatorname{Spec} A[x]/(x^p - a)$ can be given both an $\boldsymbol{\alpha}_p$ and a $\boldsymbol{\mu}_p$-torsor structure over $S = \operatorname{Spec} A$; as there are no nonzero morphisms over k between these group schemes, the identity on X does not come from a morphism $(X, \boldsymbol{\alpha}_p) \to (X, \boldsymbol{\mu}_p)$].

Proposition 4.11. *Let (S, s) be a flat pointed B-scheme. Finite fibred products exist in the category $\mathfrak{C}(S/B; s)$ if and only if they exist in $\mathfrak{Sub}(S/B; s)$.*

Proof. The if part follows from Remark 4.4: given three torsors $(Y_i, G_i, y_i) \in \mathfrak{C}(S/B; s)$, if $Y_1 \times_{Y_0} Y_2$ exists in $\mathfrak{Sub}(S/B; s)$ it is in particular flat over S, and therefore a $G_1 \times_{G_0} G_2$-torsor over the whole of S.

For the converse, let (X_i, x_i) be three submissive schemes over (S, s) and let $(Y_i, G_i, y_i) \in \mathfrak{C}(S/B; s)$ dominate (X_i, x_i). Denote by H_i the group of Y_i/X_i. Let us furthermore assume that these schemes fit in a diagram in $\mathfrak{Sub}(S/B; s)$

$$\begin{tikzcd} Y_1 \arrow[r, "\alpha_1"] \arrow[d] & Y_0 \arrow[d] & Y_2 \arrow[l, "\alpha_2"'] \arrow[d] \\ X_1 \arrow[r] & X_0 & X_2 \arrow[l] \end{tikzcd}$$

where $(\alpha_i, \beta_i) : (Y_i, G_i, y_i) \to (Y_0, G_0, y_0)$ are in $\mathfrak{C}(S/B; s)$: that such a construction is possible, will be proved in the following Lemma 4.12.
If finite fibred products exist in $\mathfrak{C}(S/B; s)$, then $Y_1 \times_{Y_0} Y_2$ is a $G_1 \times_{G_0} G_2$-torsor over S. One checks immediately that the following diagram is cartesian:

$$\begin{tikzcd} (H_1 \times_{H_0} H_2) \times_B (Y_1 \times_{Y_0} Y_2) \arrow[r, "(\mu,id)"] \arrow[d, "(\iota,id)"'] & (Y_1 \times_{Y_0} Y_2) \times_{X_1 \times_{X_0} X_2} (Y_1 \times_{Y_0} Y_2) \arrow[d, "(id,id)"] \\ (G_1 \times_{G_0} G_2) \times_B (Y_1 \times_{Y_0} Y_2) \arrow[r, "(\mu,id)"] & (Y_1 \times_{Y_0} Y_2) \times_S (Y_1 \times_{Y_0} Y_2) \end{tikzcd}$$

where μ is the group action and $\iota : H_1 \times_{H_0} H_2 \to G_1 \times_{G_0} G_2$ the inclusion. Since the bottom arrow is an isomorphism, so is the top one. Hence $Y_1 \times_{Y_0} Y_2$ is an $H_1 \times_{H_0} H_2$-torsor over $X_1 \times_{X_0} X_2$. Therefore the latter is finite and flat over S and dominated by a torsor. □

Lemma 4.12. *Let $f : X' \to X$ be a morphism of submissive S-schemes, Y a finite torsor over S dominating X. Then there exists a finite torsor Y'/S dominating both X' and Y.*

Proof. Let G be the group of Y/S. By assumption, there exists a scheme Z which is a torsor over S under a finite flat B-group scheme G' and a torsor over X' under a subgroup $H' \subseteq G'$. Put $Y' = Y \times_S Z$: by construction, it is a $G \times_B G'$-torsor over S, a G'-torsor over Y and a G-torsor over Z. Therefore, it is a $G \times_B H'$-torsor over X'.

$$\begin{tikzcd} Y' \arrow[r, "G"] \arrow[dd, "G'"'] \arrow[dr, dotted] & Z \arrow[d, "H'"] \\ & X' \arrow[d] \\ Y \arrow[r, "G"] & S. \end{tikzcd}$$

□

Theorem 4.13. *A flat B-scheme S has a fundamental group scheme if and only if the category $\mathfrak{Sub}(S/B; s)$ admits finite fibered products. The universal cover is the initial object in* $\mathrm{Pro}\,(\mathfrak{Sub}(S/B; s))$.

Proof. Nori's proof that $\mathfrak{C}(S/B;s)$ is filtered if and only if it has finite fibered products ([11], Prop. II.1) is formal and can be repeated verbatim for $\mathfrak{Sub}(S/B;s)$. By Proposition 4.11, the projective limit $(\hat{S},\hat{s})$ of $\mathfrak{Sub}(S/B;s)$ exists if and only if the universal cover $(\tilde{S},\boldsymbol{\pi}(S/B;s),\tilde{s})$, which is the projective limit of $\mathfrak{C}(S/B;s)$, exists. Since $\mathfrak{C}(S/B;s)$ is a subcategory of $\mathfrak{Sub}(S/B;s)$, there is a canonical morphism $\hat{S}\to\tilde{S}$ in $\mathrm{Pro}\,(\mathfrak{Sub}(S/B;s))$. On the other hand, any object in $\mathfrak{Sub}(S/B;s)$ receives a morphism from $\tilde{S}$ and, by Lemma 4.12, we can build a compatible system of such maps. Therefore also $\tilde{S}$ is a projective limit in $\mathfrak{Sub}(S/B;s)$, and we conclude by uniqueness of the limit. □

Remark 4.14. When B is the spectrum of a perfect field of positive characteristic, by Proposition 4.7 the category $\mathfrak{Sub}(S/B;s)$ coincides with the category of pointed F-constant S-schemes. Let $\mathfrak{FDH}(S,s)$ be the category of pointed fdh S-schemes; it contains $\mathfrak{Sub}(S/B;s)$ as a full subcategory. Then $\mathfrak{FDH}(S,s)$ has finite fibred products if and only if either $\mathfrak{Sub}(S/B;s)$ or $\mathfrak{C}(S/B;s)$ do. This is a simple consequence of Remark 4.4 (existence of products is a local problem on the base) and Proposition 2.5 (locally on the base every fdh morphism is submissive).

Remark 4.15. It would be interesting to have a characterization for submissive morphisms of arithmetic schemes. The differentially homogeneous condition is too strong: if $\Omega^1_{X/S}$ is locally free, it vanishes on the generic fibre (a submissive morphism in characteristic zero is étale), hence it is zero altogether. A necessary condition is that the fibres should be submissive (i.e., F-constant or étale).

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Hasse Principle and Cohomology of Groups

Jean-Claude Douai

Abstract. In a recent article, Colliot-Thélène, Gille and Parimala have considered fields K of cohomological dimension 2, of geometric type, analogous to totally imaginary numbers fields. One standard example is the field $\mathbb{C}((x,y))$. Using previous results of Borovoi and the author, they compute the cohomology of K in degree one and two with coefficients in a semi-simple K-group. The aim of our paper is to extend their results to fields K of cohomological dimension 2 that are not of geometric type but satisfy the Hasse principle; by Efrat, extensions of PAC fields of relative transcendence degree 1 are examples of such fields. For such fields K, we show that it is possible to calculate the non abelian cohomology in degree two with coefficients in a semi-simple K-group (the cohomology in degree one is calculated by Serre's conjecture about the fields of cohomological dimension 2). We also show, in the case that K is of transcendence degree 1 over a PAC field, that if the group is semi-simple and a direct factor of a K-rational variety, then its Shafarevitch group is trivial, thus getting an analog of a result of Sansuc for number fields. For the field $\mathbb{C}((x,y))$, the analogous result was established by Borovoi-Kunyavskii.

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1. History

Let k be a finite field, X be a smooth projective connected curve defined over k and $K = k(X)$ be its function field. The Hasse principle is valid for the function field K, that is, we have the following exact sequence where $P = P(K/k)$ is the set of all non trivial valuations on K which are trivial on k and for each $v \in P$, K_v is the completion of K for the place v.

$$0 \to \mathrm{Br}(K) \to \bigoplus_{v \in P} \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{1}$$

In fact, the exact sequence (1) corresponds to the special case where

$$\mathrm{Br}(X) = 0$$

of this exact sequence:

$$0 \to \mathrm{Br}(X) \to \mathrm{Br}(K) \to \bigoplus_{v \in P} \mathrm{Br}(K_v) \to \mathbb{Q}/\mathbb{Z} \to 0 \tag{2}$$

which itself a special case of the following theorem of Grothendieck (where P is then the set of all closed points of X).

Proposition 1 (Grothendieck [Gr]). *Let X be a noetherian, regular, integral prescheme of dimension 1, η be its generic point, $X^{(1)}$ be the set of closed points of X. If, for each point $x \in X^{(1)}$, $k(x)$ is perfect, we have an infinite exact sequence*

$$\begin{aligned} 0 \to H^2(X, G_m) \to H^2(\eta, G_m) \to & \coprod_{x \in X^{(1)}} H^1(x, \mathbb{Q}/\mathbb{Z}) \\ & \to H^3(X, G_m) \to H^3(\eta, G_m) \to \cdots \end{aligned} \tag{3}$$

Application: Let X be a smooth projective connected curve over a finite field k. We have the spectral sequence

$$H^p(k, H^q_{et}(X \otimes_k \overline{k}, G_m)) \Longrightarrow H^*_{\mathrm{et}}(X, G_m)$$

which provides the following exact sequence (note that $\mathrm{Br}(\overline{X}) = H^2(\overline{X}, G_m) = 0$)

$$\begin{array}{ccccc} \mathrm{Br}(k) \to & \mathrm{Ker}\{H^2(X, G_m) \to H^2(\overline{X}, G_m)\} & \to H^1(k, \mathrm{Pic}(\overline{X})) & \to H^3(k, G_m) \\ ||(k\,\text{finite}) & || & & ||(k\,\text{finite}) \\ 0 & H^2(X, G_m) & & 0 \end{array} \tag{4}$$

and the isomorphism

$$\mathrm{Br}(X) = H^2(X, G_m) \simeq H^1(k, \mathrm{Pic}(\overline{X})).$$

We can calculate $H^1(k, \mathrm{Pic}(\overline{X}))$ thanks to the exact sequence

$$0 \to \mathrm{Pic}^0(\overline{X}) \to \mathrm{Pic}(X) \to \mathbb{Z} \to 0.$$

We obtain

$$H^1(k, \mathrm{Pic}^0(\overline{X})) \twoheadrightarrow H^1(k, \mathrm{Pic}(\overline{X})) \to H^1(k, \mathbb{Z}) = 0.$$

As k is finite, we have $H^1(k, \mathrm{Pic}^0(\overline{X})) = 0$ by Lang's theorem, and so

$$H^1(k, \mathrm{Pic}(\overline{X})) = \mathrm{Br}(X) = 0.$$

The spectral sequence also gives

$$\begin{array}{ccccc} H^3(X, G_m) \simeq H^2(k, \mathrm{Pic}(\overline{X})) \simeq H^2(k, \mathbb{Z}) \simeq H^1(k, \mathbb{Q}/\mathbb{Z}) & \simeq & \overset{\vee}{\mathrm{Gal}}(\overline{k}/k) \\ & (k\,\text{finite}) & \wr\| \\ & & \mathbb{Q}/\mathbb{Z} \end{array}$$

where $\overset{\vee}{\mathrm{Gal}}(\overline{k}/k) = \mathrm{Hom}(\widehat{\mathbb{Z}}, \mathbb{Q}/\mathbb{Z})$ is the dual of $\mathrm{Gal}(\overline{k}/k)$. This yields the sequence

$$0 \to \mathrm{Br}(K) \to \bigoplus_{v \in P} \mathrm{Br}(K_v) \to \overset{\vee}{\mathrm{Gal}}(\overline{k}/k) \to 0. \tag{5}$$

2. Passing to infinite base fields

2.1. Quasi-finite fields

Recall that a field k is said to be *quasi-finite* if $\mathrm{Gal}(\overline{k}/k) \simeq \widehat{\mathbb{Z}}$. We consider the following two basic examples.

2.1.1. *The quasi-finite fields of type* (a) *considered by Rim and Whaples.* These are the fields k of non-zero characteristic which are algebraic over the prime subfield k_0 and have a finite p-primary degree for all prime p, *i.e.*, $[k : k_0] = \prod_p p^{\nu_p}, \nu_p < \infty$.

Then we have always $\mathrm{Br}(X) = 0$ [Do2].

2.1.2. *The field* $\mathbb{C}((t))$**.** In this case, $\mathrm{Br}(X) = H^2(X, G_m)$ is not trivial; its calculation depends on the reduction modulo (t) of the curve X (*cf.* [Do1]): we have

$$\mathrm{Br}(X)_n \simeq (\mathbb{Z}/n\mathbb{Z})^{2g-\varepsilon}$$

where g is the genus of X, ε the Ogg integer associated with the reduction of X modulo t and n any integer ≥ 1.

2.2. PAC Fields

Recall that a field k is called PAC (Pseudo Algebraically Closed) if every geometrically irreducible affine variety defined over k has a k-rational point.

Examples.

(a) Any infinite extension of a finite field is a PAC field (for instance the quasi-finite fields of type (a) considered by Rim and Whaples, *cf.* [Do2])

(b) The field $\mathbb{Q}^{tr}(\sqrt{-1})$, where $\mathbb{Q}^{tr}$ is the field of all totally real algebraic numbers, is a PAC field.

(c) For almost all n-tuples $(\sigma_1, \ldots, \sigma_n)$ of automorphisms of $\overline{\mathbb{Q}}$, the fixed field of $\sigma_1, \ldots, \sigma_n$ in $\overline{\mathbb{Q}}$ is a PAC field. Here "almost all" should be understood as "off a subset of measure 0" for the canonical Haar measure on $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})^n$.

If a perfect field is PAC, then it is infinite, non real and all its henselizations with respect to non-trivial valuations are algebraically closed. Somehow PAC fields do not carry any "essential" arithmetic objects. Furthermore if k is PAC, then $\mathrm{cd}(k) := \mathrm{cd}(\mathrm{Gal}(k_s/k)) \leq 1$.

Concerning the Brauer group $\mathrm{Br}(K)$ of the function field $K = k(X)$ over a PAC field, we have this result of Efrat.

Theorem 1 (Efrat [Ef]). *Let K be a function field in one variable over a perfect PAC field k. Then there is a natural exact sequence*

$$0 \to \mathrm{Br}(K) \to \bigoplus_{v \in P} \mathrm{Br}(K_v) \to \mathrm{Gal}(k_s/k)^{\vee} \to 0$$

where, if $\mathrm{char}(k) = q > 0$*, the* 3 *terms should be replaced by their prime-to-q part.*

Proof. As k is PAC, we have $H^1(k, \mathrm{Pic}(\overline{X})) = H^1(k, \mathrm{Pic}^0(\overline{X})) = 0$ and $\mathrm{cd}(k) \leq 1$ which implies $\mathrm{Br}(k) = 0$ and $H^3(k, G_m) = 0$. The exact sequence

$$\mathrm{Br}(k) \to \mathrm{Ker}\{H^2(X, G_m) \to H^2(\overline{X}, G_m)\} \to H^1(k, \mathrm{Pic}(\overline{X})) \to H(k, G_m) \qquad (6)$$

gives $\mathrm{Br}(X) = 0$ and we recover in this case the exact sequence (3) of Grothendieck. As in the case where k is finite, we obtain

$$H^3(X, G_m) \simeq H^2(k, \mathrm{Pic}(\overline{X})) \simeq H^2(k, Z) \simeq \widehat{\mathrm{Gal}(k_s/k)} \,. \qquad \square$$

Example. If k is a quasi-finite field of type (a) considered in [Do2], we find again the fact that

$$\mathrm{Br}(K) \to \bigoplus_{v \in P} \mathrm{Br}(K_v)$$

is injective. This fact is used in our 1986 article [Do2] (K satisfies condition (C) from there), where we show that if $\mathcal{L}$ is a "band" that is locally representable by a semi-simple simply connected group, then all classes of $H^2(K, \mathcal{L})$ are neutral. From there, we deduce the surjectivity of $\delta^1 : H^1(K, G) \to H^2(K, \mu)$ where μ is the kernel of $\widetilde{G} \to G$ and $\widetilde{G}$ is the universal covering of G.

3. Cohomology of groups

In this section we assume that K is a function field in one variable over a perfect PAC field k.

3.1. Calculation of $H^1(K, \widetilde{G})$ with $\widetilde{G}$ a semi-simple simply connected K-group

We have $\mathrm{cd}(K) \leq 2$ and by Serre's conjecture, this implies that $H^1(K, \widetilde{G}) = 0$. When k has characteristic 0 and contains all roots of unity, Serre's conjecture has been established in [JP].

3.2. Calculation of $H^2(K, .)$

Theorem 2. *Let K be a function field in one variable over a perfect PAC field k, $\mathcal{L}$ be a K-band that is locally representable by a semi-simple simply connected group $\widetilde{G}$. Then all classes of $H^2(K, \mathcal{L})$ are neutral if $|Z(\widetilde{G})|$ is prime to the characteristic q of k.* (*That is, each "gerb" locally bound by a semi-simple simply connected group over K admits a section.*)

Proof. Denote by $\mathrm{Autext}(\widetilde{G})$ the outer automorphism group of $\widetilde{G}$ and by $\widetilde{G}_{ad}$ the adjoint group of $\widetilde{G}$. The band $\mathcal{L}$ is an element of $Z^1(K, \mathrm{Autext}(\widetilde{G}))$. The sequence

$$1 \to \widetilde{G}_{ad} \to \mathrm{Aut}(\widetilde{G}) \overset{\curvearrowleft}{\to} \mathrm{Autext}(\widetilde{G}) \to 1$$

is split and $\mathcal{L}$ defines a class $[\widetilde{G}_{\mathcal{L}}]$ in $H^1(K, \mathrm{Aut}(\widetilde{G}))$: $\mathcal{L}$ is representable by $\widetilde{G}_{\mathcal{L}}$.

By Demazure (Proposition 3.13 of [SGA-D]), $\widetilde{G}_{\mathcal{L}}$ is quasi-split semi-simple simply connected and admits a Killing pair $(\widetilde{B}, \widetilde{T})$ where $\widetilde{T} \simeq \prod_{K'/K} G_{m_{K'}}$ (with K' ranging over all finite extensions of K) is an induced torus. Then the maps

(i) $\mathrm{Br}(K') \to \bigoplus_{v \in P} \mathrm{Br}(K'_v) \pmod q$,

(ii) $H^2(K', \mu_n) \to \bigoplus_{v \in P} H^2(K'_v, \mu_n),\ (n, q) = 1$,

(iii) $H^2(K, Z(\widetilde{G}_{\mathcal{L}})) \to \bigoplus_{v \in P} H^2(K_v, Z(\widetilde{G}_{\mathcal{L}})),\ (|Z(\widetilde{G}_{\mathcal{L}}|, q) = 1)$,

are injective by Theorem 1. This is obvious for (i) and (ii).

Proof of (iii): From the definition above of $\widetilde{T}$, the application

$$H^2(K, \widetilde{T}) \longrightarrow \bigoplus_{v \in P} H^2(K_v, \widetilde{T})$$

is identified with the injective application

$$\mathrm{Br}(K') \hookrightarrow \bigoplus_{v \in P'} \mathrm{Br}(K'_v)$$

(with P' the set of all non trivial valuations v on K' which are trivial on k). The image $\widetilde{T}_{ad}$ of $\widetilde{T}$ by the normal isogeny $G \longrightarrow \widetilde{G}_{ad}$ is also an induced torus (again by [SGA-D; Prop. 3.13]). Hence $H^1(K, \widetilde{T}_{ad}) = 0$ (resp. $H^1(K_v, \widetilde{T}_{ad}) = 0$ for all $v \in P$).

From this, we get the injectivity of the second vertical map in the diagram

$$\begin{tikzcd}
\overset{0}{\|} & & \\
H^1(K, \widetilde{T}_{ad}) \arrow[r] \arrow[d] & H^2(K, Z(\widetilde{G}_{\mathcal{L}})) \arrow[r] \arrow[d] & H^2(K, \widetilde{T}) \arrow[d, hook] \\
\bigoplus_{v \in P} H^1(K_v, \widetilde{T}_{ad}) \arrow[r] & \bigoplus_{v \in P} H^2(K_v, Z(\widetilde{G}_{\mathcal{L}})) \arrow[r] & \bigoplus_{v \in P} H^2(K_v, \widetilde{T}). \\
\underset{0}{\|} & &
\end{tikzcd}$$

Since $H^2(K, \tilde{G}_{\mathcal{L}})$ is a principal homogeneous space under $H^2(K, Z(\tilde{G}_{\mathcal{L}}))$, we see that $H^2(K, \tilde{G}_{\mathcal{L}}) \to \bigoplus_{v \in P} H^2(K_v, \tilde{G}_{\mathcal{L}})$ is also injective in the set-theoretic sense.

For each $v \in P$, K_v is a local field whose residue field is PAC, hence of cohomological dimension ≤ 1. Using Bruhat-Tits, we have showed [Do4; Cor. 2.6 and 2.8] that, if the residue field of K_v is of cohomological dimension ≤ 1 and if $\mathcal{L}$ is locally representable by a K_v-semi-simple simply connected group $\widetilde{G}_{\mathcal{L}}$, then each class of $H^2(K_v, \mathcal{L}) = H^2(K_v, \widetilde{G}_{\mathcal{L}})$ is neutral (we can see $\widetilde{G}$ and $\widetilde{G}_{\mathcal{L}}$ as objects of infinite dimension over the residue field of K_v).

In particular, for each $v \in P$, the map

$$(\delta^1)_v : H^1(K_v, \mathrm{Int}\ \widetilde{G}_{\mathcal{L}}) \simeq H^2(K_v, Z(\widetilde{G}_{\mathcal{L}}))$$

is a bijection (Proposition 3.2.6 (iii) of [Gir; Chap. IV], p. 255).

Index and exponent of central simple algebras over K_v coincide (if k is of characteristic 0, the field K_v is of type (sl) in the sense of Theorem 1.5 of [CGP]).

End of proof of Theorem 2: We will show that the sequence

$$0 = H^1(K, \widetilde{G}_{\mathcal{L}}) \to H^1(K, \mathrm{Int}\ \widetilde{G}_{\mathcal{L}}) \to H^2(K, \widetilde{G}_{\mathcal{L}}) \to 1$$

is exact, which will give the neutrality of each class of $H^2(K, \widetilde{G}_{\mathcal{L}})$ if $|Z(\widetilde{G}_{\mathcal{L}})|$ is prime with q.

The cohomological dimension of K is ≤ 2. For central simple algebras over K, index and exponent coincide. That follows from the injectivity of the map $\mathrm{Br}(K) \to \bigoplus_{v \in P} \mathrm{Br}(K_v)$ in theorem 1 of [Ef] together with the classical reduction to the prime degree exponent case: more precisely, as index and exponent coincide for central simple algebras over K_v ($v \in P$), the proof, written for number fields, of "Exponent= Index" in §5.4.4, p. 34 of [Ro], is still valid when K satisfies the "Hasse Principle" (in fact when $\mathrm{Br}(K) \to \bigoplus_{v \in P} \mathrm{Br}(K_v)$ is injective) and shows that, for central simple algebras over K, index and exponent also coincide.

Then we can apply Theorem 2.1 (a) of [CGP]: the boundary map

$$\delta^1 : H^1(K, \mathrm{Int}\ \widetilde{G}_{\mathcal{L}}) \simeq H^2(K, Z(\widetilde{G}_{\mathcal{L}}))$$

is a bijection. Then we can compare the exact sequence

$$\begin{tikzcd}
H^1(K, \widetilde{G}_{\mathcal{L}}) \arrow[r] & H^1(K, \mathrm{Int}\, \widetilde{G}_{\mathcal{L}}) \arrow[r] \arrow[dr, "\delta^1", "\simeq"'] & H^2(K, \tilde{G}_{\mathcal{L}}) \arrow[r] \arrow[d, "\simeq"] & 1 \\
& & H^2(K, Z(\widetilde{G}_{\mathcal{L}})) &
\end{tikzcd} \tag{7}$$

with the exact sequence

$$\begin{tikzcd}
H^1(K, \widetilde{G}_{\mathcal{L}}) \arrow[r] & H^1(K, \mathrm{Int}\, \widetilde{G}_{\mathcal{L}}) \arrow[r, "\simeq"] & H^2(K, \tilde{G}_{\mathcal{L}})' \arrow[r] \arrow[d, "\simeq"] & 1 \\
& & H^2(K, Z(\widetilde{G}_{\mathcal{L}})) &
\end{tikzcd} \tag{8}$$

(where the $'$ denotes the subset of neutral classes of $H^2(K, \widetilde{G}_{\mathcal{L}})$ given by Proposition 3.2.6 (iii) in [Gir; Chapter IV]) to conclude that each class of $H^2(K, \widetilde{G}_{\mathcal{L}})$ is neutral. One can also use the remark following Proposition 5.3 of [CGP] to conclude. $\square$

Remark 1 (*cf.* proof of Theorem 2.1 of [Do3]): We have the diagram

$$\begin{tikzcd}
H^1(K,\widetilde{G}_{\mathcal{L}}) \arrow[r] \arrow[d, equal] & H^1(K,\operatorname{Int}\widetilde{G}_{\mathcal{L}}) \arrow[r, "\delta^1", "\simeq"'] \arrow[d, hook] & H^2(K,\widetilde{G}_{\mathcal{L}}) \arrow[r, "a^{(2)}"] \arrow[d, "\wr"] & \varepsilon = [\operatorname{Tors}\widetilde{G}_{\mathcal{L}}] \\
0 \arrow[d, equal] & & H^2(K,Z(\widetilde{G}_{\mathcal{L}})) \arrow[d, hook] & \\
\bigoplus_{v\in P} H^1(K_v,\widetilde{G}_{\mathcal{L}}) \arrow[r] & \bigoplus_{v\in P} H^1(K_v,\operatorname{Int}\widetilde{G}_{\mathcal{L}}) \arrow[r, "\bigoplus_v (\delta^1)_v", "\simeq"'] & \bigoplus_{v\in P} H^2(K_v,Z(\widetilde{G}_{\mathcal{L}})) \arrow[r, "\bigoplus_v a_v^{(2)}"] & \bigoplus_{v\in P} \varepsilon_v.
\end{tikzcd}$$

From Proposition 4.2.8 (ii), p. 283, of [Gir; Chap. IV,§4]), we have the following: by the relation

$$H^2(K,Z(\widetilde{G}_{\mathcal{L}})) \xrightarrow{a^{(2)}} H^2(K,\widetilde{G}_{\mathcal{L}})$$

defined in 4.2.7.3 (7) p. 283 of *loc.citado*, $\alpha \in H^2(K,Z(\widetilde{G}_{\mathcal{L}}))$ corresponds to the class $\varepsilon = [\operatorname{Tors}\widetilde{G}_{\mathcal{L}}]$ called "unity" if and only if it belongs to the image of δ^1. In the diagram, each class of $\bigoplus_{v\in P} H^2(K_v,Z(\widetilde{G}_{\mathcal{L}}))$ corresponds to the unity class $\bigoplus_{v\in P}\varepsilon_v = \bigoplus_{v\in P}\left[\operatorname{Tors}(\widetilde{G}_{\mathcal{L}})_v\right]$ in $\bigoplus_{v\in P} H^2(K_v,\widetilde{G}_{\mathcal{L}})$ by the correspondence $\bigoplus_{v\in P} a_v^{(2)}$ in the second line.

Remark 2 (second proof of the surjectivity of δ^1 modulo Artin's conjecture in the case where k has characteristic 0, or, has positive characteristic and contains all roots of unity):

Under the assumption on k, by Lemma 2.3 of [JP], K is C_2. If we assume Artin's conjecture on K, then the exponent of every central simple K-algebra is equal to its index (the conjecture was proved by Artin for exponents of type 2^r). We can therefore directly use Theorem 2.1 of [CGP] which establishes the surjectivity of δ^1 and either conclude as in Theorem 2 or by the remark following Proposition 5.3 of [CGP] to prove that every class of $H^2(K,\widetilde{G}_{\mathcal{L}})$ is neutral. The field K is a "good" field of cohomological dimension 2 in the sense of §3.4 of [BCS].

Corollary 1. *Let G be a K-semi-simple group and μ be the kernel of $\widetilde{G}\to G$ where $\widetilde{G}$ is a universal covering of G. Assume $(|\mu|,q)=1$ and that Serre's conjecture holds for $\widetilde{G}$ and all inner forms of $\widetilde{G}$. Then the map $\delta^1 : H^1(K,G)\to H^2(K,\mu)$ is an isomorphism and all classes in $H^2(K,\widetilde{G})$ are neutral.*

Corollary 2. *With the hypotheses of Corollary 1, the Tate-Shafarevitch groups $Ш^1(K,G)$ and $Ш^2(K,\mu)$ are equal.*

4. Birational Property

In this section, we assume Serre's conjecture for $\widetilde{G}$ and all inner forms of $\widetilde{G}$.

Theorem 3. *Suppose that K is a function field in one variable over a perfect PAC field k, that G is K-semi-simple and is a direct factor of a K-rational variety (that is, there exists a k-variety Y such that $G \times Y$ is K-birational to some affine space over K) and that $(|\mu|, q) = 1$ with μ the kernel of $\widetilde{G} \to G$. Then $Ш^1(K, G) = 1$.*

Proof. (*cf.* Theorem 7.9 of [BKG] p. 327) Let X be a smooth compactification of G. Let $\overline{K}$ be an algebraic closure of K and $\Gamma = \mathrm{Gal}(\overline{K}/K)$. Because G is semi-simple, the map $\overline{K}^* \to \overline{K}[G]^*$ is a bijection. On the other hand, there is a natural Γ-isomorphism between the character group $\widehat{\mu}$ (where μ is the kernel of the map $\widetilde{G} \to G$) and the Picard group of $\overline{G} = G \times_K \overline{K}$. Therefore the natural sequence of Γ-modules

$$0 \to \overline{K}[G]^*/\overline{K}^* \to \mathrm{Div}_\infty(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{G}) \to 0$$

where $\mathrm{Div}_\infty(\overline{X})$ is the permutation module on the irreducible components of the complement of $\overline{G}$ in $\overline{X}$, rereads

$$0 \to \mathrm{Div}_\infty(\overline{X}) \to \mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{G}) \to 1. \qquad (*)$$

By assumption, there exists a Γ-module M such that the Γ-module $\mathrm{Pic}(\overline{X}) \oplus M$ is Γ-isomorphic to a permutation Γ-module. Dualizing $(*)$, we find an exact sequence $1 \to \mu \to F \to P \to 0$ with P a quasi-trivial torus and F a direct factor (as a torus) of a quasi-trivial torus. Since $Ш^2(k, \mu) = 0$ we deduce $Ш^1(K, G) = 1$. $\square$

5. Homogeneous Spaces (following Borovoi's method)

Let K be a function field in one variable over a perfect PAC field. Let X be a smooth variety over K that is a right homogeneous space of a semi-simple simply connected group H over K. Assume that the stabilizers G of X are semi-simple. Then X admits a K-rational point; namely that follows from these two facts:

- $Z^1(K, H) \xrightarrow{\circ} Z^1(K, H/G) \to \coprod_{\mathcal{L}} H^2(K, G_{\mathcal{L}}) = \coprod_{\mathcal{L}} H^2(K, G_{\mathcal{L}})'$ is exact,

  where $\xrightarrow{\circ}$ is the relation of Springer [Sp].
- $H^1(K, H) = 0$ (Serre's conjecture).

Remark 3: If $G_{\mathcal{L}}$ is only semi-simple, we consider its universal covering $\widetilde{G}_{\mathcal{L}}$. Since $\mathrm{cd}(K) \leq 2$, the map $H^2(K, \widetilde{G}_{\mathcal{L}}) \to H^2(K, G_{\mathcal{L}})$ is onto and, by Theorem 2, $H^2(K, G_{\mathcal{L}}) = H^2(K, G_{\mathcal{L}})'$.

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Periods of Mixed Tate Motives, Examples, l-adic Side

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Abstract. One hopes that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $\operatorname{Spec}\mathbb{Z}$ is generated by values of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$. These numbers are also called multiple zeta values. In this note, assuming motivic formalism, we give a proof, that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $\operatorname{Spec}\mathbb{Z}$ is generated by linear combinations with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$, which are unramified everywhere. The main subject of the paper is however the l-adic Galois analogue of the above result. We shall also discuss some other examples in the l-adic Galois setting.

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0. Introduction

One hopes that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $\operatorname{Spec}\mathbb{Z}$ is generated by values of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$. These numbers are also called multiple zeta values. In modern times these numbers first appeared in the Deligne paper [4]. In more explicit form they appeared in the article of Zagier (see [22]), though they were already studied by Euler (see [9]).

In this note we give a brief proof, assuming motivic formalism, that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $\operatorname{Spec}\mathbb{Z}$ is generated by linear combinations with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}$ of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$, which are unramified

everywhere. We explain what it means for a linear combination of such iterated integrals to be unramified everywhere. We give also a criterion when a linear combination with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,-1,\infty\}$ of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ is unramified everywhere. Such a result may be useful even if finally one shows that iterated integrals on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$ of sequences of one-forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ generate the $\mathbb{Q}$-algebra of mixed Tate motives over $\operatorname{Spec}\mathbb{Z}$.

These results have their analogues in l-adic Galois realizations. In fact we shall study l-adic situation first and in more details. The l-adic situation is easier conceptually, because the Galois group G_K of a number field K and its various weighted Tate $\mathbb{Q}_l$-completions replace the motivic fundamental group of the category of mixed Tate motives over $\operatorname{Spec}\mathcal{O}_{K,S}$, which is perhaps still a conjectural object.

Let S be a finite set of finite places of K. We shall consider weighted Tate representations of $\pi_1^{\mathrm{et}}(\operatorname{Spec}\mathcal{O}_{K,S};\operatorname{Spec}\bar{K})$ in finite-dimensional $\mathbb{Q}_l$-vector spaces. The universal proalgebraic group over $\mathbb{Q}_l$ by which such representations factorize we shall denote by $\mathcal{G}(\mathcal{O}_{K,S};l)$.

The kernel of the projection $\mathcal{G}(\mathcal{O}_{K,S};l)\to\mathbb{G}_m$ we denote by $\mathcal{U}(\mathcal{O}_{K,S};l)$. The associated graded Lie algebra of $\mathcal{U}(\mathcal{O}_{K,S};l)$ with respect of the weight filtration we denote by $L(\mathcal{O}_{K,S};l)$.

We assume that S contains all finite places of K lying over (l). Then the group $\mathcal{G}(\mathcal{O}_{K,S};l)$ is isomorphic to the conjectural motivic fundamental group of the Tannakian category of mixed Tate motives over $\operatorname{Spec}\mathcal{O}_{K,S}$ tensored with $\mathbb{Q}_l$ (see [10] and [11]).

Hain and Matsumoto also considered the case when S does not contain all finite places of K lying over (l). However the construction of the corresponding universal group is decidedly more complicated in this case and we do not understand it well. We shall present in this paper a simpler, more explicit version though only for weighted Tate representations and only on the level of graded Lie algebras. The construction is described briefly below.

Let S be a finite set of finite places of K. Every non trivial l-adic weighted Tate representation of G_K is ramified at all finite places of K which lie over (l). Therefore we must consider the weighted Tate $\mathbb{Q}_l$-completion of

$$\pi_1^{\mathrm{et}}(\operatorname{Spec}\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};\operatorname{Spec}\bar{K}),$$

where $\{\mathfrak{l}\mid l\}_K$ is the set of all finite places of K lying over (l). This has an effect that the Lie algebra $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ has more generators in degree 1 than the corresponding Lie algebra of the Tannakian category of mixed Tate motives over $\operatorname{Spec}\mathcal{O}_{K,S}$. To get rid of these additional generators in degree 1 we shall define a homogeneous Lie ideal $\langle\mathfrak{l}\mid l\rangle_{K,S}$ of $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ and then the quotient Lie algebra

$$L_l(\mathcal{O}_{K,S}):=L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)/\langle\mathfrak{l}\mid l\rangle_{K,S}\ .$$

We shall show that the Lie algebra $L_l(\mathcal{O}_{K,S})$ is also graded, i.e.,

$$L_l(\mathcal{O}_{K,S}) = \bigoplus_{i=1}^{\infty} L_l(\mathcal{O}_{K,S})_i$$

and that it has a correct number of generators.

Let us define

$$(L_l(\mathcal{O}_{K,S}))^\diamond := \oplus_{i=1}^{\infty} \mathrm{Hom}(L_l(\mathcal{O}_{K,S})_i, \mathbb{Q}_l).$$

We shall call $(L_l(\mathcal{O}_{K,S}))^\diamond$ the dual of $L_l(\mathcal{O}_{K,S})$.

The vector space $(L_l(\mathcal{O}_{K,S}))^\diamond$ is an l-adic analogue of the generators of the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $\operatorname{Spec} \mathcal{O}_{K,S}$.

Ihara in [12] and Deligne in [4] studied the action of the Galois group $G_{\mathbb{Q}}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,\infty\}; \overrightarrow{01})$. The pair $(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\}, \overrightarrow{01})$ has good reduction everywhere. Hence after passing to associated graded Lie algebras we get a Lie algebra representation

$$L\left(\mathbb{Z}\left[\frac{1}{l}\right]; l\right) \longrightarrow \mathrm{Der}^*\mathrm{Lie}(X,Y)$$

which factors through

$$L_l(\mathbb{Z}) \longrightarrow \mathrm{Der}^*\mathrm{Lie}(X,Y). \tag{0.1}$$

It is not known, at least to the author of this article, if the last morphism is injective. (This question was studied very much by Ihara and his students.) Hence we do not know if the vector space $L_l(\mathbb{Z})^\diamond$ is generated by the coefficients of the representation (0.1). This is the l-adic analogue of the problem about the multiple zeta values stated at the beginning of the section.

In [16] we have studied the action of $G_{\mathbb{Q}}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,-1,\infty\}; \overrightarrow{01})$. After the standard embedding of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus \{0,1,-1,\infty\}; \overrightarrow{01})$ into the $\mathbb{Q}_l$-algebra of non-commutative formal power series $\mathbb{Q}_l\{\{X, Y_0, Y_1\}\}$ and passing to the associated graded Lie algebra we get a Lie algebra representation

$$\Phi_{\overrightarrow{01}} : L\left(\mathbb{Z}\left[\frac{1}{2l}\right], l\right) \longrightarrow \mathrm{Der}^*\mathrm{Lie}(X, Y_0, Y_1),$$

where $\mathrm{Der}^*\mathrm{Lie}(X, Y_0, Y_1)$ is the Lie algebra of special derivations of the free Lie algebra $\mathrm{Lie}(X, Y_0, Y_1)$. The Lie ideal $\langle \mathfrak{l} \mid l \rangle_{\mathbb{Q},(2)}$ is contained in the kernel of $\Phi_{\overrightarrow{01}}$. Hence we get a morphism

$$\Phi_{\overrightarrow{01}} : L_l\left(\mathbb{Z}\left[\frac{1}{2}\right]\right) \to \mathrm{Der}^*\mathrm{Lie}(X, Y_0, Y_1).$$

Theorem 15.5.3 from [16] can be interpreted in the following way.

Theorem A. *The vector space $(L_l(\mathbb{Z}[\frac{1}{2}]))^\diamond$ is generated by the coefficients of the representation $\Phi_{\overrightarrow{01}}$.*

We shall show that the natural map

$$L_l\left(\mathbb{Z}\left[\frac{1}{2}\right]\right) \longrightarrow L_l(\mathbb{Z}),$$

induced by the inclusion $\mathbb{Z} \subset \mathbb{Z}[\frac{1}{2}]$, is a surjective morphism of Lie algebras. Let $I(\mathbb{Z}[\frac{1}{2}] : \mathbb{Z})$ be its kernel.

We say that $f \in (L_l(\mathbb{Z}[\frac{1}{2}]))^\diamond$ is unramified everywhere if $f(I(\mathbb{Z}[\frac{1}{2}] : \mathbb{Z})) = 0$. Our next result is then the immediate consequence of Theorem A.

Corollary B. *The vector space $(L_l(\mathbb{Z}))^\diamond$ is generated by these linear combinations of coefficients of the representation $\Phi_{\overrightarrow{01}}$, which are unramified everywhere.*

The result mentioned at the beginning of the section is the Hodge–de Rham analogue of Corollary B.

We shall also consider the following situation. Let L be a finite Galois extension of K. We assume that a pair (V_L, v) or a triple (V_L, z, v) is defined over L. Then we get a representation of G_L on $\pi_1(V_{\bar{L}}; v)$ or $\pi(V_{\bar{L}}; z, v)$. We shall define what it means that a coefficient of a such representation is defined over K.

Then, working in Hodge–de Rham realization and assuming motivic formalism, one can show that the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $\operatorname{Spec}\mathbb{Z}[\frac{1}{3}]$ is generated by linear combinations with rational coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus (\{0, \infty\} \cup \mu_3)$ of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$, $\frac{dz}{z-\xi_3}$, $\frac{dz}{z-\xi_3^2}$ ($\xi_3 = e^{\frac{2\pi i}{3}}$) from $\overrightarrow{01}$ to $\overrightarrow{10}$, which are defined over $\mathbb{Q}$. However in this paper we shall show only an l-adic analogue of that result.

Remark. A pair $(\mathbb{P}^1 \setminus \{0, 1, \infty\}, \overrightarrow{03})$ ramifies only at (3), hence periods of a mixed Tate motive associated with $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}; \overrightarrow{03})$ are periods of mixed Tate motives over $\operatorname{Spec}\mathbb{Z}[\frac{1}{3}]$. However one can easily show that in this way we shall not get all such periods.

The final aim is to show that the vector space $(L_l(\mathcal{O}_{K,S}))^\diamond$ is generated by linear combinations of coefficients, which are unramified outside S and defined over K of representations of G_L – for various L finite Galois extensions of K – on fundamental groups or on torsors of paths of a projective line minus a finite number of points or perhaps some other algebraic varieties. This will imply (by the very definition) that all mixed Tate representations of $L_l(\mathcal{O}_{K,S})$ are of geometric origin. We are however very far from this aim.

Then we must pass from Lie algebra representations of $L_l(\mathcal{O}_{K,S})$ to the representation of the corresponding group in order to show that any mixed Tate representation of G_K is of geometric origin. This part of the problem is not studied here.

The results of this paper where presented in a seminar talk in Lille in May 2009 and then at the end of my lectures at the summer school at Galatasaray University in Istanbul in June 2009.

In the first version of this paper some results of Section 2 (in particular Proposition 2.3) were proved under the assumption that l does not divide the order of $\mathrm{Gal}(L/K)$ and that $K(\mu_{l^\infty}) \cap L = K$. After the suggestion of the referee we removed these restrictive assumptions.

While finishing this paper the author has a delegation in CNRS in Lille at the *Laboratoire*, Paul Painlevé and he would like to thank very much the director, Professor Jean D'Almeida for accepting him in the Painlevé Laboratory. Thanks are also due to Professor J.-C. Douai who helped me to get this delegation.

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1. Weighted Tate completions of Galois groups

Let K be a number field and let S be a finite set of finite places of K. Let $\mathcal{O}_{K,S}$ be the ring of S-integers in K, i.e.,

$$\mathcal{O}_{K,S} := \left\{ \frac{a}{b} \mid a, b \in \mathcal{O}_K,\, b \notin \mathfrak{p} \text{ for all } \mathfrak{p} \notin S \right\}.$$

Let us fix a rational prime l. We denote by $\{\mathfrak{l} \mid l\}_K$ the set of finite places of K lying over the prime ideal (l) of $\mathbb{Z}$.

We introduce here some standard notation concerning Lie algebras that we shall use frequently.

Let L be a Lie algebra. The Lie subalgebras $\Gamma^n L$ of the lower central series of L are defined recursively by $\Gamma^1 L := L$, $\Gamma^{n+1} L := [\Gamma^n L, L]$, $n = 1, 2, 3, \ldots$. If L is graded then $L^{ab} = L/[L, L]$, $\Gamma^n L$ and $L/\Gamma^n L$ are also graded.

Let $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ be the weighted Tate $\mathbb{Q}_l$-completion of the étale fundamental group $\pi_1^{\mathrm{et}}(\mathrm{Spec}\, \mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; \mathrm{Spec}\bar{K})$. The group $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ is an affine, proalgebraic group over $\mathbb{Q}_l$ equipped with the homomorphism

$$\pi_1^{\mathrm{et}}(\mathrm{Spec}\, \mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; \mathrm{Spec}\bar{K}) \longrightarrow \mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)(\mathbb{Q}_l)$$

with a Zariski dense image, such that any weighted Tate finite-dimensional $\mathbb{Q}_l$-representation of $\pi_1^{\mathrm{et}}(\mathrm{Spec}\, \mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; \mathrm{Spec}\bar{K})$ factors through $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$. We point out that weighted Tate finite-dimensional $\mathbb{Q}_l$-representations of

$$\pi_1^{\mathrm{et}}(\mathrm{Spec}\, \mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; \mathrm{Spec}\bar{K})$$

provide weighted Tate finite-dimensional $\mathbb{Q}_l$- representations of G_K unramified outside $S \cup \{\mathfrak{l} \mid l\}_K$ and vice versa.

There is an exact sequence

$$1 \to \mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathbb{G}_m \to 1.$$

The kernel $\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ is a prounipotent proalgebraic affine group over $\mathbb{Q}_l$ equipped with the weight filtration $\{W_{-2i}\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)\}_{i\in\mathbb{N}}$ (see [10] and [11].)

Let us define

$$L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_i := W_{-2i}\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)/W_{-2(i+1)}\mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$$

and

$$L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l) := \bigoplus_{i=1}^{\infty} L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_i.$$

The Lie algebra $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ is a free Lie algebra. In degree 1 there are functorial isomorphisms

$$\mathrm{Hom}(L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_1;\mathbb{Q}_l) \approx H^1(\mathrm{Spec}\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};\mathbb{Q}_l(1)) \approx \mathcal{O}^{\times}_{K,S\cup\{\mathfrak{l}|l\}_K}\otimes\mathbb{Q}_l \tag{1.1.a}$$

and

$$L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_1 \approx \mathrm{Hom}(\mathcal{O}^{\times}_{K,S\cup\{\mathfrak{l}|l\}_K};\mathbb{Q}_l). \tag{1.1.b}$$

In degree $i>1$ there are functorial isomorphisms

$$\mathrm{Hom}\big((L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)^{ab})_i;\mathbb{Q}_l\big) \approx H^1(G_K;\mathbb{Q}_l(i)) \tag{1.1.c}$$

(see [10] Theorem 7.2.).

Let us assume that a pair (V,v) is defined over K and has good reduction outside S. The representation of G_K on the pro-l quotient of $\pi_1^{\mathrm{et}}(V_{\bar{K}};v)$ is unramified outside $S\cup\{\mathfrak{l}\mid l\}_K$ and if it is non-trivial, it is ramified at all finite places of K, which lie over (l). This has an effect that the Lie algebra $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ has more generators in degree 1 than the corresponding Lie algebra of the Tannakian category of mixed Tate motives over $\mathrm{Spec}\mathcal{O}_{K,S}$.

We shall show below how to kill these additional generators corresponding to finite places of K lying over (l), which are not in S.

Let $u\in\mathcal{O}^{\times}_{K,S\cup\{\mathfrak{l}|l\}_K}$ and let $\kappa(u): G_K\to\mathbb{Z}_l$ be the l-adic Kummer character of u. We denote by $\chi: G_K\to\mathbb{Z}_l^{\times}$ the l-adic cyclotomic character. The representation

$$G_K\ni\sigma\longrightarrow\begin{pmatrix}1 & 0\\ \kappa(u)(\sigma) & \chi(\sigma)\end{pmatrix}\in GL_2(\mathbb{Q}_l)$$

is an l-adic weighted Tate representation of G_K unramified outside $S\cup\{\mathfrak{l}\mid l\}_K$, i.e., it is an l-adic weighted Tate representation of $\pi_1^{\mathrm{et}}(\mathrm{Spec}\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};\mathrm{Spec}\bar{K})$. By (1.1.a) the Kummer character $\kappa(u)$ we can view also as a homomorphism

$$\kappa(u): L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_1\to\mathbb{Q}_l.$$

Let us set

$$(\mathfrak{l}\mid l)_{K,S} := \bigcap_{u\in\mathcal{O}^{\times}_{K,S}}\big(\mathrm{Ker}\big(\kappa(u): L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_1\to\mathbb{Q}_l\big)\big).$$

Let $\langle\mathfrak{l}\mid l\rangle_{K,S}$ be the Lie ideal of $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ generated by elements of $(\mathfrak{l}\mid l)_{K,S}$.

Definition 1.2. We set

$$L_l(\mathcal{O}_{K,S}) = L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)/\langle \mathfrak{l} \mid l\rangle_{K,S}.$$

Observe that $L_l(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}) = L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$.

Proposition 1.3.

i) *The Lie algebra $L_l(\mathcal{O}_{K,S})$ is graded.*

ii) *For i greater than 1 there are functorial isomorphisms*

$$\mathrm{Hom}\big((L_l(\mathcal{O}_{K,S})^{ab})_i; \mathbb{Q}_l\big) \approx H^1(G_K; \mathbb{Q}_l(i)).$$

iii) *In degree 1 there is a functorial isomorphism*

$$\mathrm{Hom}(L_l(\mathcal{O}_{K,S})_1; \mathbb{Q}_l) \approx \mathcal{O}_{K,S}^\times \otimes \mathbb{Q}_l.$$

iv) *The Lie algebra $L_l(\mathcal{O}_{K,S})$ is free, freely generated by $n_1 = \dim_{\mathbb{Q}}(\mathcal{O}_{K,S}^\times \otimes \mathbb{Q})$ elements in degree 1 and by $n_i = \dim_{\mathbb{Q}_l}(H^1(G_K; \mathbb{Q}_l(i))$ elements in degree $i > 1$.*

Proof. The Lie ideal $\langle \mathfrak{l} \mid l\rangle_{K,S}$ of the Lie algebra $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ is generated by elements of degree 1, hence it is homogeneous. Therefore the quotient Lie algebra $L_l(\mathcal{O}_{K,S})$ has a natural grading induced from that of $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$.

Let us choose $u_1, \ldots, u_p \in \mathcal{O}_{K,S}^\times$ ($p = \dim \mathcal{O}_{K,S}^\times \otimes \mathbb{Q}$) such that $u_1 \otimes 1, \ldots, u_p \otimes 1$ is a base of $\mathcal{O}_{K,S}^\times \otimes \mathbb{Q}$. Let $z_1, \ldots, z_q \in \mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}^\times$ be such that $u_1 \otimes 1, \ldots, u_p \otimes 1, z_1 \otimes 1, \ldots, z_q \otimes 1$ is a base of $(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}^\times) \otimes \mathbb{Q}$. Let $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ be the base of $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)_1$ dual to the Kummer characters $\kappa(u_1), \ldots, \kappa(u_p)$, $\kappa(z_1), \ldots, \kappa(z_q)$. Then $\beta_1, \ldots, \beta_q$ generate the Lie ideal $\langle \mathfrak{l} \mid l\rangle_{K,S}$. The points ii), iii) and iv) follow now immediately from the fact that the Lie algebra $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ is free, freely generated by the elements $\alpha_1, \ldots, \alpha_p, \beta_1, \ldots, \beta_q$ in degree 1 and by n_i generators in degrees $i > 1$ (see [10] Theorem 7.2.) and from the functorial isomorphisms (1.1.a) and (1.1.c). □

Definition 1.4. Let $L = \bigoplus_{i=1}^\infty L_i$ be a graded Lie algebra over a field k such that $\dim L_i < \infty$ for every i. We define

$$L^\diamond := \bigoplus_{i=1}^\infty \mathrm{Hom}(L_i, k).$$

We call $L^\diamond$ the dual of L. The vector space $L^\diamond$ is graded and

$$(L^\diamond)_i = (L_i)^\diamond := \mathrm{Hom}(L_i, k).$$

The Lie bracket $[\,,\,]$ of the Lie algebra L induces a morphism

$$d : L^\diamond \to L^\diamond \otimes L^\diamond,$$

whose image is contained in the subspace of $L^\diamond \otimes L^\diamond$ generated by all anti-symmetric tensors of the form $a \otimes b - b \otimes a$.

Definition 1.5. The $\mathbb{Q}_l$-vector space $(L_l(\mathcal{O}_{K,S}))^\diamond$ we shall call the vector space of coefficients on $L_l(\mathcal{O}_{K,S})$.

Remark 1.5.1. We consider the $\mathbb{Q}_l$-vector space $(L_l(\mathcal{O}_{K,S}))^\diamond$ as an analogue of generators of the $\mathbb{Q}$-algebra of periods of mixed Tate motives over $\mathrm{Spec}\mathcal{O}_{K,S}$.

The morphism

$$d : (L_l(\mathcal{O}_{K,S}))^\diamond \to (L_l(\mathcal{O}_{K,S}))^\diamond \otimes (L_l(\mathcal{O}_{K,S}))^\diamond$$

induced by the Lie bracket of $L_l(\mathcal{O}_{K,S})$ we denote by $d_{\mathcal{O}_{K,S}}$. We set

$$\mathcal{L}(\mathcal{O}_{K,S}; l) := \mathrm{Ker}(d_{\mathcal{O}_{K,S}}).$$

Observe that

$$\mathcal{L}(\mathcal{O}_{K,S}; l) = \{f \in (L_l(\mathcal{O}_{K,S}))^\diamond \ \mid \ f(\Gamma^2 L_l(\mathcal{O}_{K,S})) = 0\} \approx (L_l(\mathcal{O}_{K,S})^{ab})^\diamond.$$

The vector space $\mathcal{L}(\mathcal{O}_{K,S}; l)$ inherits grading from $(L_l(\mathcal{O}_{K,S}))^\diamond$ and we have

$$\mathcal{L}(\mathcal{O}_{K,S}; l) = \bigoplus_{i=1}^{\infty} \mathcal{L}(\mathcal{O}_{K,S}; l)_i.$$

It follows from Proposition 1.3 that there are natural isomorphisms

$$\mathcal{L}(\mathcal{O}_{K,S}; l)_i = \mathrm{Ker}(d_{\mathcal{O}_{K,S}})_i \approx H^1(G_K; \mathbb{Q}_l(i)) \quad \text{for} \quad i > 1 \tag{1.5.a}$$

and

$$\mathcal{L}(\mathcal{O}_{K,S}; l)_1 = \mathrm{Ker}(d_{\mathcal{O}_{K,S}})_1 = (L_l(\mathcal{O}_{K,S})_1)^\diamond \approx \mathcal{O}_{K,S}^\times \otimes \mathbb{Q}_l. \tag{1.5.b}$$

We finish this section with the study of the dual of the Lie bracket of the Lie algebra $L_l(\mathcal{O}_{K,S})$. To simplify the notation we denote $d_{\mathcal{O}_{K,S}}$ by d. The operators

$$d^{(n)} : (L_l(\mathcal{O}_{K,S}))^\diamond \longrightarrow \bigotimes_{i=1}^{n+1} (L_l(\mathcal{O}_{K,S}))^\diamond$$

are defined recursively by $d^{(1)} := d$, $\quad d^{(n+1)} := (d \otimes (\otimes_{i=1}^n Id_{(L_l(\mathcal{O}_{K,S}))^\diamond})) \circ d^{(n)}$, $\quad n = 1, 2, 3, \ldots$. The linear maps

$$pr_{n+1} : \otimes_{i=1}^{n+1} L_l(\mathcal{O}_{K,S}) \longrightarrow L_l(\mathcal{O}_{K,S})$$

are defined recursively by $pr_1(u_1) := u_1$, $\quad pr_{n+1}(u_1 \otimes u_2 \otimes \ldots \otimes u_n \otimes u_{n+1}) := [pr_n(u_1 \otimes u_2 \otimes \ldots \otimes u_n), u_{n+1}]$, $\quad n = 1, 2, 3, \ldots$.

Lemma 1.6. *We have:*

i) $(pr_{n+1})^\diamond = d^{(n)}$.

ii) $f \in (L_l(\mathcal{O}_{K,S}))^\diamond$ *vanishes on* $\Gamma^{n+1}(L_l(\mathcal{O}_{K,S}))$ *if and only if* $d^{(n)}(f) = 0$.

iii) *Let* $f \in (L_l(\mathcal{O}_{K,S}))^\diamond$ *be such that* $d^{(k+1)}(f) = 0$. *Then*

$$d^{(k)}(f) \in \bigotimes_{i=1}^{k+1} \mathcal{L}(\mathcal{O}_{K,S}; l).$$

Proof. The point i) is clear and ii) follows from i). It rests to show the point iii). It follows from ii) that f vanishes on $\Gamma^{k+2}L_l(\mathcal{O}_{K,S})$ hence it factors by $L_l(\mathcal{O}_{K,S})/\Gamma^{k+2}L_l(\mathcal{O}_{K,S})$. The map $d^{(k)}f = f \circ pr_{k+1}$ is then equal to the composition of the following two maps

$$\otimes_{i=1}^{k+1} L_l(\mathcal{O}_{K,S}) \to \otimes_{i=1}^{k+1} L_l(\mathcal{O}_{K,S})^{ab} \to \Gamma^{k+1}L_l(\mathcal{O}_{K,S})/\Gamma^{k+2}L_l(\mathcal{O}_{K,S})$$

and

$$\Gamma^{k+1}L_l(\mathcal{O}_{K,S})/\Gamma^{k+2}L_l(\mathcal{O}_{K,S}) \hookrightarrow L_l(\mathcal{O}_{K,S})/\Gamma^{k+2}L_l(\mathcal{O}_{K,S}) \xrightarrow{f} \mathbb{Q}_l.$$

The isomorphism $\mathcal{L}(\mathcal{O}_{K,S};l) \approx (L_l(\mathcal{O}_{K,S})^{ab})^{\diamond}$ implies that

$$d^{(k)}(f) \in \bigotimes_{i=1}^{k+1} \mathcal{L}(\mathcal{O}_{K,S};l). \qquad \square$$

2. Functorial properties of weighted Tate completions

Let K be a number field and let L be a finite extension of K. Let S be a set of finite places of K and let T be a set of finite places of L containing all places lying over elements of S. The inclusion of fields $K \subset L$ induces the inclusion of rings

$$\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K} \hookrightarrow \mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}. \tag{2.1}$$

The morphism of rings (2.1) induces a morphism of groups

$$\pi_1^{\mathrm{et}}(\mathrm{Spec}\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};\mathrm{Spec}\bar{L}) \to \pi_1^{\mathrm{et}}(\mathrm{Spec}\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};\mathrm{Spec}\bar{K}).$$

Therefore we get morphisms of affine proalgebraic groups over $\mathbb{Q}_l$

$$\pi_{K,S\cup\{\mathfrak{l}|l\}_K}^{L,T\cup\{\mathfrak{l}|l\}_L} : \mathcal{G}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l) \longrightarrow \mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$$

and

$$\pi_{K,S\cup\{\mathfrak{l}|l\}_K}^{L,T\cup\{\mathfrak{l}|l\}_L} : \mathcal{U}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l) \longrightarrow \mathcal{U}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l).$$

Passing to associated graded Lie algebras we get a morphism of graded Lie algebras

$$L(\pi_{K,S\cup\{\mathfrak{l}|l\}_K}^{L,T\cup\{\mathfrak{l}|l\}_L};l) : L(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l) \longrightarrow L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l).$$

Lemma 2.1. *For each $i > 1$ we have the following commutative diagram*

$$\begin{array}{ccc} \mathcal{L}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_i & \longrightarrow & \mathcal{L}(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)_i \\ \approx\big\downarrow & & \approx\big\downarrow \\ H^1(K;\mathbb{Q}_l(i)) & \longrightarrow & H^1(L;\mathbb{Q}_l(i)). \end{array}$$

In degree 1 there is the following commutative diagram

$$\begin{array}{ccc} (L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)_1)^{\diamond} & \longrightarrow & (L(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)_1)^{\diamond} \\ \approx\big\downarrow & & \approx\big\downarrow \\ \mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}^{\times} \otimes \mathbb{Q}_l & \longrightarrow & \mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L}^{\times} \otimes \mathbb{Q}_l. \end{array}$$

Proof. The lemma follows from the existence of the functorial isomorphisms (1.1.a) and (1.1.c) and from the functoriality of weighted Tate completions. □

Lemma 2.2. *The morphism of graded Lie algebras*

$$L(\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K};l) : L(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l) \longrightarrow L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l).$$

maps the Lie ideal $\langle \mathfrak{l} \mid l\rangle_{L,T}$ *of* $L(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)$ *into the Lie ideal* $\langle \mathfrak{l} \mid l\rangle_{K,S}$ *of* $L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$.

Proof. The Lie ideal $\langle \mathfrak{l} \mid l\rangle_{L,T}$ is generated by all elements $z \in L(\mathcal{O}_{L,T\cup\{\mathfrak{l}|l\}_L};l)_1$ satisfying $\kappa(u)(z) = 0$ for all $u \in \mathcal{O}^\times_{L,T}$. We have $\mathcal{O}^\times_{K,S} \subset \mathcal{O}^\times_{L,T}$. Hence it follows from the second part of Lemma 2.1 that $\kappa(u)(L(\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K};l)(z)) = 0$ for all $u \in \mathcal{O}^\times_{K,S}$. Hence $L(\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K};l)(z)$ belongs to the set $(\mathfrak{l} \mid l)_{K,S}$ of generators of the Lie ideal $\langle \mathfrak{l} \mid l\rangle_{K,S}$. □

It follows from Lemma 2.2 that $L(\pi^{L,T\cup\{\mathfrak{l}|l\}_L}_{K,S\cup\{\mathfrak{l}|l\}_K};l)$ induces

$$L_l(\pi^{L,T}_{K,S}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S}).$$

Proposition 2.3. *We have:*

i) *The morphism*

$$L_l(\pi^{L,T}_{K,S}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})$$

is a surjective morphism of graded Lie algebras.

ii) *For each* $i > 1$ *there is the following commutative diagram*

$$\begin{array}{ccc} \mathcal{L}(\mathcal{O}_{K,S};l)_i & \longrightarrow & \mathcal{L}(\mathcal{O}_{L,T};l)_i \\ \approx\big\downarrow & & \approx\big\downarrow \\ H^1(K;\mathbb{Q}_l(i)) & \longrightarrow & H^1(L;\mathbb{Q}_l(i)). \end{array}$$

iii) *In degree* 1 *there is the following commutative diagram*

$$\begin{array}{ccc} (L_l(\mathcal{O}_{K,S})_1)^\diamond & \longrightarrow & (L_l(\mathcal{O}_{L,T})_1)^\diamond \\ \approx\big\downarrow & & \approx\big\downarrow \\ \mathcal{O}^\times_{K,S}\otimes\mathbb{Q}_l & \longrightarrow & \mathcal{O}^\times_{L,T}\otimes\mathbb{Q}_l. \end{array}$$

Proof. By the very definition the ideals $\langle \mathfrak{l} \mid l\rangle_{K,S}$ and $\langle \mathfrak{l} \mid l\rangle_{L,T}$ are generated by elements of degree 1. Hence it follows from Lemma 2.2 that $L_l(\pi^{L,T}_{K,S}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})$ is a morphism of graded Lie algebras. The points ii) and iii) follow from Lemma 2.1. It rests to show that the morphism of graded Lie algebras

$$L_l(\pi^{L,T}_{K,S}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})$$

is surjective. The inclusion of number fields $K \subset L$ induces injective morphisms in Galois cohomology

$$H^1(G_K;\mathbb{Q}_l(i)) \to H^1(G_L;\mathbb{Q}_l(i))$$

for $i > 1$. It follows from this fact and from the parts ii) and iii) of the proposition already proved that the map

$$\mathcal{L}(\mathcal{O}_{K,S}; l) \to \mathcal{L}(\mathcal{O}_{L,T}; l)$$

is injective. Hence the homomorphism

$$L_l(\mathcal{O}_{L,T})^{ab} \to L_l(\mathcal{O}_{K,S})^{ab}$$

is surjective. Therefore the morphism of graded Lie algebras

$$L_l(\pi_{K,S}^{L,T}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})$$

is surjective. □

Definition 2.4. We define

$$I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S}) := \mathrm{Ker}\big(L_l(\pi_{K,S}^{L,T}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})\big).$$

Proposition 2.5. *We have:*

i) *The Lie ideal $I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})$ is generated by homogeneous elements.*
ii) *The quotient Lie algebra $L_l(\mathcal{O}_{L,T})/I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})$ is a graded Lie algebra.*
iii) *The induced morphism*

$$L_l(\mathcal{O}_{L,T})/I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S}) \to L_l(\mathcal{O}_{K,S})$$

is an isomorphism of graded Lie algebras.

Proof. The morphism $L_l(\pi_{K,S}^{L,T}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})$ is a surjective morphism of graded Lie algebras. Therefore $\mathrm{Ker}(L_l(\pi_{K,S}^{L,T})) = I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})$ is a graded Lie ideal. Hence one can choose homogeneous set of generators of $I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})$. Therefore the points ii) and iii) are clear. □

The surjective morphism of graded Lie algebras

$$L_l(\pi_{K,S}^{L,T})_l : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S})$$

induces an injective map of graded vector spaces

$$\Pi_{L,T}^{K,S} : L_l(\mathcal{O}_{K,S})^\diamond \to L_l(\mathcal{O}_{L,T})^\diamond.$$

Hence we get the following description of coefficients on $L_l(\mathcal{O}_{K,S})$.

Corollary 2.6. *The map $\Pi_{L,T}^{K,S}$ induces an isomorphism*

$$(L_l(\mathcal{O}_{K,S}))^\diamond \approx \{f \in (L_l(\mathcal{O}_{L,T}))^\diamond \mid f(I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})) = 0\}.$$

We indicate two important special cases.

Let S and S_1 be finite disjoint sets of finite places of K. The inclusion of rings

$$\mathcal{O}_{K,S} \hookrightarrow \mathcal{O}_{K,S\cup S_1}$$

induces the surjective morphism of graded Lie algebras

$$\pi_{K,S}^{K,S\cup S_1} : L_l(\mathcal{O}_{K,S\cup S_1}) \longrightarrow L_l(\mathcal{O}_{K,S}).$$

Definition 2.7. Let S and S_1 be finite disjoint sets of finite places of K. We say that $f \in (L_l(\mathcal{O}_{K,S\cup S_1}))^\diamond$ is unramified outside S_1 if $f(I(\mathcal{O}_{K,S\cup S_1} : \mathcal{O}_{K,S})) = 0$.

Corollary 2.6 in this special case can be formulated in the following suggestive form.

Corollary 2.8. *The vector space of coefficients on $L_l(\mathcal{O}_{K,S})$ is the subspace of the vector space of coefficients on $L_l(\mathcal{O}_{K,S\cup S_1})$ consisting of elements which are unramified outside S_1.*

The following observation will be useful.

Lemma 2.9. *The Lie ideal $I(\mathcal{O}_{K,S\cup S_1} : \mathcal{O}_{K,S})$ is generated by elements of degree* 1.

The second important case is the following one. Let K be a number field and let S be a set of finite places of K. Let L be a finite Galois extension of K and let T be a set of finite places of L lying over elements of S. The inclusion of rings of algebraic integers

$$\mathcal{O}_{K,S} \hookrightarrow \mathcal{O}_{L,T}$$

induces the surjective morphism of graded Lie algebras

$$L_l(\pi_{K,S}^{L,T}) : L_l(\mathcal{O}_{L,T}) \to L_l(\mathcal{O}_{K,S}).$$

Definition 2.10. We say that $f \in (L_l(\mathcal{O}_{L,T}))^\diamond$ is defined over K if

$$f(I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})) = 0.$$

In this special case we reformulate Corollary 2.6 in the following way.

Corollary 2.11. *The vector space of coefficients on $L_l(\mathcal{O}_{K,S})$ is the subspace of the vector space of coefficients on $L_l(\mathcal{O}_{L,T})$ consisting of elements which are defined over K.*

3. Geometric coefficients

Let $a_1, \ldots, a_n \in K$ and let $V := \mathbb{P}^1_K \setminus \{a_1, \ldots, a_n, \infty\}$. Let v and z be K-points of V or tangential points defined over K. Let S be a finite set of finite places of K. Let l be a fixed rational prime.

We denote by $\pi_1(V_{\bar{K}}; v)$ the pro-l completion of the étale fundamental group of $V_{\bar{K}}$ based at v and by $\pi(V_{\bar{K}}; z, v)$ the $\pi_1(V_{\bar{K}}; v)$-torsor of pro-l paths from v to z.

The Galois group G_K acts on $\pi_1(V_{\bar{K}}; v)$ and on $\pi(V_{\bar{K}}; z, v)$. After the standard embedding of $\pi_1(V_{\bar{K}}; v)$ into the $\mathbb{Q}_l$-algebra $\mathbb{Q}_l\{\{X_1, \ldots, X_n\}\}$ of formal power series in non-commuting variables we get two Galois representations

$$\varphi_v = \varphi_{V,v} : G_K \longrightarrow \mathrm{Aut}(\mathbb{Q}_l\{\{X_1, \ldots, X_n\}\})$$

and

$$\psi_{z,v} = \psi_{V,z,v} : G_K \longrightarrow GL(\mathbb{Q}_l\{\{X_1, \ldots, X_n\}\})$$

deduced from actions of G_K on π_1 and on the π_1-torsor (see [14], Section 4).

Let us assume that a pair (V, v) and a triple (V, z, v) have good reduction outside S. Then the representations $\varphi_{V,v}$ and $\psi_{V,z,v}$ factor through the weighted Tate $\mathbb{Q}_l$-completion $\mathcal{G}(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l)$ of $\pi_1^{\mathrm{et}}(\mathrm{Spec}\,\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; \mathrm{Spec}\bar{K})$ because the representations $\varphi_{V,v}$ and $\psi_{V,z,v}$ are weighted Tate $\mathbb{Q}_l$-representations unramified outside $S \cup \{\mathfrak{l} \mid l\}_K$ (see [18] Proposition 1.0.3). Passing to associated graded Lie algebras with respect to the weight filtrations we get morphisms of graded Lie algebras

$$\mathrm{gr}_W\mathrm{Lie}\varphi_v : L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n)$$

and

$$\mathrm{gr}_W\mathrm{Lie}\psi_{z,v} : L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathrm{Lie}(X_1, \ldots, X_n)\tilde{\times}\mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n),$$

where $\mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n)$ is the Lie algebra of special derivations of $\mathrm{Lie}(X_1, \ldots, X_n)$ (see the definition of the Lie algebra $\mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n)$ and the semi-direct product $\mathrm{Lie}(X_1, \ldots, X_n)\tilde{\times}\mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n)$ in [14], p. 134).

Theorem 3.1. *Let $a_1, \ldots, a_{n+1}$ be K-points of $\mathbb{P}^1_K$ and let $V := \mathbb{P}^1_K \setminus \{a_1, \ldots, a_{n+1}\}$. Let z and v be K-points of V or tangential points defined over K. Let us assume that the pair (V, v) (resp. the triple (V, z, v)) has good reduction outside S. Then the morphisms of graded Lie algebras*

$$\mathrm{gr}_W\mathrm{Lie}\varphi_{V,v} : L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathrm{Der}^*Lie(X_1, \ldots, X_n)$$

and

$$\mathrm{gr}_W\mathrm{Lie}\psi_{V,z,v} : L(\mathcal{O}_{K,S\cup\{\mathfrak{l}|l\}_K}; l) \to \mathrm{Lie}(X_1, \ldots, X_n)\tilde{\times}\mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n)$$

deduced from the action of G_K on $\pi_1(V_{\bar{K}}; v)$ and on $\pi(V_{\bar{K}}; z, v)$ respectively factor through the Lie algebra $L_l(\mathcal{O}_{K,S})$.

Proof. Let us assume that a pair (V, v) (resp. a triple (V, z, v)) has good reduction outside S. We shall show in the next lemma that then the morphism $\mathrm{gr}_W\mathrm{Lie}\varphi_{V,v}$ (resp. $\mathrm{gr}_W\mathrm{Lie}\psi_{V,z,v}$) in degree 1 is given by Kummer characters of elements belonging to $\mathcal{O}^\times_{K,S}$. This implies that the morphism vanishes on $(\mathfrak{l} \mid l)_{K,S}$, hence it vanishes on $\langle \mathfrak{l} \mid l\rangle_{K,S}$. Hence the theorem follows immediately. □

Lemma 3.1.1. *Let us assume that a pair (V, v) (resp. a triple (V, z, v)) has good reduction outside S. Then the morphism $\mathrm{gr}_W\mathrm{Lie}\varphi_{V,v}$ (resp. $\mathrm{gr}_W\mathrm{Lie}\psi_{V,z,v}$) in degree 1 is given by the Kummer characters of elements belonging to $\mathcal{O}^\times_{K,S}$.*

Proof. For simplicity we shall consider only a pair (V, v), where v is a K-point. The definition of good reduction at a finite place $\mathfrak{p}$ depends only on the isomorphism class of (V, v) over K (see [17], Definition 17.5), hence we can assume that $a_1 = 0$, $a_2 = 1$ and $a_{n+1} = \infty$.

The morphism $\mathrm{gr}_W\mathrm{Lie}\varphi_{V,v}$ is given in degree 1 by the Kummer characters $\kappa(\frac{a_i - a_k}{v - a_k})$ for $i \neq k$ and $i, k \in \{1, 2, \ldots, n\}$ (see [17], 17.10.a). Let $\mathcal{S}(V, v)$ be a set of finite places $\mathfrak{p}$ of K such that there exists a pair (i, k) satisfying $i \neq k$ and such that $\mathfrak{p}$ valuation of $\frac{a_i - a_k}{v - a_k}$ is different from 0. Then clearly $\frac{a_i - a_k}{v - a_k} \in \mathcal{O}^\times_{K,\mathcal{S}(V,v)}$ for all pair (i, k) with $i \neq k$.

For the pair (V, v) the notion of good reduction at $\mathfrak{p}$ and strong good reduction at $\mathfrak{p}$ coincide (see [17], Definitions 17.4, 17.5 and Corollary 17.18). It follows from Lemma 17.15 in [17] that $\mathfrak{p} \notin S$ implies $\mathfrak{p} \notin \mathcal{S}(V, v)$. Hence $\mathcal{S}(V, v) \subset S$. Therefore $\frac{a_i - a_k}{v - a_k} \in \mathcal{O}_{K,S}^{\times}$ for all pairs (i, k) with $i \neq k$. $\square$

We shall denote by

$$L_l(\varphi_v) : L_l(\mathcal{O}_{K,S}) \to \mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n)$$

and by

$$L_l(\psi_{z,v}) : L_l(\mathcal{O}_{K,S}) \to \mathrm{Lie}(X_1, \ldots, X_n) \tilde{\times} \mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n)$$

the morphisms induced by

$$\mathrm{gr}_W \mathrm{Lie} \varphi_{V,v} : L(\mathcal{O}_{K,S \cup \{\mathfrak{l} | l\}_K}; l) \to \mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n)$$

and by

$$\mathrm{gr}_W \mathrm{Lie} \psi_{V,z,v} : L(\mathcal{O}_{K,S \cup \{\mathfrak{l} | l\}_K}; l) \to \mathrm{Lie}(X_1, \ldots, X_n) \tilde{\times} \mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n)$$

respectively.

Let $\langle X_i \rangle$ be a one-dimensional vector subspace of $\mathrm{Lie}(X_1, \ldots, X_n)$ generated by X_i. The Lie algebra $\mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n)$ is isomorphic as a vector space to the direct sum $\bigoplus_{i=1}^n \mathrm{Lie}(X_1, \ldots, X_n)/\langle X_i \rangle$ (see [14], p. 138). The Lie bracket of $\mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n)$ induces the new Lie bracket, denoted by $\{,\}$, on the direct sum. The vector space $\bigoplus_{i=1}^n \mathrm{Lie}(X_1, \ldots, X_n)/\langle X_i \rangle$ equipped with the Lie bracket $\{,\}$ we shall denote by $(\bigoplus_{i=1}^n \mathrm{Lie}(X_1, \ldots, X_n)/\langle X_i \rangle; \{\,\})$. Passing to dual vector spaces and substituting $\mathrm{Der}^*\mathrm{Lie}(X_1, \ldots, X_n)$ by $(\bigoplus_{i=1}^n \mathrm{Lie}(X_1, \ldots, X_n)/\langle X_i \rangle; \{\,\})$ we get morphisms

$$\Phi^v := (L_l(\varphi_v))^\diamond : \left(\bigoplus_{i=1}^n \mathrm{Lie}(X_1, \ldots, X_n)/\langle X_i \rangle; \{\,\} \right)^\diamond \to (L_l(\mathcal{O}_{K,S}))^\diamond$$

and

$$\Psi^{z,v} := L_l(\psi_{z,v})^\diamond : \left(\mathrm{Lie}(X_1, \ldots, X_n) \tilde{\times} \left(\bigoplus_{i=1}^n \mathrm{Lie}(X_1, \ldots, X_n)/\langle X_i \rangle; \{\,\} \right) \right)^\diamond \to (L_l(\mathcal{O}_{K,S}))^\diamond.$$

Definition 3.2. We set

$$\mathrm{GeomCoeff}^l_{\mathcal{O}_{K,S}}(V, v) := \mathrm{Image}\,(\Phi^v)$$

and

$$\mathrm{GeomCoeff}^l_{\mathcal{O}_{K,S}}(V, z, v) := \mathrm{Image}\,(\Psi^{z,v}).$$

The vector subspace $\mathrm{GeomCoeff}^l_{\mathcal{O}_{K,S}}(V, v)$ (resp. $\mathrm{GeomCoeff}^l_{\mathcal{O}_{K,S}}(V, z, v)$) of $(L_l(\mathcal{O}_{K,S}))^\diamond$ we shall call the vector space of geometric coefficients on $L_l(\mathcal{O}_{K,S})$ coming from (V, v) (resp. (V, z, v)).

Let us fix a Hall base $\mathcal{B}$ of the free Lie algebra $\mathrm{Lie}(X_1,\dots,X_n)$. If $e\in\mathcal{B}$ then e^* denotes the dual linear form in $\mathrm{Lie}(X_1,\dots,X_n)^\diamond$ with respect to the base $\mathcal{B}$. Let

$$pr_{i_0}:\bigoplus_{i=1}^{n}\mathrm{Lie}(X_1,\dots,X_n)/\langle X_i\rangle\longrightarrow\mathrm{Lie}(X_1,\dots,X_n)/\langle X_{i_0}\rangle$$

be the projection on the i_0th component. Let

$$p:\mathrm{Lie}(X_1,\dots,X_n)\tilde{\times}\left(\bigoplus_{i=1}^{n}\mathrm{Lie}(X_1,\dots,X_n)/\langle X_i\rangle\right)\longrightarrow\mathrm{Lie}(X_1,\dots,X_n)$$

be the projection on the first factor.

We set

$$\{z,v\}_{e^*}:=e^*\circ p\circ L_l(\psi_{z,v})=\Psi^{z,v}(e^*\circ p). \tag{3.3}$$

Let $e\in\mathcal{B}$ be different from X_i. Let $\overrightarrow{a_i}$ be any tangential point defined over K at a_i. Then we have

$$\{\overrightarrow{a_i},v\}_{e^*}=e^*\circ pr_i\circ L_l(\varphi_v)=\Phi^v(e^*\circ pr_i). \tag{3.4}$$

The geometric coefficients $\{z,v\}_{e^*}$ considered here are the l-adic iterated integrals from [14]. We use here the notation $\{z,v\}_{e^*}$ because it is more convenient for our study.

If $\psi\in\left(\mathrm{Lie}(X_1,\dots,X_n)\tilde{\times}(\bigoplus_{i=1}^{n}\mathrm{Lie}(X_1,\dots,X_n)/\langle X_i\rangle)\right)^\diamond$ then $\Psi^{z,v}(\psi)=\psi\circ L_l(\psi_{z,v})$ is a linear combination of symbols (3.3) and (3.4).

Elements of $(L_l(\mathcal{O}_{K,S}))^\diamond$ which belong to $\mathrm{GeomCoeff}^l_{\mathcal{O}_{K,S}}(V,v)$ are of geometric origin, hence they are motivic. For few rings of algebraic integers one can show that

$$(L_l(\mathcal{O}_{K,S}))^\diamond=\mathrm{GeomCoeff}^l_{\mathcal{O}_{K,S}}(V,v) \tag{3.5}$$

for a convenable choice of a pair (V,v). In the next sections we shall indicate these examples. They follow easily from our paper [16]. The Hodge–de Rham side for the ring $\mathbb{Z}[\frac{1}{2}]$ was presented by P. Deligne on the conference in Schloss Ringberg (see [5]). The talk delivered by P. Deligne on this conference motivated our study in [16]. The result of Deligne is in his recent preprint (see [6]).

One cannot expect to show the equality (3.5) for all rings $\mathcal{O}_{K,S}$. Examples in Zagier paper [21] suggests a way to follow.

Let K be a number field and let L be a finite extension of K. Let S be a finite set of finite places of K and let T be a finite set of finite places of L containing all places lying over elements of S. The inclusion of rings

$$\mathcal{O}_{K,S}\hookrightarrow\mathcal{O}_{L,T}$$

induces the surjective morphism

$$L_l\left(\pi^{L,T}_{K,S}\right):L_l(\mathcal{O}_{L,T})\to L_l(\mathcal{O}_{K,S}),$$

whose kernel we have denoted by $I(\mathcal{O}_{L,T}:\mathcal{O}_{K,S})$.

Definition 3.6. Let $g \in (L_l(\mathcal{O}_{K,S}))^\diamond$. We say that g is geometric if there exists $f \in (L_l(\mathcal{O}_{L,T}))^\diamond$ such that

i) f is a geometric coefficient coming from some pair (V, v) or triple (V, z, v);
ii) $f\big(I(\mathcal{O}_{L,T} : \mathcal{O}_{K,S})\big) = 0$;
iii) $g \circ L_l(\pi^{L,T}_{K,S}) = f$.

We shall usually denote f and g by the same letter f.

Let $\mathcal{O}_{F,R}$ be a subring of $\mathcal{O}_{K,S}$. Corollary 2.6, which we recall here in the form

$$(L_l(\mathcal{O}_{F,R}))^\diamond = \{f \in (L_l(\mathcal{O}_{K,S}))^\diamond \ \mid \ f(I(\mathcal{O}_{K,S} : \mathcal{O}_{F,R})) = 0\},$$

implies that for subrings $\mathcal{O}_{F,R}$ of the ring $\mathcal{O}_{K,S}$ satisfying (3.5) we have

$$(L_l(\mathcal{O}_{F,R}))^\diamond = \{f \in \mathrm{GeomCoeff}^l_{\mathcal{O}_{K,S}}(V, v) \ \mid \ f(I(\mathcal{O}_{K,S} : \mathcal{O}_{F,R})) = 0\}.$$

Examples of such rings we shall also discuss in the next sections. In particular we shall show that

$$(L_l(\mathbb{Z}))^\diamond = \left\{ f \in \mathrm{GeomCoeff}^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1 \setminus \{0,1,-1,\infty\}, \overrightarrow{01}) \mid f\left(I\left(\mathbb{Z}\left[\frac{1}{2}\right] : \mathbb{Z}\right)\right) = 0 \right\}.$$

Hence we shall show that all elements of $\big(L_l(\mathbb{Z})\big)^\diamond$ are geometric in the sense of Definition 3.6.

We hope that for any ring $\mathcal{O}_{K,S}$, all coefficients on $L_l(\mathcal{O}_{K,S})$ are geometric in the sense of Definition 3.6.

Remark 3.7. In [18] we were studying related questions. Starting from the torsor of paths $\pi(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus 0, 1, \infty\}; \xi_p, \overrightarrow{01})$ we have constructed all coefficient on $L_l(\mathbb{Z}[\frac{1}{p}])$. However we have not proved that they are geometric in the sense of Definition 3.6. In the moment of publishing [18] we were thinking that it was obvious. But this is not the case.

Remark 3.8. The geometric coefficients $\{z, v\}_{e^*}$ coming from (V, z, v) are l-adic Galois analogues of iterated integrals from v to z on $\mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \ldots, a_n, \infty\}$ of sequences of one-forms $\frac{dz}{z-a_1}, \ldots, \frac{dz}{z-a_n}$. Geometric coefficients in the sense of Definition 3.6 correspond to linear combinations of such iterated integrals. For example $Li_n(\xi_p^k)$ for $1 \le k \le p-1$ are periods of a mixed Tate motive over $\mathrm{Spec}\mathbb{Q}(\mu_p)$, but $\sum_{k=1}^{p-1} Li_n(\xi_p^k)$ is a period of a mixed Tate motive over $\mathrm{Spec}\mathbb{Q}$.

4. From $\mathbb{P}^1 \setminus \{0, 1, -1, \infty\}$ to periods of mixed Tate motives over $\mathrm{Spec}\mathbb{Z}$

Let $V := \mathbb{P}^1_{\mathbb{Q}} \setminus \{0, 1, -1, \infty\}$. In [16], 15.5 we have studied the Galois representation

$$\varphi_{\overrightarrow{01}} : G_{\mathbb{Q}} \to \mathrm{Aut}(\pi_1(V_{\bar{\mathbb{Q}}}; \overrightarrow{01})). \tag{4.0}$$

Observe that the pair $(V, \overrightarrow{01})$ has good reduction outside the prime ideal (2) of $\mathbb{Z}$ (see [18], Definition 2.0). Hence the representation (4.0) is unramified outside

prime ideals (2) and (l) (see [17], Corollary 17.17). After the standard embedding of $\pi_1(V_{\mathbb{Q}};\overrightarrow{01})$ into the $\mathbb{Q}_l$-algebra of formal power series in non-commuting variables $\mathbb{Q}_l\{\{X,Y_0,Y_1\}\}$ (see [16], 15.2) we get a representation

$$\varphi_{\overrightarrow{01}} : G_{\mathbb{Q}} \to \mathrm{Aut}(\mathbb{Q}_l\{\{X,Y_0,Y_1\}\}). \tag{4.1}$$

It follows from the universal properties of the weighted Tate $\mathbb{Q}_l$-completion that the morphism (4.1) factors through

$$\varphi_{\overrightarrow{01}} : \mathcal{G}\left(\mathbb{Z}\left[\frac{1}{2l}\right];l\right) \to \mathrm{Aut}(\mathbb{Q}_l\{\{X,Y_0,Y_1\}\}).$$

Passing to associated graded Lie algebras we get a morphism of graded Lie algebras studied in [16], 15.5,

$$\mathrm{gr}_W\mathrm{Lie}\varphi_{\overrightarrow{01}} : L\left(\mathbb{Z}\left[\frac{1}{2l}\right];l\right) \to (\mathrm{Lie}(X,Y_0,Y_1),\{\,\}). \tag{4.2}$$

It follows from Theorem 3.1 that the morphism (4.2) induces a morphism of graded Lie algebras

$$L_l(\varphi_{\overrightarrow{01}}) : L_l\left(\mathbb{Z}\left[\frac{1}{2}\right]\right) \to (\mathrm{Lie}(X,Y_0,Y_1),\{\,\}). \tag{4.3}$$

Proposition 4.4. *The morphism of graded Lie algebras*

$$L_l(\varphi_{\overrightarrow{01}}) : L_l\left(\mathbb{Z}\left[\frac{1}{2}\right]\right) \to (\mathrm{Lie}(X,Y_0,Y_1),\{\,\})$$

deduced from the action of $G_{\mathbb{Q}}$ on $\pi_1(\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,-1,\infty\};\overrightarrow{01})$ is injective.

Proof. The proposition follows from [16], Theorem 15.5.3. Below we give a more detailed proof.

We recall that $\{G_i(V,\overrightarrow{01})\}_{i\in\mathbb{N}}$ is a filtration of $G_{\mathbb{Q}}$ associated with the representation (4.0) (see [14], Section 3). The pair $(V,\overrightarrow{01})$ has good reduction outside the prime ideal (2) of $\mathbb{Z}$. Hence the natural morphism of graded Lie algebras

$$L\left(\mathbb{Z}\left[\frac{1}{2l}\right];l\right) \to \bigoplus_{i=1}^{\infty}(G_i(V,\overrightarrow{01})/G_{i+1}(V,\overrightarrow{01}))\otimes\mathbb{Q} \tag{4.4.1}$$

is surjective (see [17], Proposition 19.1). Moreover the natural morphism

$$\bigoplus_{i=1}^{\infty}(G_i(V,\overrightarrow{01})/G_{i+1}(V,\overrightarrow{01}))\otimes\mathbb{Q} \to (\mathrm{Lie}(X,Y_0,Y_1),\{\ \}) \tag{4.4.2}$$

is injective (see [17], Proposition 19.2). The morphism (4.2) is the composition of morphisms (4.4.1) and (4.4.2). It follows from Theorem 3.1 that the morphism (4.2) induces a morphism (4.3) Hence the morphism (4.3) induces a surjective morphism of graded Lie algebras

$$L_l\left(\mathbb{Z}\left[\frac{1}{2}\right]\right) \to \bigoplus_{i=1}^{\infty}(G_i(V,\overrightarrow{01})/G_{i+1}(V,\overrightarrow{01}))\otimes\mathbb{Q}. \tag{4.4.3}$$

The graded Lie algebra $L_l(\mathbb{Z}[\frac{1}{2}])$ is free, freely generated by elements dual to $\kappa(2)$ and $l_{2n+1}(-1)$ for $n > 0$. It follows from [16], Theorem 15.5.3 that the elements dual to $\kappa(2)$ and $l_{2n+1}(-1)$ for $n > 0$ are generators of a free Lie subalgebra of $\bigoplus_{i=1}^{\infty}(G_i(V,\overrightarrow{01})/G_{i+1}(V,\overrightarrow{01}))\otimes\mathbb{Q}$. Therefore the morphism (4.4.3) is an isomorphism. This implies that the morphism

$$L_l(\varphi_{\overrightarrow{01}}) : L_l\left(\mathbb{Z}\left[\frac{1}{2}\right]\right) \to (\mathrm{Lie}(X,Y_0,Y_1),\{\,\})$$

is injective. □

The immediate consequence of Proposition 4.4 is the following corollary.

Corollary 4.5. *All coefficients on $L_l(\mathbb{Z}[\frac{1}{2}])$ are geometric, more precisely*

$$\left(L_l\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)\right)^{\diamond} = \mathrm{GeomCoeff}^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,-1,\infty\},\overrightarrow{01}).$$

We recall that the morphism of graded Lie algebras

$$L_l(\pi^{\mathbb{Q},(2)}_{\mathbb{Q},\emptyset}) : L_l\left(\mathbb{Z}\left[\frac{1}{2}\right]\right) \to L_l(\mathbb{Z})$$

induced by the inclusion of rings $\mathbb{Z}\hookrightarrow\mathbb{Z}[\frac{1}{2}]$ is surjective by Proposition 2.3 and its kernel is by the very definition the Lie ideal $I(\mathbb{Z}[\frac{1}{2}]:\mathbb{Z})$.

Corollary 4.6. *We have*

$$(L_l(\mathbb{Z}))^{\diamond} = \left\{ f\in\mathrm{GeomCoeff}^l_{\mathbb{Z}[\frac{1}{2}]}\left(\mathbb{P}^1_{\mathbb{Q}}\setminus\{0,1,-1,\infty\},\overrightarrow{01}\right) \mid f\left(I\left(\mathbb{Z}\left[\frac{1}{2}\right]:\mathbb{Z}\right)\right)=0\right\},$$

i.e., the vector space of coefficients on $L_l(\mathbb{Z})$ is equal to the vector subspace of $\mathrm{GeomCoeff}^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}}\setminus 0,1,-1,\infty\},\overrightarrow{01})$ *consisting of all coefficients unramified everywhere.*

Proof. The corollary follows from Corollary 4.5 and Corollary 2.6. □

Remark 4.6.1. The corresponding statement in Hodge–de Rham realization says that all periods of mixed Tate motives over Spec$\mathbb{Z}$ are unramified everywhere $\mathbb{Q}$-linear combinations of iterated integrals on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,-1,\infty\}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ in one forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$. It will be proved in Section 7.

Now we shall look more carefully at geometric coefficients to see which are unramified everywhere.

The Lie algebra $L_l(\mathbb{Z}[\frac{1}{2}])$ is free, freely generated by one generator z_i in each odd degree. The Lie ideal $I(\mathbb{Z}[\frac{1}{2}]:\mathbb{Z})$ is generated by the generator in degree 1. This generator z_1 can be chosen to be dual to the Kummer character $\kappa(2)$, i.e., $\kappa(2)(z_1)=1$.

Let us choose a Hall base $\mathcal{B}$ of the free Lie algebra $\mathrm{Lie}(X,Y_0,Y_1)$. Then the geometric coefficients, elements of the $\mathbb{Q}_l$-vector space $\mathrm{GeomCoeff}^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}}\setminus$

$\{0,1,-1,\infty\},\overrightarrow{01})$ are of the form $\{\overrightarrow{10},\overrightarrow{01}\}_{e^*}$ and $\{\overrightarrow{10},\overrightarrow{01}\}_{\psi}$, where $\psi=\sum_{i=1}^{k} n_i e_i^*$ and $e, e_i \in \mathcal{B}$.

Proposition 4.7. *Let $e \in \mathcal{B}$ be a Lie bracket in X and Y_0 only. Then the coefficient $\{\overrightarrow{10},\overrightarrow{01}\}_{e^*}$ is unramified everywhere.*

Proof. Let $j : \mathbb{P}^1 \setminus \{0,1,-1,\infty\} \hookrightarrow \mathbb{P}^1 \setminus \{0,1,\infty\}$ be the inclusion. Then j induces

$$j_* : \pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,-1,\infty\};\overrightarrow{01}) \to \pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,\infty\};\overrightarrow{01}).$$

After the standard embeddings of the fundamental groups into the $\mathbb{Q}_l$-algebras of non-commutative formal power series $\mathbb{Q}_l\{\{X,Y_0,Y_1\}\}$ and $\mathbb{Q}_l\{\{X,Y\}\}$ we get a morphism of $\mathbb{Q}_l$-algebras

$$j_* : \mathbb{Q}_l\{\{X,Y_0,Y_1\}\} \to \mathbb{Q}_l\{\{X,Y\}\}$$

induced by the morphism of fundamental groups such that $j_*(X)=(X)$, $j_*(Y_0)=Y$ and $j_*(Y_1)=0$.

Then we have $\{\overrightarrow{10},\overrightarrow{01}\}_{e(X,Y_0)^*} = \{\overrightarrow{10},\overrightarrow{01}\}_{e(X,Y)^*\circ j_*} = \{j(\overrightarrow{10}),j(\overrightarrow{01})\}_{e(X,Y)^*} = \{\overrightarrow{10},\overrightarrow{01}\}_{e(X,Y)^*}$ (see [15] (10.0.6)). The pair $(\mathbb{P}^1 \setminus \{0,1,\infty\},\overrightarrow{01})$ is unramified everywhere, hence the coefficient $\{\overrightarrow{10},\overrightarrow{01}\}_{e(X,Y_0)^*}$ belonging to $\mathrm{GeomCoeff}^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1 \setminus \{0,1,-1,\infty\},\overrightarrow{01})$ is unramified everywhere. □

There are however coefficients in the $\mathbb{Q}_l$-vector space $\mathrm{GeomCoeff}^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1 \setminus \{0,1,-1,\infty\},\overrightarrow{01})$ which contain Y_1 and which also are unramified everywhere. These coefficients are of course the most interesting in view of Corollary 4.6 as we perhaps still do not know if the inclusion

$$\mathrm{GeomCoeff}^l_{\mathbb{Z}}(\mathbb{P}^1 \setminus \{0,1,\infty\},\overrightarrow{01}) \subset (L_l(\mathbb{Z}))^{\diamond}$$

is the equality. For example we have the following result.

Proposition 4.8. *We have*

$$\{\overrightarrow{10},\overrightarrow{01}\}_{[Y_1,X^{(n-1)}]^*} = \frac{1-2^{n-1}}{2^{n-1}} \cdot \{\overrightarrow{10},\overrightarrow{01}\}_{[Y_0,X^{(n-1)}]^*}.$$

Proof. It follows immediately from the definition of coefficients $\{\overrightarrow{10},\overrightarrow{01}\}_{e^*}$ and the definition of l-adic polylogarithms (see [15], Definition 11.0.1) that

$$\{\overrightarrow{10},\overrightarrow{01}\}_{[Y_0,X^{(n-1)}]^*} = l_n(1).$$

It follows from [16], Lemma 15.3.1 that $\{\overrightarrow{10},\overrightarrow{01}\}_{[Y_1,X^{(n-1)}]^*} = l_n(-1)$. The proposition now follows from the distribution relation $2^{n-1}(l_n(-1)+l_n(1)) = l_n(1)$ (see [15] Corollary 11.2.3). □

Below we shall give an inductive procedure to decide which coefficients are unramified everywhere. Let us denote for simplicity

$$\mathcal{L} := \mathcal{L}\left(\mathbb{Z}\left[\frac{1}{2}\right]; l\right), \quad \mathcal{L}_i := \mathcal{L}\left(\mathbb{Z}\left[\frac{1}{2}\right]; l\right)_i \quad \text{and} \quad \mathcal{L}_{>1} := \bigoplus_{i=2}^{\infty} \mathcal{L}_i.$$

Lemma 4.9. *We have*

i) $\mathcal{L}_i = \mathbb{Q}_l$ *for* i *odd and* $\mathcal{L}_i = 0$ *for* i *even;*
ii) $\mathcal{L}_1$ *is generated by the Kummer character* $\kappa(2)$*;*
iii) $\mathcal{L}_{2k+1}$ *is generated by* $l_{2k+1}(-1)$ *for* $k > 0$.

Proof. It follows from (1.5.b) that $\mathcal{L}_1 = (L_l(\mathbb{Z}[\frac{1}{2}]))^{\diamond}_1 \approx \mathbb{Z}[\frac{1}{2}]^{\times} \otimes \mathbb{Q}_l \approx \mathbb{Q}_l$. Hence $\mathcal{L}_1$ is generated by the Kummer character $\kappa(2)$.

For $i > 1$ it follows from (1.5.a) that $\mathcal{L}_i \approx H^1(G_{\mathbb{Q}}; \mathbb{Q}_l(i))$.

The group $H^1(G_{\mathbb{Q}}; \mathbb{Q}_l(i)) = 0$ for i even and $H^1(G_{\mathbb{Q}}; \mathbb{Q}_l(i)) \approx \mathbb{Q}_l$ for i odd by the result of Soulé (see [13]) combined with the theorem of A. Borel (see [2]). The cohomology group $H^1(G_{\mathbb{Q}}; \mathbb{Q}_l(2k+1))$ is generated by a Soulé class, which is a rational multiple of $l_{2k+1}(-1)$. □

If $e \in \mathcal{B}$ then $\deg_{Y_i} e$ denotes degree of e with respect to Y_i. We define

$$\deg_Y e := \deg_{Y_0} e + \deg_{Y_1} e.$$

Lemma 4.10. *Let* $\varphi \in \mathrm{GeomCoeff}^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1 \setminus \{0, 1, -1, \infty\}, \overrightarrow{01})$ *be homogeneous of degree* k.

i) *If* $k = 1$ *then* $d\varphi = 0$ *and* φ *is a* $\mathbb{Q}_l$*-multiple of* $\kappa(2)$*. Hence if* $\varphi \neq 0$ *then* φ *ramifies at* (2).
ii) *If* $k > 1$ *and* $d\varphi = 0$ *then* φ *is unramified everywhere.*
iii) *If* $k > 1$ *and* $\varphi = \sum_{i=1}^m a_i e_i^*$*, where* $e_i \in \mathcal{B}$ *and* $\deg_Y e_i^* = 1$ *for each* i *then* $d\varphi = 0$ *and* φ *is unramified everywhere.*

Proof. In degree 1 there are the following geometric coefficients $\{\overrightarrow{10}, \overrightarrow{01}\}_X = 0$, $\{\overrightarrow{10}, \overrightarrow{01}\}_{Y_0} = 0$ and $\{\overrightarrow{10}, \overrightarrow{01}\}_{Y_1} = \kappa(2)$ – the Kummer character of 2, which ramifies at (2).

If $\deg\varphi = k > 1$ and $d\varphi = 0$ then φ is a $\mathbb{Q}_l$-multiple of $l_k(-1)$ by Lemma 4.9 iii). Hence φ is unramified everywhere by Propositions 4.8 and 4.7.

If $\deg_Y e = 1$ then $e = [Y_0, X^{(k-1)}]$ or $e = [Y_1, X^{(k-1)}]$. In both cases it is clear that $d(e^*) = 0$. Hence it follows the part iii) of the lemma. □

Proposition 4.11. *Let* $\varphi \in \mathrm{GeomCoeff}^l_{\mathbb{Z}[\frac{1}{2}]}(\mathbb{P}^1 \setminus \{0, 1, -1, \infty\}, \overrightarrow{01})$ *be homogeneous of degree greater than* 1.

i) *If* $d^{(k+1)}\varphi = 0$ *then* $d^{(k)}\varphi \in \otimes_{i=1}^k \mathcal{L}$.
ii) *Let us assume that* $d^{(k+1)}\varphi = 0$*. Then* φ *is unramified everywhere if and only if* $d^{(k)}\varphi \in \otimes_{i=1}^k \mathcal{L}_{>1}$ *and* $d^{(j)}\varphi$ *is unramified everywhere for* $0 < j < k$*, i.e.,* $d^{(j)}\varphi \in \otimes_{i=1}^j (L_l(\mathbb{Z}))^{\diamond}$ *for* $0 < j < k$.

iii) *Let* $\varphi = \sum_{i=1}^{m} n_i \{\overrightarrow{10}, \overrightarrow{01}\}_{e_i^*}$, *where* $e_i \in \mathcal{B}$ *and* $\deg_Y e_i \leq k+1$ *for each* $i = 1, 2, \ldots, m$. *Then* $d^{(k+1)}(\varphi) = 0$.

Proof. Let us write $d^{(k)}\varphi$ in the form $\sum_{i \in I} \beta_i^1 \otimes \alpha_i \otimes \beta_i^2$, where

$$\beta_i^1 \in \otimes_{t=1}^{s} \big(L\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big)\big)^{\diamond}, \quad \alpha_i \in \big(L\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big)\big)^{\diamond} \quad \text{and} \quad \beta_i^2 \in \otimes_{t=1}^{k-s} \big(L_l\big(\mathbb{Z}\big[\tfrac{1}{2}\big]\big)\big)^{\diamond}.$$

We can assume that elements $\beta_i^1 \otimes \beta_i^2$, $i \in I$ are linearly independent. Observe that the condition $d^{(k+1)}\varphi = 0$ implies that $\big((\otimes_{t=1}^{s} id) \otimes d \otimes (\otimes_{t=1}^{k-s} id)\big) \circ d^{(k)}\varphi = 0$. Hence we get $d\alpha_i = 0$ for $i \in I$. Therefore $\alpha_i \in \mathcal{L}$ for $i \in I$. We have chosen s arbitrary, hence $d^{(k)}\varphi \in \otimes_{i=1}^{k} \mathcal{L}$.

Now we shall prove the part ii) of the proposition. If $d^{(j)}\varphi \in \otimes_{i=1}^{j} (L_l(\mathbb{Z}))^{\diamond}$ for $0 < j < k$ and $d^{(k)}\varphi \in \otimes_{i=1}^{k} \mathcal{L}_{>1}$ then φ vanishes on all Lie brackets containing z_1 of length d for $2 \leq d \leq k+1$. The linear form φ has degree greater than 1, hence it vanishes on z_1. The assumption $d^{(k+1)}\varphi = 0$ implies that φ vanishes on $\Gamma^{k+2} L_l(\mathbb{Z}[\frac{1}{2}])$. Hence φ vanishes on the Lie ideal $I(\mathbb{Z}[\frac{1}{2}] : \mathbb{Z})$. Therefore φ is unramified everywhere. The implication in the opposite direction is clear. The part iii) of the proposition is also clear. □

5. $\mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0, \infty\} \cup \mu_3)$ and periods of mixed Tate motives over Spec$\mathbb{Z}\left[\frac{1}{3}\right]$ and Spec$\mathbb{Z}[\mu_3]$

In this section and the next one we present more examples when $(L_l(\mathcal{O}_{K,S}))^{\diamond}$ is given by geometric coefficients though without detailed proofs.

Let $U := \mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0, \infty\} \cup \mu_3)$. In [16] we have also studied the Galois representation

$$\varphi_{U,\overrightarrow{01}} : G_{\mathbb{Q}(\mu_3)} \longrightarrow \mathrm{Aut}\big(\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0, \infty\} \cup \mu_3); \overrightarrow{01})\big).$$

The pair $(U, \overrightarrow{01})$ has good reduction outside the prime ideal $(1 - \xi_3)$ of $\mathcal{O}_{\mathbb{Q}(\mu_3)}$, where ξ_3 is a primitive 3rd root of 1. Observe that we have the equality of ideals $(1 - \xi_3)^2 = (3)$. Hence we get a morphism of graded Lie algebras

$$\mathrm{gr}_W \mathrm{Lie}\varphi_{U,\overrightarrow{01}} : L\left(\mathbb{Z}[\mu_3]\left[\frac{1}{3l}\right]; l\right) \longrightarrow (\mathrm{Lie}(X, Y_0, Y_1, Y_2), \{\ \}). \tag{5.0}$$

It follows from Theorem 3.1 that the morphism (5.0) induces

$$L_l(\varphi_{U,\overrightarrow{01}}) : L_l\left(\mathbb{Z}[\mu_3]\left[\frac{1}{3}\right]\right) \longrightarrow (\mathrm{Lie}(X, Y_0, Y_1, Y_2), \{\ \}). \tag{5.1}$$

Proposition 5.2. *The morphism of graded Lie algebras*

$$L_l(\varphi_{U,\overrightarrow{01}}) : L_l\left(\mathbb{Z}[\mu_3]\left[\frac{1}{3}\right]\right) \longrightarrow (\mathrm{Lie}(X, Y_0, Y_1, Y_2), \{\ \}).$$

deduced from the action of $G_{\mathbb{Q}(\mu_3)}$ *on* $\pi_1(U_{\bar{\mathbb{Q}}}; \overrightarrow{01})$ *is injective.*

Proof. The proposition follows from [16], Theorem 15.4.7. □

Corollary 5.3. *All coefficients on $L_l(\mathbb{Z}[\mu_3][\frac{1}{3}])$ are geometric. More precisely we have*

$$\left(L_l\left(\mathbb{Z}[\mu_3]\left[\frac{1}{3}\right]\right)\right)^{\diamond} = \mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01}).$$

Proof. The result follows immediately from Proposition 5.2. □

The rings of algebraic S-integers $\mathbb{Z}[\mu_3]$, $\mathbb{Z}[\frac{1}{3}]$ and $\mathbb{Z}$ are subrings of the ring $\mathbb{Z}[\mu_3][\frac{1}{3}]$. The following result follows immediately from Corollaries 2.6 and 5.3.

Corollary 5.4. *Let us denote by $I(\mu_3)$ the Lie ideal $I(\mathbb{Z}[\mu_3][\frac{1}{3}] : \mathbb{Z}[\mu_3])$ and by $I(\frac{1}{3})$ the Lie ideal $I(\mathbb{Z}[\mu_3][\frac{1}{3}] : \mathbb{Z}[\frac{1}{3}])$. We have:*

i) *The vector space $(L_l(\mathbb{Z}[\mu_3]))^{\diamond}$ is equal to the vector subspace of these elements of $\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01})$, which are unramified everywhere, i.e.,*

$$(L_l(\mathbb{Z}[\mu_3]))^{\diamond} = \{f \in \mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01}) \mid f\big(I(\mu_3)\big) = 0\}.$$

ii) *The vector space $(L_l(\mathbb{Z}[\frac{1}{3}]))^{\diamond}$ is equal to the vector subspace of*

$$\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01})$$

consisting of coefficients which are defined over $\mathbb{Q}$, i.e.,

$$\left(L_l\left(\mathbb{Z}\left[\frac{1}{3}\right]\right)\right)^{\diamond} = \{f \in \mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01}) \mid f\big(I(\frac{1}{3})\big) = 0\}.$$

iii) *The vector space $(L_l(\mathbb{Z}))^{\diamond}$ is equal to the vector subspace of these elements of $\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01})$, which are defined over $\mathbb{Q}$ and unramified everywhere, i.e.,*

$$(L_l(\mathbb{Z}))^{\diamond} = \left\{f \in \mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_3][\frac{1}{3}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_3)} \setminus (\{0,\infty\} \cup \mu_3), \overrightarrow{01}) \mid f\left(I\left(\mathbb{Z}[\mu_3]\left[\frac{1}{3}\right] : \mathbb{Z}\right)\right) = 0\right\}.$$

6. More examples

Let us set $W = \mathbb{P}^1_{\mathbb{Q}(\mu_4)} \setminus (\{0,\infty\} \cup \mu_4)$ and $Z = \mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8)$. The pair $(W, \overrightarrow{01})$ (resp. $(Z, \overrightarrow{01})$) has good reduction outside the prime ideal $(1-i)$ of $\mathbb{Z}[\mu_4]$ (resp. $(1 - e^{\frac{2\pi i}{8}})$ of $\mathbb{Z}[\mu_8]$) lying over (2). Hence it follows from Theorem 3.1 and

from [16], Corollary 15.6.4 and Proposition 15.6.5 that morphisms of graded Lie algebras

$$L_l(\varphi_{W,\overrightarrow{01}}) : L_l\left(\mathbb{Z}[\mu_4]\left[\frac{1}{2}\right]\right) \to (\mathrm{Lie}(X, Y_0, Y_1, Y_2, Y_3), \{\ \})$$

and

$$L_l(\varphi_{Z,\overrightarrow{01}}) : L_l\left(\mathbb{Z}[\mu_8]\left[\frac{1}{2}\right]\right) \to (\mathrm{Lie}(X, Y_0, Y_1, \ldots, Y_8), \{\ \})$$

deduced from the action of $G_{\mathbb{Q}(\mu_4)}$ (resp. $G_{\mathbb{Q}(\mu_8)}$) on $\pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_4); \overrightarrow{01})$ (resp. $\pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus (\{0,\infty\} \cup \mu_8); \overrightarrow{01})$) are injective. Hence we get the following theorem.

Theorem 6.1. *All coefficients on $L_l(\mathbb{Z}[\mu_4][\frac{1}{2}])$ and on $L_l(\mathbb{Z}[\mu_8][\frac{1}{2}])$ are geometric, more precisely*

$$\left(L_l\left(\mathbb{Z}[\mu_4]\left[\frac{1}{2}\right]\right)\right)^{\diamond} = \mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_4][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_4)} \setminus (\{0,\infty\} \cup \mu_4), \overrightarrow{01})$$

and

$$\left(L_l\left(\mathbb{Z}[\mu_8]\left[\frac{1}{2}\right]\right)\right)^{\diamond} = \mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \overrightarrow{01}).$$

The rings of algebraic S-integers $\mathbb{Z}[\mu_4]$, $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}$ are subrings of $\mathbb{Z}[\mu_4][\frac{1}{2}]$, while $\mathbb{Z}[\mu_8]$, $\mathbb{Z}[\sqrt{2}][\frac{1}{2}]$, $\mathbb{Z}[\sqrt{2}]$, $\mathbb{Z}[\sqrt{-2}][\frac{1}{2}]$, $\mathbb{Z}[\sqrt{-2}]$ and also $\mathbb{Z}[\mu_4][\frac{1}{2}]$, $\mathbb{Z}[\mu_4]$, $\mathbb{Z}[\frac{1}{2}]$ and $\mathbb{Z}$ are subrings of $\mathbb{Z}[\mu_8][\frac{1}{2}]$. Hence we get the following result.

Corollary 6.2. *Let us denote by $I(\mu_4)$ the Lie ideal $I(\mathbb{Z}[\mu_4][\frac{1}{2}] : \mathbb{Z}[\mu_4])$. We have*

i) *The vector space $(L_l(\mathbb{Z}[\mu_4]))^{\diamond}$ is equal to the vector subspace of*

$$\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_4][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_4)} \setminus (\{0,\infty\} \cup \mu_4), \overrightarrow{01})$$

consisting of the coefficients which are unramified everywhere, i.e.,

$$(L_l(\mathbb{Z}[\mu_4]))^{\diamond}$$
$$= \{f \in \mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_4][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_4)} \setminus (\{0,\infty\} \cup \mu_4), \overrightarrow{01}) \mid f\big(I(\mu_4)\big) = 0\}.$$

ii) *The vector space $(L_l(\mathbb{Z}[\mu_8]))^{\diamond}$ is equal to the vector subspace of*

$$\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \overrightarrow{01})$$

consisting of the coefficients which are unramified everywhere.

iii) *The vector space $(L_l(\mathbb{Z}[\sqrt{2}][\frac{1}{2}]))^{\diamond}$ is equal to the vector subspace of*

$$\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \overrightarrow{01})$$

consisting of coefficients which are defined over $\mathbb{Q}(\sqrt{2})$.

iv) *The vector space* $(L_l(\mathbb{Z}[\sqrt{2}]))^\diamond$ *is equal to the vector subspace of*

$$\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \overrightarrow{01})$$

consisting of coefficients which are unramified everywhere and defined over $\mathbb{Q}(\sqrt{2})$.

v) *The vector space* $(L_l(\mathbb{Z}[\sqrt{-2}][\frac{1}{2}]))^\diamond$ *is equal to the vector subspace of*

$$\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \overrightarrow{01})$$

consisting of coefficients which are defined over $\mathbb{Q}(\sqrt{-2})$.

vi) *The vector space* $(L_l(\mathbb{Z}[\sqrt{-2}]))^\diamond$ *is equal to the vector subspace of*

$$\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_8][\frac{1}{2}]}(\mathbb{P}^1_{\mathbb{Q}(\mu_8)} \setminus (\{0,\infty\} \cup \mu_8), \overrightarrow{01})$$

consisting of coefficients which are unramified everywhere and defined over $\mathbb{Q}(\sqrt{-2})$.

7. Periods of mixed Tate motives

Assuming the motivic formalism as in [1], we shall show here the result announced at the beginning of the paper.

Theorem 7.1. *The* $\mathbb{Q}$*-algebra of periods of mixed Tate motives over* $\mathrm{Spec}\mathbb{Z}$ *is generated by these linear combinations with* $\mathbb{Q}$*-coefficients of iterated integrals on* $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\}$ *of sequences of one forms* $\frac{dz}{z}$, $\frac{dz}{z-1}$ *and* $\frac{dz}{z+1}$ *from* $\overrightarrow{01}$ *to* $\overrightarrow{10}$, *which are unramified everywhere.*

Before giving the proof of the theorem we recall some facts about mixed Tate motives.

As in [1] we assume that the category $\mathcal{MT}_{\mathcal{O}_{K,S}}$ of mixed Tate motives over $\mathrm{Spec}\mathcal{O}_{K,S}$ exists and has all good properties. In particular the category $\mathcal{MT}_{\mathcal{O}_{K,S}}$ is a tannakian category over $\mathbb{Q}$. Let $\mathcal{G}(\mathcal{O}_{K,S})$ be the motivic fundamental group of the category $\mathcal{MT}_{\mathcal{O}_{K,S}}$ and let $\mathcal{U}(\mathcal{O}_{K,S}) := \mathrm{Ker}(\mathcal{G}(\mathcal{O}_{K,S}) \to \mathbb{G}_m)$. We have various realization functors from the category $\mathcal{MT}_{\mathcal{O}_{K,S}}$. In particular we have the Hodge–de Rham realization functor to the category of mixed Hodge structures over $\mathrm{Spec}\mathcal{O}_{K,S}$;

$$\mathrm{real}_{H-DR} : \mathcal{MT}_{\mathcal{O}_{K,S}} \to MHS_{\mathcal{O}_{K,S}}, \;\; M \to \Big((M_{DR}, W, F),\ (M_{B,\sigma}, W)_{\sigma:K\to\mathbb{C}},$$
$$(\mathrm{comp}_{M,\sigma} : (M_{B,\sigma}\otimes\mathbb{C}, W) \xrightarrow{\approx} (M_{DR}\otimes_\sigma\mathbb{C}, W))_{\sigma:K\to\mathbb{C}}\Big).$$

Let V be a smooth quasi-projective algebraic variety over $\mathrm{Spec}K$. Let us assume that V has good reduction outside S. Let M be a mixed motive determined by V. Then $M_{DR} = H^*_{DR}(V)$ equipped with weight and Hodge filtrations. For any $\sigma : K \subset \mathbb{C}$, let $V_\sigma := V \times_\sigma \mathrm{Spec}\mathbb{C}$. Let $V_\sigma(C)$ be the set of $\mathbb{C}$-points of V_σ. Then $M_{B,\sigma} = H^*(V_\sigma(\mathbb{C}); \mathbb{Q})$ equipped with weight filtration. The isomorphism $\mathrm{comp}_{M,\sigma}$ is the comparison isomorphism $H^*(V_\sigma(\mathbb{C}); \mathbb{Q})\otimes\mathbb{C} \to H^*_{DR}(V)\otimes_\sigma\mathbb{C}$.

From now on we assume that $K = \mathbb{Q}$ and S is a finite set of finite places of $\mathbb{Q}$. Then the ring $\mathcal{O}_{\mathbb{Q},S} = \mathbb{Z}[\frac{1}{m}]$ for some $m \in \mathbb{Z}$. Hence we shall write $\mathbb{Z}[\frac{1}{m}]$ instead of $\mathcal{O}_{\mathbb{Q},S}$.

We have two fiber functors on $\mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]}$ with values in vector spaces over $\mathbb{Q}$: the Betti realization functor

$$F_B : \mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]} \to \mathrm{Vect}_{\mathbb{Q}};\ M \to M_B$$

and the de Rham realization functor

$$F_{DR} : \mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]} \to \mathrm{Vect}_{\mathbb{Q}},\ M \to M_{DR}.$$

These two fiber functors are isomorphic. Let $(s_M)_{M \in Ob\mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]}} \in \mathrm{Iso}^{\otimes}(F_{DR}, F_B)$ be an isomorphism between the fiber functors F_{DR} and F_B. For each $M \in \mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]}$ let α_M be the composition

$$M_{DR}\otimes\mathbb{C} \xrightarrow{s_M \otimes id_{\mathbb{C}}} M_B\otimes\mathbb{C} \xrightarrow{\mathrm{comp}_M} M_{DR}\otimes\mathbb{C}.$$

Then $\alpha := (\alpha_M)_{M \in Ob\mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]}}$ is an automorphism of the fiber functor

$$F_{DR}\otimes\mathbb{C} : \mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]} \to \mathrm{Vect}_{\mathbb{C}};\ \text{ given by } \ (F_{DR}\otimes\mathbb{C})(M) = M_{DR}\otimes\mathbb{C}.$$

Hence $\alpha \in \mathrm{Aut}^{\otimes}(F_{DR}\otimes\mathbb{C})$, the group of automorphisms of the fiber functor $F_{DR}\otimes\mathbb{C}$. The group $\mathrm{Aut}^{\otimes}(F_{DR}\otimes\mathbb{C})$ is the group of $\mathbb{C}$-points of $\mathcal{G}_{DR}(\mathbb{Z}\left[\frac{1}{m}\right]) = \mathrm{Aut}^{\otimes}(F_{DR})$. Observe that the group $\mathcal{G}_{DR}(\mathbb{Z}\left[\frac{1}{m}\right])(\mathbb{C})$ acts on $M_{DR}\otimes\mathbb{C}$ for any $M \in \mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]}$ and

$$\alpha(M_{DR}) = \mathrm{comp}_M(M_B) \subset M_{DR}\otimes\mathbb{C}. \tag{7.2}$$

We denote the element α by $\alpha_{\mathbb{Z}[\frac{1}{m}]}$. Observe that $\mathrm{comp}_M(M_B)$ is the Betti lattice in $M_{DR}\otimes\mathbb{C}$ and its coordinates with respect to any base of the $\mathbb{Q}$-vector space M_{DR} are periods of the mixed Tate motive M.

Definition 7.3. We denote by $\mathrm{Periods}(M)$ the $\mathbb{Q}$-algebra generated by periods of a mixed Tate motive M.

It is clear that the $\mathbb{Q}$-algebra $\mathrm{Periods}(M)$ does not depend on a choice of a base of M_{DR}.

The element $\alpha_{\mathbb{Z}[\frac{1}{m}]} \in \mathcal{G}_{DR}(\mathbb{Z}\left[\frac{1}{m}\right])(\mathbb{C})$. The group scheme $\mathcal{G}_{DR}(\mathbb{Z}\left[\frac{1}{m}\right])$ is an affine group scheme over $\mathbb{Q}$, hence

$$\mathcal{G}_{DR}\left(\mathbb{Z}\left[\frac{1}{m}\right]\right) = \mathrm{Spec}(\mathcal{A}_{\mathbb{Z}[\frac{1}{m}]}),$$

where $\mathcal{A}_{\mathbb{Z}[\frac{1}{m}]}$ is the $\mathbb{Q}$-algebra of polynomial functions on $\mathcal{G}_{DR}(\mathbb{Z}\left[\frac{1}{m}\right])$.

Definition 7.4. We set

$$\mathrm{UnivPeriods}\left(\mathbb{Z}\left[\frac{1}{m}\right]\right) := \left\{ f(\alpha_{\mathbb{Z}[\frac{1}{m}]}) \in \mathbb{C} \ \mid \ f \in \mathcal{A}_{\mathbb{Z}[\frac{1}{m}]} \right\}.$$

The set $\mathrm{UnivPeriods}(\mathbb{Z}\left[\frac{1}{m}\right])$ is a $\mathbb{Q}$-algebra. Observe that we have a surjective morphism of $\mathbb{Q}$-algebras

$$\mathcal{A}_{\mathbb{Z}[\frac{1}{m}]} \longrightarrow \mathrm{UnivPeriods}\left(\mathbb{Z}\left[\frac{1}{m}\right]\right);\ f \to f(\alpha_{\mathbb{Z}[\frac{1}{m}]}).$$

The usual conjecture about periods is that this morphism of $\mathbb{Q}$-algebras is an isomorphism.

Proposition 7.5. *For any mixed Tate motive M over* $\mathrm{Spec}\mathbb{Z}\left[\frac{1}{m}\right]$*, the $\mathbb{Q}$-algebra* $\mathrm{Periods}(M)$ *is a $\mathbb{Q}$-subalgebra of the $\mathbb{Q}$-algebra* $\mathrm{UnivPeriods}(\mathbb{Z}\left[\frac{1}{m}\right])$.

Proof. It follows immediately from the formula (7.2). □

Another easy observation is the following one.

Proposition 7.6. *We have*

$$\mathrm{UnivPeriods}\left(\mathbb{Z}\left[\frac{1}{m}\right]\right) = \bigcup_{M\in\mathcal{MT}_{\mathbb{Z}[\frac{1}{m}]}} \mathrm{Periods}(M).$$

Now we shall study relations between periods of mixed Tate motives over different subrings of $\mathbb{Q}$.

Proposition 7.7. *For any relatively prime positive integers m and n, the $\mathbb{Q}$-algebra* $\mathrm{UnivPeriods}(\mathbb{Z}\left[\frac{1}{m}\right])$ *is a $\mathbb{Q}$-subalgebra of the $\mathbb{Q}$-algebra*

$$\mathrm{UnivPeriods}\left(\mathbb{Z}\left[\frac{1}{m\cdot n}\right]\right).$$

Proof. Let M be a mixed Tate motive over $\mathrm{Spec}\mathbb{Z}\left[\frac{1}{m}\right]$. Then M is also a mixed Tate motive over $\mathrm{Spec}\mathbb{Z}[\frac{1}{m\cdot n}]$. But in both cases the Betti and the De Rham lattices in $M_{DR}\otimes\mathbb{C}$ are the same. Hence the proposition follows from Proposition 7.6. □

Definition 7.8. Let m and n be relatively prime, positive integers. We say that $\lambda \in \mathrm{UnivPeriods}(\mathbb{Z}[\frac{1}{m\cdot n}])$ is unramified outside m if $\lambda \in \mathrm{UnivPeriods}(\mathbb{Z}\left[\frac{1}{m}\right])$.

Examples 7.9. Let $z \in \mathbb{Q}^\times$ be such that $1-z \in \mathbb{Q}^\times$. The triple $(\mathbb{P}^1\setminus\{0,1,\infty\}, z, \overrightarrow{01})$ has good reduction outside the set S of primes which appear in the decomposition of the product $z(1-z)$. The mixed Hodge structure of the torsor of paths $\pi(\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}; z, \overrightarrow{01})$ is described by iterated integrals of sequences of one-forms $\frac{dz}{z}$ and $\frac{dz}{z-1}$ from $\overrightarrow{01}$ to z and from $\overrightarrow{01}$ to $\overrightarrow{10}$ on $\mathbb{P}^1(\mathbb{C})\setminus\{0,1,\infty\}$. Hence the numbers $\log z$, $\log(1-z)$, $Li_2(z),\dots,Li_n(z),\dots$ belong to $\mathrm{UnivPeriods}(\mathcal{O}_{\mathbb{Q},S})$.

Let p be a prime number. The pair $(\mathbb{P}^1\setminus\{0,1,\infty\}, \overrightarrow{0p})$ has good reduction outside p. Using the definition of iterated integrals starting from tangential points (see [20]) one gets that $\int_{\overrightarrow{0p}}^{\overrightarrow{10}} \frac{dz}{z} = \log p$. Hence $\log p \in \mathrm{UnivPeriods}(\mathbb{Z}[\frac{1}{p}])$.

Now we restrict our attention to $\mathbb{Z}$ and $\mathbb{Z}\left[\frac{1}{2}\right]$. First we present the result of Deligne from the conference in Schloss Ringberg (see [5]). The result of Deligne is also in his recent preprint (see [6]).

The mixed Hodge structure on $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\};\overrightarrow{01})$ is entirely described by the formal power series $\Lambda_{\overrightarrow{01}}(\overrightarrow{10})$ belonging to $\mathbb{C}\{\{X,Y_0,Y_1\}\}$, whose coefficients are iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\}$ of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$. Observe that the pair $(\mathbb{P}^1 \setminus \{0,1,-1,\infty\}, \overrightarrow{01})$ has good reduction outside (2). Hence the mixed Tate motive associated with $\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\};\overrightarrow{01})$ is over $\mathrm{Spec}\mathbb{Z}\left[\frac{1}{2}\right]$. The result of Deligne can be formulated in the following way.

Theorem 7.10. *The morphism*

$$\mathcal{G}_{DR}\left(\mathbb{Z}\left[\frac{1}{2}\right]\right)(\mathbb{C}) \longrightarrow \mathrm{Aut}\big(\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,-1,\infty\};\overrightarrow{01})_{DR}\otimes\mathbb{C}\big)$$

is injective.

The following corollary is an immediate consequence of the theorem.

Corollary 7.11. *The $\mathbb{Q}$-algebra* $\mathrm{UnivPeriods}(\mathbb{Z}\left[\frac{1}{2}\right])$ *is generated by all iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\}$ of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$.*

Proof. Let us denote by $\mathrm{Motive}(\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\};\overrightarrow{01}))$ the mixed Tate motive over $\mathrm{Spec}\mathbb{Z}\left[\frac{1}{2}\right]$ associated with the fundamental group

$$\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\};\overrightarrow{01}).$$

It follows from Theorem 7.10 that

$$\mathrm{Periods}\big(\mathrm{Motive}(\pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,-1,\infty\};\overrightarrow{01}))\big) = \mathrm{UnivPeriods}\left(\mathbb{Z}\left[\frac{1}{2}\right]\right).$$

By (7.2) the Betti lattice of $\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,-1,\infty\};\overrightarrow{01})_{DR}\otimes\mathbb{C}$ is given by

$$\alpha_{\mathbb{Z}\left[\frac{1}{2}\right]}\big(\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus \{0,1,-1,\infty\};\overrightarrow{01})_{DR}\big).$$

But on the other side it is explicitly given by the formal power series $\Lambda_{\overrightarrow{01}}(\overrightarrow{10}) \in \mathbb{C}\{\{X,Y_0,Y_1\}\}$. Hence it follows that the algebra $\mathrm{UnivPeriods}(\mathbb{Z}\left[\frac{1}{2}\right])$ is generated by the coefficients of the formal power series $\Lambda_{\overrightarrow{01}}(\overrightarrow{10})$. □

Proof of Theorem 7.1. It follows from Proposition 7.7 that $\mathrm{UnivPeriods}(\mathbb{Z})$ is a $\mathbb{Q}$-subalgebra of $\mathrm{UnivPeriods}(\mathbb{Z}\left[\frac{1}{2}\right])$. Hence it follows from Corollary 7.11 that the $\mathbb{Q}$-algebra $\mathrm{UnivPeriods}(\mathbb{Z})$ is generated by certain products of sums of some iterated integrals of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ on

$\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}$. A product of iterated integrals is a sum of iterated integrals by the formula of Chen (see [3]), which is also valid for iterated integrals from tangential points to tangential points (see [20]). Hence the $\mathbb{Q}$-algebra UnivPeriods($\mathbb{Z}$) is generated by certain linear combinations with $\mathbb{Q}$-coefficients of iterated integrals of sequences of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$ and $\frac{dz}{z+1}$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, -1, \infty\}$. By the very definition (see Definition 7.8) such linear combinations are unramified everywhere. □

8. Relations in the image of the Galois representations on fundamental groups

Let p be an odd prime. The pair $(\mathbb{P}^1_{\mathbb{Q}(\mu_p)} \setminus (\{0, \infty\} \cup \mu_p), \overrightarrow{01})$ has good reduction outside (p). The Galois group $G_{\mathbb{Q}(\mu_p)}$ acts on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0, \infty\} \cup \mu_p); \overrightarrow{01})$. After the standard embedding of $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0, \infty\} \cup \mu_p); \overrightarrow{01})$ into $\mathbb{Q}_l\{\{X, Y_0, \ldots, Y_{p-1}\}\}$ we get the Galois representation

$$\varphi_{\overrightarrow{01}} : G_{\mathbb{Q}(\mu_p)} \to \mathrm{Aut}\big(\mathbb{Q}_l\{\{X, Y_0, \ldots, Y_{p-1}\}\}\big)$$

(see [16]). It follows from Theorem 3.1 that $\varphi_{\overrightarrow{01}}$ induces the morphism of graded Lie algebras

$$L_l(\varphi_{\overrightarrow{01}}) : L_l\left(\mathbb{Z}[\mu_p]\left[\frac{1}{p}\right]\right) \longrightarrow \mathrm{Der}^*_{\mathbb{Z}/p}(\mathrm{Lie}(X, Y_0, \ldots, Y_{p-1})).$$

(See [16], where the Lie algebra of special derivations $\mathrm{Der}^*_{\mathbb{Z}/p}(\mathrm{Lie}(X, Y_0, \ldots, Y_{p-1}))$ is defined.) The following result generalizes our partial results for $p = 5$ (see [17], Proposition 20.5) and for $p = 7$ (see [7], Theorem 4.1).

Proposition 8.1. *Let p be an odd prime.*

i) *In the image of the morphism of graded Lie algebras*

$$L_l(\varphi_{\overrightarrow{01}}) : L_l\left(\mathbb{Z}[\mu_p]\left[\frac{1}{p}\right]\right) \longrightarrow \mathrm{Der}^*_{\mathbb{Z}/p}(\mathrm{Lie}(X, Y_0, \ldots, Y_{p-1}))$$

there are linearly independent over $\mathbb{Q}_l$ derivations τ_i for $1 \le i \le \frac{p-1}{2}$ such that

$$\tau_i(Y_0) = [Y_0, Y_i + Y_{p-i}].$$

ii) *There are the following relations between commutators*

$$\mathcal{R}_k : \quad \left[\tau_k; \sum_{i=1}^{\frac{p-1}{2}} \tau_i\right] = 0 \quad \text{for} \quad 1 \le k \le \frac{p-1}{2}$$

and between relations

$$\sum_{i=k}^{\frac{p-1}{2}} \mathcal{R}_k = 0.$$

Proof. The equality $\xi_p^i(1-\xi_p^{p-i}) = -(1-\xi_p^i)$ implies that $l(1-\xi_p^{p-i}) = l(1-\xi_p^i)$ on $L_l(\mathbb{Z}[\mu_p][\frac{1}{p}])$. Elements $1-\xi_p^i$ for $1 \le i \le \frac{p-1}{2}$ are linearly independent in the $\mathbb{Z}$-module $\mathbb{Z}[\mu_p]^\times$. Hence the point i) of the proposition follows from [16], Lemma 15.3.2.

To show the point ii) we need to calculate the Lie bracket

$$\left[\tau_k; \sum_{i=1}^{\frac{p-1}{2}} \tau_i\right]$$

in the Lie algebra of special derivations $\mathrm{Der}^*_{\mathbb{Z}/p}(\mathrm{Lie}(X, Y_0, \ldots, Y_{p-1}))$. We recall that the Lie algebra $\mathrm{Der}^*_{\mathbb{Z}/p}(\mathrm{Lie}(X, Y_0, \ldots, Y_{p-1}))$ is isomorphic to the Lie algebra $(\mathrm{Lie}(X, Y_0, \ldots, Y_{p-1}), \{\,\})$ (see [16]), hence we can do all the calculations in the Lie algebra $(\mathrm{Lie}(X, Y_0, \ldots, Y_{p-1}), \{\,\})$. We have

$$\begin{aligned}
\left[\tau_k; \sum_{i=1}^{\frac{p-1}{2}} \tau_i\right] &= \left\{Y_k + Y_{p-k}, \sum_{i=1}^{\frac{p-1}{2}} (Y_i + Y_{p-i})\right\} = \left\{Y_k + Y_{p-k}, \sum_{i=0}^{p-1} Y_i\right\} \\
&= \left[Y_k, \sum_{i=0}^{p-1} Y_i\right] + \sum_{i=0}^{p-1} [Y_i, Y_{i+k}] - \sum_{i=0}^{p-1} [Y_k, Y_{k+i}] \\
&\quad + \left[Y_{p-k}, \sum_{i=0}^{p-1} Y_i\right] + \sum_{i=0}^{p-1} [Y_i, Y_{i+p-k}] - \sum_{i=0}^{p-1} [Y_{p-k}, Y_{i+p-k}] = 0.
\end{aligned}$$

The relation $[\sum_{k=1}^{\frac{p-1}{2}} \tau_k, \sum_{i=1}^{\frac{p-1}{2}} \tau_i] = 0$ holds in any Lie algebra, hence we have a relation $\sum_{k=1}^{\frac{p-1}{2}} \mathcal{R}_k = 0$ between the relations. □

9. An example of a missing coefficient

We finish our paper with an example showing that one can deal with a single coefficient. We shall use notations and results from our papers [16] and [17].

Let p be an odd prime. It follows from Proposition 1.3 that

$$\left(L_l\left(\mathbb{Z}[\mu_p]\left[\frac{1}{p}\right]\right)_1\right)^\diamond \approx \left(\mathbb{Z}[\mu_p]\left[\frac{1}{p}\right]\right)^\times \otimes \mathbb{Q}_l.$$

Observe that the elements $1-\xi_p^i$, $1 \le i \le \frac{p-1}{2}$ generate freely a free $\mathbb{Z}$-module of maximal rank in $\mathbb{Z}[\mu_p][\frac{1}{p}]^\times$. Hence $\dim(L_l(\mathbb{Z}[\mu_p][\frac{1}{p}])_1) = \frac{p-1}{2}$ and elements $T_1, \ldots, T_{\frac{p-1}{2}}$ dual to the Kummer characters $\kappa(1-\xi_p^1), \ldots, \kappa(1-\xi_p^{\frac{p-1}{2}})$ form a base of $L_l(\mathbb{Z}[\mu_p][\frac{1}{p}])_1$. The elements $T_1, \ldots, T_{\frac{p-1}{2}}$ generate freely a free Lie subalgebra of $L_l(\mathbb{Z}[\mu_p][\frac{1}{p}])$.

The elements $\tau_1, \dots, \tau_{\frac{p-1}{2}}$ from Proposition 8.1 are also dual to the Kummer characters $\kappa(1-\xi_p^1), \dots, \kappa(1-\xi_p^{\frac{p-1}{2}})$ by the very construction. Hence we have

$$L_l(\varphi_{\overrightarrow{01}})(T_i) = \tau_i \quad \text{for} \quad 1 \le i \le \frac{p-1}{2},$$

where $L_l(\varphi_{\overrightarrow{01}}) : L_l(\mathbb{Z}[\mu_p][\frac{1}{p}]) \to \mathrm{Der}^*_{\mathbb{Z}/p}(\mathrm{Lie}(X, Y_0, \dots, Y_{p-1}))$ is the morphism from Proposition 8.1. However we have the relations

$$\left[\tau_k; \sum_{i=1}^{\frac{p-1}{2}} \tau_i\right] = 0 \quad \text{for} \quad 1 \le k \le \frac{p-1}{2}.$$

Therefore in degree 2 we have

$$\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_p][\frac{1}{p}]}\big(\mathbb{P}^1_{\mathbb{Q}(\mu_p)} \setminus (\{0,\infty\} \cup \mu_p), \overrightarrow{01}\big)_2 \subset \left(L_l\left(\mathbb{Z}[\mu_p]\left[\frac{1}{p}\right]\right)\right)^\diamond_2$$

but

$$\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_p][\frac{1}{p}]}\big(\mathbb{P}^1_{\mathbb{Q}(\mu_p)} \setminus (\{0,\infty\} \cup \mu_p), \overrightarrow{01}\big)_2 \neq \left(L_l\left(\mathbb{Z}[\mu_p]\left[\frac{1}{p}\right]\right)\right)^\diamond_2$$

for $p > 3$.

The obvious question is how to construct geometric coefficients in degree 2 (or periods of mixed Tate motives over $\mathrm{Spec}\mathbb{Z}[\mu_p][\frac{1}{p}]$ in degree 2) which are dual to Lie brackets $[T_i, T_j]$ for $(i < j)$. It is clear from Proposition 8.1 that there is not enough coefficients in $\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_p][\frac{1}{p}]}\big(\mathbb{P}^1_{\mathbb{Q}(\mu_p)} \setminus (\{0,\infty\} \cup \mu_p), \overrightarrow{01}\big)$ if $p > 3$.

We consider only the simplest case $p = 5$. It follows from Proposition 8.1 (see also [17], Proposition 20.5) that there is a coefficient of degree 2 in $(L_l(\mathbb{Z}[\mu_5][\frac{1}{5}]))^\diamond$, which does not belong to $\mathrm{GeomCoeff}^l_{\mathbb{Z}[\mu_5][\frac{1}{5}]}\big(\mathbb{P}^1_{\mathbb{Q}(\mu_5)} \setminus (\{0,\infty\} \cup \mu_5), \overrightarrow{01}\big)$. We shall construct this missing coefficient using the action of $G_{\mathbb{Q}(\mu_{10})} = G_{\mathbb{Q}(\mu_5)}$ on $\pi_1(\mathbb{P}^1_{\mathbb{Q}} \setminus (\{0,\infty\} \cup \mu_{10}); \overrightarrow{01})$. The pair $(\mathbb{P}^1_{\mathbb{Q}(\mu_{10})} \setminus (\{0,\infty\} \cup \mu_{10}), \overrightarrow{01})$ has good reduction outside prime divisors of (10).

Observe that $\dim(\mathbb{Z}[\mu_{10}][\frac{1}{10}]^\times \otimes \mathbb{Q}) = 3$. Hence $\dim L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])_1 = 3$. There are the following relations in $\mathbb{Z}[\mu_{10}][\frac{1}{10}]^\times$ modulo torsion

$$(1-\xi_{10}^i) = (1-\xi_{10}^{-i}), \quad (1-\xi_{10}^i)(1-\xi_{10}^{5+i}) = (1-\xi_5^i) \quad \text{and} \quad (1-\xi_{10}^5) = 2\,. \tag{9.1.a}$$

Hence we get

$$(1-\xi_{10}^1) = (1-\xi_{10}^3)^{-1} = (1-\xi_5^1)(1-\xi_5^2)^{-1}\,. \tag{9.1.b}$$

Therefore the Kummer characters $l(1-\xi_5^1)$, $l(1-\xi_5^2)$ and $l(2)$ form a base of $(L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}]))^\diamond_1$ and $l(1-\xi_5^1)$, $l(1-\xi_5^2)$ form a base of $(L_l(\mathbb{Z}[\mu_5][\frac{1}{5}]))^\diamond_1$. Let S_1, S_2, N (resp. s_1, s_2) be the base of $L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])_1$ (resp. $L_l(\mathbb{Z}[\mu_5][\frac{1}{5}])_1$) dual to the base $l(1-\xi_5^1)$, $l(1-\xi_5^2)$ and $l(2)$ (resp. $l(1-\xi_5^1)$, $l(1-\xi_5^2)$). Then the morphism

$$\Pi := \pi^{\mathbb{Q}[\mu_{10}],\{(5),(2)\}}_{\mathbb{Q}[\mu_5],(5)} : L_l\left(\mathbb{Z}[\mu_{10}]\left[\frac{1}{10}\right]\right) \longrightarrow L_l\left(\mathbb{Z}[\mu_5]\left[\frac{1}{5}\right]\right)$$

is given in degree 1 by the formulas

$$\Pi(S_1) = s_1, \quad \Pi(S_2) = s_2, \quad \Pi(N) = 0.$$

Hence it follows the following result.

Lemma 9.2 *The Lie ideal $I(\mathbb{Z}[\mu_{10}][\frac{1}{10}] : \mathbb{Z}[\mu_5][\frac{1}{5}])$ of the Lie algebra $L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])$ is generated by the element N.*

Let us fix a Hall base $\mathcal{B}$ of the free Lie algebra $\mathrm{Lie}(X, Y_0, \ldots, Y_9)$. If $e \in \mathcal{B}$ we denote by $e^\diamond$ the dual linear form on $\mathrm{Lie}(X, Y_0, \ldots, Y_9)$ with respect to $\mathcal{B}$.

We have the following result.

Proposition 9.3. *We have:*

i) *In degree* 1 *the image of the morphism*

$$L_l(\varphi_{\overrightarrow{01}}) : L_l\left(\mathbb{Z}[\mu_{10}]\left[\frac{1}{10}\right]\right) \longrightarrow (\mathrm{Lie}(X, Y_0, \ldots, Y_9), \{\ \})$$

induced by the action of $G_{\mathbb{Q}(\mu_{10})}$ on $\pi_1(\mathbb{P}^1_{\bar{\mathbb{Q}}} \setminus (\{0, \infty\} \cup \mu_{10}), \overrightarrow{01})$ is generated by $\sigma_1 := Y_1 + Y_9 + Y_2 + Y_8 - Y_3 - Y_7$, $\sigma_2 := -Y_1 - Y_9 + Y_4 + Y_6 + Y_3 + Y_7$ and $\eta := Y_5$.

ii) *The Lie bracket*

$$\begin{aligned}\{\sigma_1, \sigma_2\} = {} & [Y_1, 2Y_4 + Y_6 + 2Y_8] + [Y_9, 2Y_2 + Y_4 + 2Y_6] \\ & - [Y_3, 2Y_2 + 2Y_4 + Y_8] - [Y_7, Y_2 + 2Y_6 + 2Y_8] \\ & + [-Y_2 - Y_8 + Y_4 + Y_6 - Y_1 - Y_9 + Y_3 + Y_7, Y_5] \\ & + 2[Y_3 + Y_7, Y_1 + Y_9].\end{aligned}$$

iii) *Let $\mathcal{F} := [Y_1, Y_8]^\diamond \circ L_l(\varphi_{\overrightarrow{01}})$. Then $\mathcal{F} \neq 0$ and $\mathcal{F}$ vanishes on the Lie ideal $I(\mathbb{Z}[\mu_{10}][\frac{1}{10}] : \mathbb{Z}[\mu_5][\frac{1}{5}])$. Hence $\mathcal{F}$ defines a non trivial linear form of degree 2 on $L_l(\mathbb{Z}[\mu_5][\frac{1}{5}])$ non vanishing on $\Gamma^2 L_l(\mathbb{Z}[\frac{1}{5}])$, i.e., $\mathcal{F}([s_1, s_2]) \neq 0$.*

Proof. Let $S \in L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])_1$. Then it follows from [16] that

$$L_l(\varphi_{\overrightarrow{01}})(S) = \sum_{i=1}^{9} l(1 - \xi_{10}^{-i})(S) Y_i\ .$$

It follows from the relations (9.1.a) and (9.1.b) and the very definition of the elements S_1, S_2 and N of $L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])_1$ that $\sigma_1 := L_l(\varphi_{\overrightarrow{01}})(S_1) = Y_1 + Y_9 + Y_2 + Y_8 - Y_3 - Y_7$, $\sigma_2 := L_l(\varphi_{\overrightarrow{01}})(S_2) = -Y_1 - Y_9 + Y_4 + Y_6 + Y_3 + Y_7$ and $\eta := L_l(\varphi_{\overrightarrow{01}})(N) = Y_5$. The elements S_1, S_2 and N form a base of $L_l(\mathbb{Z}[\mu_{10}][\frac{1}{10}])_1$. Hence σ_1, σ_2, η generate the image of $L_l(\varphi_{\overrightarrow{01}})$ in degree 1.

To show the point ii) one calculates the Lie bracket $\{\sigma_1, \sigma_2\}$.

Let $\mathcal{F} := [Y_1, Y_8]^\diamond \circ L_l(\varphi_{\overrightarrow{01}})$. Then $\mathcal{F}([S_1, S_2]) = [Y_1, Y_8]^\diamond(\{\sigma_1, \sigma_2\}) = 2$. Therefore we have $\mathcal{F} \neq 0$.

The Lie ideal $I(\mathbb{Z}[\mu_{10}][\frac{1}{10}] : \mathbb{Z}[\mu_5][\frac{1}{5}])$ has a base $[S_1, N]$, $[S_2, N]$ in degree 2. Observe that $\mathcal{F}([S_i, N]) = [Y_1, Y_8]^\diamond(\{\sigma_i, \eta\}) = 0$ because the Lie brackets $[Y_a, Y_b]$

appearing in $\{\sigma_i, \eta\}$ contain Y_5 or the difference $a-b$ is 5 or -5. Therefore $\mathcal{F}$ vanishes on the Lie ideal $I(\mathbb{Z}[\mu_{10}][\frac{1}{10}] : \mathbb{Z}[\mu_5][\frac{1}{5}])$. Hence it follows that $\mathcal{F}$ defines a linear form $\bar{\mathcal{F}}$ on $L_l(\mathbb{Z}[\mu_5][\frac{1}{5}])$ such that $\bar{\mathcal{F}}([s_1, s_2]) = 2$. □

Corollary 9.4. *Any element of* $\left(L_l(\mathbb{Z}[\mu_5][\frac{1}{5}])\right)_i^{\diamond}$ *for* $i \leq 2$ *is geometric.*

Remark 9.5. There are three linearly independent over $\mathbb{Q}$ periods of mixed Tate motives over $\mathrm{Spec}\mathbb{Z}[\mu_5][\frac{1}{5}]$ in degree 2, $Li_2(\xi_5^1)$, $Li_2(\xi_5^2)$ and the third one, which we denote by λ_2. One cannot get this third period λ_2 as an iterated integral on $\mathbb{P}^1(\mathbb{C}) \setminus (\{0, \infty\} \cup \mu_5)$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ of a sequence of length two of one-forms $\frac{dz}{z}$, $\frac{dz}{z-1}$, $\frac{dz}{z-\xi_5^k}$ for $k = 1, 2, 3, 4$. One gets λ_2 as a linear combination with $\mathbb{Q}$-coefficients of iterated integrals on $\mathbb{P}^1(\mathbb{C}) \setminus (\{0, \infty\} \cup \mu_{10})$ from $\overrightarrow{01}$ to $\overrightarrow{10}$ of sequences of length two of one-forms $\frac{dz}{z}$ and $\frac{dz}{z-\xi_{10}^k}$ for $k = 0, 1, 2, \ldots, 9$.

Note added 9.6. The formula ii) of Proposition 8.1 was also communicated by P. Deligne to H. Nakamura in his letter of August 31, 2009.

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On Totally Ramified Extensions of Discrete Valued Fields

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Abstract. We give a simple characterization of the totally wild ramified valuations in a Galois extension of fields of characteristic p. This criterion involves the valuations of Artin-Schreier cosets of the $\mathbb{F}_{p^r}^\times$-translation of a single element. We apply the criterion to construct some interesting examples.

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1. Introduction

Let F/E be a Galois extension of fields of characteristic p of degree q, a power of p. This work gives a simple criterion that classifies the totally ramified discrete valuations of F/E.

The classical case where F/E is a p-extension, hence generated by a root of an Artin-Schreier polynomial $X^p - X - a$ with $a \in E$, is well known: a discrete valuation v of E totally ramifies in F if and only if the maximum of the valuation in the coset $a + \wp(E)$ is negative, where $\wp(x) = x^p - x$, i.e., $m_{a,v} = \max\{v(b) \mid b \in a + \wp(E)\} < 0$. A standard Frattini argument reduces the general case to finitely many p-extensions, or in other words to a criterion with finitely many elements. More precisely, there exist $a_1, \ldots, a_n \in E$ such that v totally ramifies in F if and only if $m_{a_i,v} < 0$ for all i (n being the minimal number of generators of the Frattini quotient).

The goal of this work is to simplify this criterion and show that there exists (a single) $a \in E\mathbb{F}_q$ such that v totally ramifies in F if and only if $m_{\gamma a,v} < 0$, for all $\gamma \in \mathbb{F}_q^\times$ (see Theorem 3.2), where $\mathbb{F}_q$ is the finite field with q elements.

We apply our criterion to construct somewhat surprising examples: Assume $\mathbb{F}_q \subseteq E$ and that F/E is generated by a degree q Artin-Schreier polynomial

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$\wp_q(X) - a$, $a \in E$, where $\wp_q(X) = X^q - X$. For a discrete valuation v of E let $M_{a,v} = \max\{v(b) \mid b \in a + \wp_q(E)\}$ be the maximum of the valuation of the q-Artin-Schreier coset of a. It is an easy exercise to show that if $M_{a,v} < 0$ and $\gcd(p, M_{a,v}) = 1$, then v totally ramifies in F. So one might suspect that $M_{a,v}$ encodes the information whether v totally ramifies in F as in the case $q = p$. However this is false: We construct two extensions with the same $M_{a,v} < 0$. In the first example v totally ramifies in F although $p \mid M_{a,v}$. In the second example v does not totally ramify although it does ramify in F.

Notation. Let F/E be a Galois extension of fields of characteristic p of degree a power of p. We write $q = p^r$ for the degree $[F : E]$ of the extension. We let $\wp(x) = x^p - x$ and $\wp_q = \wp^r$, so $\wp_q(x) = x^q - x$. The symbol v denotes a discrete valuation of E, and w a valuation of F lying above v. We denote by $\mathbb{F}_{p^r}$ the finite field with p^r elements. Sometimes we identify $\mathbb{F}_{p^r}$ with its additive group. The multiplicative group of a field K is denoted by $K^\times$.

For an element $a \in E$ and discrete valuation v of E we denote

$$m_{a,v} = m(a, E, v) = \max\{v(b) \mid b \in a + \wp(E)\} \tag{1}$$

if the valuation set of the elements in the coset is bounded, and $m_{a,v} = \infty$ otherwise.

2. Classical theory

Let us start this discussion by recalling the well-known case $q = p$. In this case Artin-Schreier theory tells us that $F = E(\alpha)$, where α satisfies an equation $\wp(X) = a$, for some $a \in E$. Furthermore, one can replace α with a solution of $\wp(X) = b$, for any $b \in a + \wp(E)$. We have the following classical result (cf. [3, Proposition III.7.8]).

Theorem 2.1. *Assume $F = E(\alpha)$, for some $\alpha \in F$ satisfying an equation $\wp(X) = a$, $a \in E$. Then the following conditions are equivalent for a discrete valuation v of E.*

(a) *v totally ramifies in F.*
(b) *there exists $b \in a + \wp(E)$ such that $\gcd(p, v(b)) = 1$ and $v(b) < 0$.*
(c) *$m_{a,v} < 0$.*

If these conditions hold, then $v(b) = m_{a,v}$, and in particular $v(b)$ is independent of the choice of b. Moreover, if β is another Artin-Schreier generator, i.e., $F = E(\beta)$, and $\wp(\beta) = a_\beta \in E$, then $m_{a_\beta,v} = m_{a,v}$.

We return to the case of an arbitrary $q = p^r$. Then a standard Frattini argument reduces the question of when a discrete valuation v of E totally ramifies in F to extensions with p-elementary Galois group. Here a group G is p-elementary if G is abelian and of exponent p; equivalently $G \cong \mathbb{F}_q$. For the sake of completeness, we provide a formal proof of the reduction.

Proposition 2.2. *There exists $\bar{F} \subseteq F$ such that $\mathrm{Gal}(\bar{F}/E)$ is p-elementary and a discrete valuation v of E totally ramifies in F if and only if v totally ramifies in $\bar{F}$.*

Proof. Prolong v to a valuation w of F. Let $G = \mathrm{Gal}(F/E)$, let $\Phi = \Phi(G) = G^p[G,G]$ be the Frattini subgroup of G, and let $\bar{F} = F^\Phi$ be the fixed field of Φ in F. Let $\bar{w}$ be the restriction of w to $\bar{F}$. Then $\mathrm{Gal}(\bar{F}/E) \cong G/\Phi$ is p-elementary.

Let $I_{w/v}$, $I_{\bar{w}/v}$ be the inertia groups of w/v, $\bar{w}/v$, respectively. Consider the restriction map $r\colon \mathrm{Gal}(F/E) \to \mathrm{Gal}(\bar{F}/E)$. Then $r(I_{w/v}) = I_{\bar{w}/v}$ [2, Proposition I.8.22]. This implies that $I_{w/v} = G$ if and only if $I_{\bar{w}/v} = r(I_{w/v}) = \mathrm{Gal}(\bar{F}/E)$ (recall that a subgroup H of a finite group G satisfies $H\Phi(G) = G$ if and only if $H = G$). □

Remark 2.3. The Frattini subgroup is the intersection of all maximal subgroups. Therefore $\bar{F}$, as its fixed field, is the compositum of all minimal sub-extensions of F/E.

Applying Theorem 2.1 for $\bar{F}$ gives the following

Corollary 2.4. *Let F/E be a Galois extension of degree $q = p^r$. Then there exist $a_1, \ldots, a_n \in E$ such that for any discrete valuation v of E we have that v totally ramifies in F if and only if $m_{a_i,v} < 0$ for all i.*

3. Criterion for total ramification using one element

In this section we strengthen Corollary 2.4 and prove that it suffices to take $\mathbb{F}_q^\times$-translation of a single element. For this we need the following lemma.

Lemma 3.1. *Let p be a prime and $q = p^r$ a power of p. Consider a tower of extensions $\mathbb{F}_q \subset E \subseteq F$ with $q = [F : E]$. Assume $F = E(x)$ for some $x \in F$ that satisfies $a := \wp_q(x) \in E$. Then the family of fields generated over E by roots of $\wp(X) - \gamma a$, where γ runs over $\mathbb{F}_q^\times$ coincides with the family of all minimal sub-extensions of F/E.*

Proof. Since $\wp_q(X) - a = \prod_{\alpha \in \mathbb{F}_q}(X - (x + \alpha))$, the extension F/E is Galois. Let $G = \mathrm{Gal}(F/E)$, then the map

$$\phi\colon \begin{cases} G & \to & \mathbb{F}_q \\ \sigma & \mapsto & \sigma(x) - x \end{cases}$$

is well defined. Moreover it is immediate to verify that ϕ is an isomorphism.

Let C be the kernel of the trace map $\mathrm{Tr}\colon \mathbb{F}_q \to \mathbb{F}_p$; $\mathrm{Tr}(u) = u^{p^{r-1}} + \cdots + u$. It is well known that T is a non-trivial linear transformation [1, Theorem VI.5.2] over $\mathbb{F}_p$. This implies that T is surjective, so C is a hyper-space of $\mathbb{F}_q$ (as a vector space over $\mathbb{F}_p$).

The minimal sub-extensions of F/E are the fixed fields of maximal subgroups of $\mathrm{Gal}(F/E)$, which correspond to hyper-spaces of $\mathbb{F}_q$ via ϕ. Let C' be a hyper-space in $\mathbb{F}_q$. Then there exists an automorphism $M\colon \mathbb{F}_q \to \mathbb{F}_q$ under which $M(C') = C$. But $\mathrm{Aut}(\mathbb{F}_q) = \mathbb{F}_q^\times$, so M acts by multiplying by some $\gamma \in \mathbb{F}_q^\times$. Hence $\gamma C' = C$. Vice-versa, if $\gamma \in \mathbb{F}_q^\times$, then $\gamma^{-1}C$ is a hyper-space. Therefore, it suffices to show, for

an arbitrary $\gamma \in \mathbb{F}_q^\times$, that the fixed field F' of $H := \phi^{-1}(\gamma^{-1}C)$ in F is generated by a root of $\wp(X) - \gamma a$.

Let $y = \gamma x$. Then $F = E(y)$ and

$$y^q - y = \gamma^q x^q - \gamma x = \gamma(x^q - x) = \gamma a.$$

Let $z = y^{p^{r-1}} + \cdots + y^p + y$. Then $z \notin E$, hence $[F : E(z)] \le p^{r-1}$. We have

$$z^p - z = y^{p^r} + \cdots + y^{p^2} + y^p - (y^{p^{r-1}} + \cdots + y^p + y) = y^q - y = \gamma a.$$

Thus $[E(z) : E] \le p$, and we get $[F : E(z)] = p^{r-1}$. To complete the proof we need to show that $F' = E(z)$, so it suffices to show that H fixes z. Indeed, let $\sigma \in H = \phi^{-1}(\gamma^{-1}C)$. Then $\beta := \sigma(y) - y = \gamma(\sigma(x) - x) = \gamma\phi(\sigma) \in C$. We have

$$\begin{aligned}\sigma(z) - z &= \sigma(y^{p^{r-1}} + \cdots + y^p + y) - (y^{p^{r-1}} + \cdots + y^p + y)\\ &= (\sigma(y) - y)^{p^{r-1}} + \cdots + (\sigma(y) - y)^p + (\sigma(y) - y)\\ &= \beta^{p^{r-1}} + \cdots + \beta = \mathrm{Tr}(\beta) = 0,\end{aligned}$$

as needed. □

We are now ready for the main result that classifies totally ramified discrete valuations of Galois extensions in characteristic p.

Theorem 3.2. *Assume F/E is a Galois extension of fields of characteristic p of degree a power of p and with Galois group G. Let $d = d(G)$ be the minimal number of generators of G and let $q = p^r$, for some $r \ge d$ (e.g., $q = [F : E]$). Let $F' = F\mathbb{F}_q$ and $E' = E\mathbb{F}_q$. If v is a valuation of E, we denote by v' its (unique) extension to E'. Then there exists $a \in E'$ such that for every discrete valuation v of E the following is equivalent.*

(a) *v totally ramifies in F.*
(b) *v' totally ramifies in F'.*
(c) *$m(\gamma a, E', v') < 0$, for every $\gamma \in \mathbb{F}_q^\times$.*
(d) *There exists $b_\gamma \in \gamma a + \wp(E')$ such that $\gcd(p, v'(b_\gamma)) = 1$ and $v'(b_\gamma) < 0$, for every $\gamma \in \mathbb{F}_q^\times$.*

Remark 3.3. In the above conditions (c) and (d) it suffices that γ runs over representatives of $\mathbb{F}_q^\times/\mathbb{F}_p^\times$.

Proof. Since finite fields admit only trivial valuations, we get that both F'/F and E'/E are unramified, so (a) and (b) are equivalent. Theorem 2.1 implies that (c) and (d) are equivalent. So it remains to proof that (b) and (c) are equivalent. For simplicity of notation, we replace F, E with F', E' and assume that $\mathbb{F}_q \subseteq E$.

Let $\bar{F} \subseteq F$ be the extension given in Proposition 2.2. Let $\bar{d}$ be the minimal number of generators of $\mathrm{Gal}(\bar{F}/E)$. Then $\bar{q} = p^{\bar{d}} = [\bar{F} : E]$ and $\bar{d} \le d$. By Proposition 2.2 we may replace $\bar{F}$ with F, and assume that $\mathrm{Gal}(F/E) \cong \mathbb{F}_q$.

By Artin-Schreier Theory, $F = E(x)$, where x satisfies the equation $\wp_q(x) = a$, for some $a \in E$. Lemma 3.1 implies that all the minimal sub-extensions of F/E are generated by roots of $\wp(X) - \gamma a$, where γ runs over $\mathbb{F}_q^\times$. Note that v totally

ramifies in F if and only if v totally ramifies in all the minimal sub-extensions of F/E (since if the inertia group is not the whole group, it fixes some minimal sub-extension, so v does not ramify in this sub-extension). This finishes the proof, since, by Theorem 2.1, v totally ramifies in all the minimal sub-extensions of F/E if and only if $m(\gamma a, E, v) < 0$, for all $\gamma \in \mathbb{F}_q^\times$. □

4. An application

We come back to the case where $\mathbb{F}_q \subseteq E \subseteq F$, and F/E is a Galois extension with Galois group isomorphic to $\mathbb{F}_q$. By Artin-Schreier Theory $F = E(x)$, where $x \in F$ satisfies an equation $\wp_q(X) = a$, for some $a \in E$. This a can be replaced by any element of the coset $a + \wp_q(E)$. If there exists $b \in a + \wp_q(E)$ such that $v(b) < 0$ and $\gcd(q, v(b)) = 1$, then v totally ramifies in F. It is reasonable to suspect that the converse also holds, as in the case $q = p$. We bring two interesting examples. The first is a totally ramified extension such that there exists no b as above. The other construction is of an extension which is not totally ramified, although Condition (c) of Theorem 3.2 holds for $\gamma = 1$.

Let p be a prime, $d \geq 1$ prime to p, $q = p^r$, and let $E = \mathbb{F}_q(t)$. Consider the t-adic valuation, i.e., $v(t) = 1$. Let $\gamma \neq 1$ be an element of $\mathbb{F}_q$ with norm 1 (w.r.t. the extension $\mathbb{F}_q/\mathbb{F}_p$). Consider an element

$$a(t) = \frac{1}{t^{dp}} - \frac{\gamma}{t^d} + f(t) \in E$$

and let $F = E(x)$, where x satisfies $\wp_q(x) = a$.

If $f(t) = \frac{1}{t}$ and $d > 1$, then $\mathrm{Gal}(F/E) \cong \mathbb{F}_q$, v totally ramifies in F, but there is no $b \in a + \wp_q(E)$ whose valuation is prime to p.

Indeed, let $\delta \in \mathbb{F}_q^\times$. For $\epsilon \in \mathbb{F}_q$ with $\epsilon^p = \delta$ we set

$$b_\delta(t) = \delta a(t) - \wp\Big(\frac{\epsilon}{t^d}\Big) = \delta a(t) - \Big(\frac{\epsilon}{t^d}\Big)^p + \frac{\epsilon}{t^d} = \frac{\epsilon - \delta\gamma}{t^d} + \delta f(t). \tag{2}$$

Take $f(t) = \frac{1}{t}$. Then $v(b_\delta(t))$ is either $-d$ if $\epsilon \neq \delta\gamma$ or -1 if $\epsilon = \gamma\delta$, so $p \nmid v(b_\delta) < 0$. By Theorem 3.2, v totally ramifies in F.

To this end assume there exists $b \in a + \wp_q(E)$ with $p \nmid v(b) < 0$, and let $-m = v(b)$. By Lemma 3.1 the minimal sub-extensions of F/E are generated by roots of $\wp(X) - \delta b$, where $\delta \in \mathbb{F}_q^\times$. Since $\gamma \neq 1$ has norm 1, $\gamma = \frac{\delta_0}{\delta_0^p}$, for some $\delta_0 \in \mathbb{F}_q$ (Hilbert 90). But since $v(\delta b) = v(b)$, we get $-d = m(b_{\delta_0}, E, v) = m(b, E, v) = m(b_1, E, v) = -1$ (Theorem 2.1). This contradiction implies that such a b does not exist.

If $f(t) = t$, then $\max\{v(b) \mid b \in a + \wp_q(E)\} < 0$ but v does not totally ramify in F.

Indeed, assume that $f(t) = t$, then since $\gamma = \frac{\delta_0}{\delta_0^p}$, (2) implies that $v(b_{\delta_0^p}) = v(f(t)) = 1$. So v is not totally ramified in F (Theorem 3.2). Assume there was $b \in a + \wp_q(E)$ with $v(b) \geq 0$. Then all the minimal sub-extensions F' of F/E were

generated by $\wp(X) - \delta b$, where $\delta \in \mathbb{F}_q^\times$. But $v(\delta b) = v(b) \geq 0$, so all the F' are unramified (Theorem 2.1). This conclusion contradicts the fact that the extension generated by $X^p - X - b_1$ is ramified. So $\max\{v(b) \mid b \in a + \wp_q(E)\} < 0$, as claimed.

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An Octahedral Galois-Reflection Tower of Picard Modular Congruence Subgroups

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Abstract. Between tradition (Hilbert's 12th Problem) and actual challenges (coding theory) we attack infinite two-dimensional Galois theory. From a number theoretic point of view we work over $\mathbb{Q}(x)$. Geometrically, one has to do with towers of Shimura surfaces and Shimura curves on them. We construct and investigate a tower of rational Picard modular surfaces with Galois groups isomorphic to the (double) octahedron group and of their (orbitally) uniformizing arithmetic groups acting on the complex 2-dimensional unit ball $\mathbb{B}$.

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1. Introduction

The main results are dedicated to a natural congruence subgroup $\Gamma(2)$ of the full Picard modular group Γ of Gauß numbers. From the number theoretic side it is interesting, that this infinite group is finitely generated by special elements of order two. More precisely we can choose as generator system a (finite) set of reflections. In number theory such elements are comparable with "inertia elements" generating inertia groups of a Galois covering. The proof is based on a strong geometric result: We need the fine classification of the (Baily-Borel compactified) quotient surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$. It turns out, that it is a nice blowing up of the projective plane at triple and quadruple points of the very classical harmonic configuration of lines. We mention that this is the first precise classification of a Picard modular surface of a natural congruence subgroup. Along an easy correspondence the harmonic configuration changes to the globe configuration with equator and two meridians meeting each

other in six (elliptic) cusp singularities, see the picture at the end of Section 6. On this way we visualized the octahedral action of the factor Galois group $\Gamma/\Gamma(2)$. In Galois towers between the surfaces $\widehat{\Gamma(2)\backslash\mathbb{B}}$ and $\widehat{\Gamma\backslash\mathbb{B}}$ we discover a classical orbital ball quotient surface of the PTDM-list (Picard, Terada, Mostow, Deligne), which was also published in Hirzebruch's (and other's) monograph [BHH]. On the one hand we need this del Pezzo surface for proving our results. On the other hand we found the arithmetic group uniformizing this orbital surface. It is a Picard modular congruence subgroup. The precise description is important for the further work with the Picard modular forms of this group found by H. Shiga and his team, see [KS], [Mat]. In the same manner we find also the uniformizing arithmetic group of the first surface (with a new line configuration) sitting in the infinite Galois-tower of orbital (plane) ball quotient surfaces constructed by Uludag [Ul]. It allows to work with algebraic equations for Shimura curves, which are important in coding theory.

2. Picard modular varieties and Galois-Reflection towers

Let V be the vector space $\mathbb{C}^{n+1}$ endowed with hermitian metric $\langle .,. \rangle$ of signature $(n,1)$. Explicitly we will work with the diagonal representation

$$\begin{pmatrix} 1 & 0 & \cdots & & \\ 0 & 1 & \cdots & & \\ & & \ddots & & \\ & & & 1 & \\ \cdot & \cdot & & & -1 \end{pmatrix}.$$

For $v \in V$ we call $\langle v, v \rangle$ the *norm* of v. The space of all vectors with negative (positive) norms is denoted by V^- (V^+). The image $\mathbb{P}V^-$ of V^- in $\mathbb{P}V = \mathbb{P}^n$ is the complex n-dimensional *unit ball* denoted by $\mathbb{B}^n$. The unitary group $\mathbb{U}((n,1),\mathbb{C})$ acts transitively on it.

Now let K be an imaginary quadratic number field, $\mathcal{O}_K$ its ring of integers.

Definition 2.1. The arithmetic subgroup $\Gamma_K = \mathbb{U}((n,1),\mathcal{O}_K)$ is called the *full Picard modular group* (*of K, of dimension n*). Each subgroup Γ of $\mathbb{U}((n,1),\mathbb{C})$ commensurable with Γ_K is called *Picard modular group.*

Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$, closed under complex conjugation. Then, over the finite factor ring $A = \mathcal{O}_K/\mathfrak{a}$, the finite unitary group $\Gamma_A = \mathbb{U}((n,1),\mathcal{O}_K/\mathfrak{a})$ is well defined together with the reduction (group) morphism $\rho_{\mathfrak{a}} : \Gamma_K \longrightarrow \Gamma_A$. The kernel of $\rho_{\mathfrak{a}}$ is denoted by $\Gamma_K(\mathfrak{a})$.

Definitions 2.2. This group is called the *congruence subgroup of the ideal* $\mathfrak{a}$ *in* Γ_K. A subgroup Γ of Γ_K is called a (*Picard modular*) *congruence subgroup,* iff it contains a congruence subgroup $\Gamma_K(\mathfrak{a})$. If $\mathfrak{a}$ is a principal ideal (α), then we get a *principal congruence subgroup* $\Gamma_K(\alpha)$. For any natural number a we call $\Gamma_K(a)$ a *natural congruence subgroup of* Γ_K. Intersecting the above subgroups with a given Picard modular group Γ, we get (*principal, natural*) *congruence subgroups* $\Gamma(\mathfrak{a})$, $\Gamma(\alpha)$, $\Gamma(a)$ *of* Γ.

Remark 2.3. *The full Picard modular group appears also as* $\Gamma_K(1)$ *now. More generally, we have to identify the groups* $\Gamma(1)$ *and* Γ.

The ball quotients $\Gamma\backslash\mathbb{B}^n$ are quasiprojective. They have a minimal algebraic compactification $\widehat{\Gamma\backslash\mathbb{B}^n}$ constructed by Baily and Borel in [BB]. The authors proved that these compactifications are normal projective complex varieties. We call them *Baily-Borel compactifications.* In the Picard modular cases the Baily-Borel compactifications consist of finitely many points, called *cusp singularities* or *cusp points.* It may happen that such point is a regular one.

The Picard modular groups of a fixed imaginary quadratic number field K act also on the hermitian $\mathcal{O}_K$-lattice $\Lambda = (\mathcal{O}_K)^{n+1} \subset V$.

Definition 2.4. Let $c \in \Lambda$ be a primitive positive vector and $c^\perp$ its orthogonal complement in V. It is a hermitian subspace of V of signature $(n-1, 1)$. The intersection

$$\mathbb{D}_c := \mathbb{P}c^\perp \cap \mathbb{B}^n$$

is isomorphic to $\mathbb{B}^{n-1}$. We call it an *arithmetic hyperball of* $\mathbb{B}^n$. Arithmetic hyperballs of $\mathbb{B}^2$ are called *arithmetic subdiscs.*

Take all elements of Γ acting on $\mathbb{D}_c$:

$$\Gamma_c := \{\gamma \in \Gamma;\ \gamma(\mathbb{D}_c) = \mathbb{D}_c\}.$$

This is an arithmetic group. The image $p(\mathbb{D}_c)$ along the quotient projection $p : \mathbb{B}^n \longrightarrow \Gamma\backslash\mathbb{B}^n$ is an algebraic subvariety H_c of $\Gamma\backslash\mathbb{B}^n$ of codimension 1.

Definition 2.5. The algebraic subvarieties H_c are called *arithmetic hypersurfaces* of the Picard modular variety $\Gamma\backslash\mathbb{B}^n$. The same notion is used for the compactifications. The *norm* $n(H_c)$ *of* H_c is defined as $n(c)$.

The analytic closure of H_c on the Baily-Borel compactification $\widehat{\Gamma\backslash\mathbb{B}^n}$ is denoted by $\hat{H}_c$. Around general points the quotient variety $\Gamma_c\backslash\mathbb{D}_c$ coincides with $H_c = \Gamma\backslash\mathbb{D}_c$. More precisely, we have normalizations

$$\Gamma_c\backslash\mathbb{D}_c \longrightarrow \Gamma\backslash\mathbb{D}_c = H_c$$
$$\widehat{\Gamma_c\backslash\mathbb{D}_c} \longrightarrow \widehat{\Gamma\backslash\mathbb{D}_c} = \widehat{H}_c.$$

For the proof we refer to [BSA] IV.4, where it is given for the surface case $n = 2$. It is easily seen, that it works also in general dimensions n.

Definition 2.6. A non-trivial element of finite order $\sigma \in \mathbb{U}((n,1),\mathbb{C})$ is called a *reflection* iff there is a positive vector $c \in V$ such that $V_c := c^\perp$ is the eigenspace of σ of eigenvalue 1. If σ belongs to the Picard modular group Γ, then we call it a Γ*-reflection.*

Remark 2.7. Some authors call them "quasi reflections". Only in the order 2 cases they omit "quasi".

Looking at the characteristic polynomial of σ we see that the eigenvector c belongs to K^{n+1} in the Picard case in 2.6. We can and will choose c primitive in $\Lambda = \mathcal{O}^{n+1}$. Then it is clear that σ acts identically on the arithmetic hyperball

$$\mathbb{D}_\sigma := \mathbb{D}_c = \mathbb{P}V_c \cap \mathbb{B}^n$$

of $\mathbb{B}^n$. We call such $\mathbb{D}_c$ a Γ-*reflection subball* of $\mathbb{B}^n$, or a Γ-*reflection disc* in the surface case $n = 2$.

Definition 2.8. The hypersurface H_c of the primitive eigenvector $c = c(\sigma)$ of a Γ-reflection σ is called a Γ-*reflection hypersurface.* In the two-dimensional case we call it Γ-*reflection curve.*

Fact. The irreducible hypersurface components of the branch locus of the quotient projection $p : \mathbb{B}^n \to \Gamma\backslash\mathbb{B}^n$ are precisely the Γ-reflection hypersurfaces.

Let Γ' be a normal subgroup of finite index of the Picard modular group Γ. We do not change notations, if such lattices doesn't act effectively on $\mathbb{B}^n$. We keep the effectivization (= projectivization) in mind. We do the same for the Galois group $G := \Gamma/\Gamma'$ of the covering

$$\Gamma'\backslash\mathbb{B} \longrightarrow \Gamma\backslash\mathbb{B}. \tag{1}$$

Definition 2.9. This finite morphism (1) is called a *Galois-Reflection covering* iff G is generated by Γ'-cosets of some Γ-reflections. We call G in this case a *Galois-Reflection group.*

In pure ball lattice terms this means that

$$\Gamma = \langle \Gamma', \sigma_1, \ldots, \sigma_k \rangle \tag{2}$$

for suitable reflections σ_i, i=1,...,k.

We want to prove

Proposition 2.10. *If $\Gamma\backslash\mathbb{B}$ is simply-connected and smooth, then* (1) *is a Galois-Reflection covering for each normal sublattice Γ' of Γ.*

This can be easily deduced from the following

Theorem 2.11. *If $\Gamma\backslash\mathbb{B}$ is simply-connected, then Γ is generated by finitely many elements of finite order* (torsion elements)*. If, moreover, the Picard modular variety $\Gamma\backslash\mathbb{B}$ is smooth, then Γ is generated by finitely many reflections.*

For the proof we need first the following

Theorem 2.12 ((Armstrong, [Ar] 1968)). *Let G be a discrete group of homeomorphisms acting on a path-wise connected, simply-connected, locally compact metric space X and H the* (*normal*) *subgroup generated by the stabilizer groups G_x of all points $x \in X$. Then G/H is the fundamental group of the* (*topological*) *quotient space X/G.*

Proof of Theorem 2.11. We substitute $\Gamma, \mathbb{B}, Tor\Gamma$ for G, X, H in Armstrong's Theorem. It follows that $\Gamma/Tor\Gamma$ is the fundamental group of the quotient variety $\Gamma\backslash\mathbb{B}$. If it is 1, then $\Gamma/Tor\,\Gamma = 1$. This means that Γ is generated by all its torsion elements. These elements are finite order. Now we remember that each arithmetic group is finitely generated, by a theorem of Borel [Bo]. All generators are products of finitely many torsion elements. So we can generate Γ by finitely many torsion elements. This proves the first part of Theorem 2.11.

For the second part, we look at the stabilizers Γ_x, $x \in \mathbb{B}^n$. These are finite groups. Claude Chevalley proved in [Ch] that the image point $p(x) \in \Gamma\backslash\mathbb{B}^n$ is regular, if and only if Γ_x is generated by reflections. On the other hand, each torsion element of Γ has a fixed point $x \in \mathbb{B}^n$. Therefore Tor Γ is generated by reflections, if $\Gamma\backslash\mathbb{B}^n$ is smooth. So the second part of Theorem 2.11 follows now from the first. □

Definition 2.13. Let

$$\Gamma_N \lhd \cdots \lhd \Gamma_{i+1} \lhd \Gamma_i \lhd \cdots \lhd \Gamma_1 \subseteq \Gamma \tag{3}$$

be a normal series of subgroups of finite index of the Picard modular group Γ. We call it a Γ-*reflection series,* if Γ_i is generated by Γ_{i+1} and finitely many reflections for each in (3) occurring pair $(i+1, i)$. The corresponding Galois tower of finite Galois coverings

$$\Gamma_N\backslash\mathbb{B}^n \to \cdots \to \Gamma_{i+1}\backslash\mathbb{B}^n \longrightarrow \Gamma_i\backslash\mathbb{B}^n \to \cdots \to \Gamma_1\backslash\mathbb{B}^n, \tag{4}$$

with the normal factors Γ_i/Γ_{i+1} as Galois groups, is then called a *Galois-Reflection tower* (attached to the normal series (3)).

In this case each map of the sequence is a Galois-Reflection covering with the normal factors Γ_i/Γ_{i+1} as Galois groups. The extension of the definition to (Baily-Borel or other) compactificatons should be clear. It is left to the reader.

Theorem 2.14. *If all members, except for* $\Gamma_N\backslash\mathbb{B}^n$*, in the covering tower* (4) *attached to* (3) *are simply-connected smooth varieties, then it is a Galois-Reflection tower.*

Proof. We have to show that each covering of the tower has the Galois-Reflection property. We refer to Proposition 2.10. □

Moreover, we call an infinite tower

$$\mathbb{B}^n \to \cdots \to \Gamma_{i+1}\backslash\mathbb{B}^n \longrightarrow \Gamma_i\backslash\mathbb{B}^n \to \cdots \to \Gamma_1\backslash\mathbb{B}^n, \tag{5}$$

a *Galois-Reflection tower*, if all occurring ball lattices Γ_i are generated by reflections.

Example 2.15. Uludag constructed in [Ul] an infinite tower

$$\cdots \to \mathbb{P}^2 \to \mathbb{P}^2 \to \cdots \to \mathbb{P}^2 \to \mathbb{P}^2 \tag{6}$$

of ball quotient planes $\mathbb{P}^2 = \widehat{\Gamma_i\backslash\mathbb{B}^2}$. It is not clear until now that the Γ_i's can be chosen as infinite normal series. We know only the existence of the ball lattices

Γ_i, $i = 1, 2, 3, \ldots$, and that the successive coverings in (6) have the Klein's 4-group $Z_2 \times Z_2$ as Galois group. The last member is the orbital $\mathbb{P}^2 = \Gamma(\widehat{1-i})\backslash\mathbb{B}$ with "Apollonius divisor", supported by a quadric and three tangents as orbital branch divisor of the ball covering. We refer to [HPV] or [BMG], first appearance of the Appolonius picture in [SY]. In [HPV], [BMG] we proved that the congruence subgroup $\Gamma(1-i)$ is the uniformizing ball lattice, with the *full Picard-Gauß lattice* $\Gamma = \Gamma(1) := \mathbb{SU}((2,1), \mathbb{Z}[i])$. By Theorem 2.11 it is true that all ball lattices Γ_i in this example are generated by reflections.

We consider a Γ-reflection covering as in 2.9. We want to construct a set of reflections whose Γ'-cosets generate the Galois group $G = \Gamma/\Gamma'$. For this purpose we consider all *K-arithmetic subballs* $\mathbb{D}$ of $\mathbb{B}^n$. By definition, these are the arithmetic subballs for our fixed imaginary-quadratic field K, see Definition 2.4. Such $\mathbb{D}$ is a Γ-reflection if and only if the finite cyclic group

$$Z_\Gamma(\mathbb{D}) = \{\sigma \in \Gamma; \sigma|_{\mathbb{D}} = id_{\mathbb{D}}\},$$

called *centralizer group of* Γ *at* $\mathbb{D}$, is not trivial. In this case the image H of $\mathbb{D}$ on $\Gamma\backslash\mathbb{B}^n$ belongs to the branch divisor, and the ramification index there coincides with $\#Z_\Gamma(\mathbb{D})$.

Now let Γ' be a subgroup of finite index of Γ. Then we dispose on a commutative diagram

$$\begin{tikzcd} \mathbb{B}^n \arrow[r, "="] \arrow[d, "p'"'] & \mathbb{B}^n \arrow[d, "p"] \\ \Gamma'\backslash\mathbb{B}^n \arrow[r, "f"'] & \Gamma\backslash\mathbb{B}^n \end{tikzcd}$$

of analytic maps, where f is finite, and the verticals are locally finite. With $H' := p'(\mathbb{D})$, it restricts to

$$\begin{tikzcd} \mathbb{D} \arrow[r, "="] \arrow[d] & \mathbb{D} \arrow[d] \\ H' \arrow[r] & H\,. \end{tikzcd}$$

The covering f is branched along H, if and only if $Z' := Z_{\Gamma'}(\mathbb{D})$ is a honest (cyclic) subgroup of Z. The ramification order of f at H' is equal to the index $[Z : Z']$.

Now we see a practical way to get generating reflection elements σ_i of the Galois group f, if it is a Galois-reflection covering as described in (2). We have to know the components H of the branch divisor of f. Then we must find a reflection subball $\mathbb{D} = \mathbb{D}_\sigma \subset \mathbb{B}^n$ projecting onto H along p as above. Then σ is one of the generating σ_i you look for. Now we change to the next branch divisor component to find the next of the generating reflections. It is helpful to know the order of the Galois group G of f. Then one can compare group orders of $G = \Gamma/\Gamma'$ (assumed to

be known) and of $G'_i := \Gamma/\langle\Gamma', \sigma_1, \ldots, \sigma_i\rangle$ using all the reflections already found. One has to stop the procedure, if both group orders coincide. If $\Gamma' = \Gamma(\mathfrak{a})$ is a congruence subgroup of Γ, then we calculate the orders of G'_i modulo the ideal $\mathfrak{a}$ by a computer program, e.g., MAPLE.

3. The level 2 Reflection tower

From now on we restrict ourselves to the second (complex) dimension $n = 2$. We write $\mathbb{B}$ for the complex 2-dimensional unit ball $\mathbb{B}^2$. Moreover we concentrate our attention to the Gauß number field $K = \mathbb{Q}(i)$.

A) The Galois-Reflection covering of $\Gamma(1-i) \subset \Gamma$

For $\Gamma = \mathbb{SU}((2,1), \mathbb{Z}[i])$ we want to construct reflection generators of

$$\Gamma(1)/\Gamma(1-i) \subseteq \mathbb{O}(3, \mathbb{F}_2) \cong S_3, \tag{7}$$

where $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z}$ denotes the primitive field of characteristic 2. We take two primitive elements of $\Lambda = \mathbb{Z}[i]^3$ of norm 2, namely

$$a = (1+i, 1, 1),\ b = (1, i, 0).$$

We look for a reflection with eigenvector a of eigenvalue -1. It sends each $z \in V = \mathbb{C}^3$ to $z - < z, a > a$. For explicit Γ-representations we refer to the appendix Section 7. It turns out that both reflections generate a subgroup Σ_3 of $\mathbb{SU}((2,1), \mathbb{Z}[i])$ isomorphic to S_3. Especially, the inclusion in (7) is an equality.

It is easy to find $\mathbb{C}$-bases of the orthogonal complements $a^\perp$ or $b^\perp$ in V, respectively. Via projectivization we get explicitly the Γ-reflection discs

$$\mathbb{D}_a = \mathbb{P}a^\perp \cap \mathbb{B}\ ,\ \mathbb{D}_b = \mathbb{P}b^\perp \cap \mathbb{B}.$$

These linear discs go through $(1:0:1-i)$ or $(0:0:1)$ in $\mathbb{B} \subset \mathbb{P}^2$, respectively, and intersect each other in $P = (i:1:2+i)$. This is the common fixed point of Σ_3. Conversely, Σ_3 is the isotropy group of Γ at P.

The Baily-Borel compactification $\widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is $\mathbb{P}^2$. It has been determined in [HPV], [BMG]. More precisely, this *orbital* quotient surface is a pair $(\mathbb{P}^2;\ 4C_0 + \cdots + 4C_3)$, where C_0 is an S_3-invariant quadric, and C_1, C_2, C_3 are three of its tangent lines. The three (Baily-Borel) compactifying cusp points are the touch points of the tangents and the quadric. Look at Picture 5 in the later Section 5. The coefficients 4 denote the branch indices of each curve C_i along the locally finite quotient covering $\mathbb{B} \to \mathbb{P}^2 \backslash \{3 \text{ points}\}$. Especially, $\Gamma(1-i)\backslash\mathbb{B}$ is smooth. From Theorem 2.11 it follows now that $\Gamma(1-i)$ is generated by finitely many reflections. Together with 7 and the above reflection representation of S_3-generators, we see altogether that Γ itself is generated by finitely many reflections. This doesn't follow directly from Theorem 2.11, because $\widehat{\Gamma\backslash\mathbb{B}}$ has a surface singularity, namely the image point of $P = (i:1:2+i) \in \mathbb{B}$ on the quotient surface. This is the only singularity there, see [BSA], Chapter V, §5.3 (especially, point P_2 in Figure 5.3.7).

This shows that surface smoothness is not necessary for the existence of finitely many reflections generating the corresponding ball lattice.

B) The Galois-Reflection covering of $\Gamma(2) \lhd \Gamma(1-i)$

We continue the above Γ-example with the consideration of the natural congruence subgroup $\Gamma(2)$. In [HPV], Theorem 7.2 we proved that all torsion elements of $\Gamma(2)$ have order 2. Moreover, they all are squares of $\Gamma(1-i)$-elements of order 4. Each isotropy group of $\Gamma(1-i)$-elliptic points is generated by two $\Gamma(1-i)$-reflections of order 4. Each non-reflection torsion element $\tau \in \Gamma(1-i)$ of order 4 fixes a(n *elliptic*) point, say $Q \in \mathbb{B}$. It turns out that τ is the product of two $\Gamma(1-i)_Q$-generating reflections. So we have

$$\Gamma(1-i)_Q \cong Z_4 \times Z_4, \text{ with } Z_d := (\mathbb{Z}/d\mathbb{Z}, +).$$

Conversely, all squares of order 4 elements belong to $\Gamma(2)$. In [HPV], Proposition 8.3, we determined the index as $[\Gamma(1-i) : \Gamma(2)] = 8$. The diagonal reflections

$$\sigma_1 := \mathrm{diag}(i, 1, 1),\ \sigma_2 := \mathrm{diag}(1, i, 1)$$

have the *coordinate reflection discs* $\mathbb{D}_2 :\ z_2 = 0$ or $\mathbb{D}_1 :\ z_1 = 0$, respectively. They generate the isotropy group $\Gamma(1-i)_O$, O the zero coordinate point. Reduction mod $(1-i)$ yields the exact sequence

$$1 \longrightarrow Z_2 \times Z_2 = \Gamma(2)_O \longrightarrow Z_4 \times Z_4 = \Gamma(1-i)_O \longrightarrow \Gamma(1-i)/\Gamma(2).$$

The image group on the right has the same structure as the kernel, namely

$$K4 := Z_2 \times Z_2 \subset \Gamma(1-i)/\Gamma(2) \text{ (Klein's Vierer-Gruppe).}$$

Observe that the norm 1 vectors, whose ortho-complements determine the coordinate reflection discs, are $\mathfrak{n}_1 = (0, 1, 0)$ or $\mathfrak{n}_2 = (1, 0, 0)$, respectively. We determine a third reflection σ_0, which is incongruent mod 2 to the elements of $\langle \sigma_1, \sigma_2 \rangle$. For this purpose we take the norm 1 vector $\mathfrak{n}_0 := (1, 1, 1)$. Then σ_0 is the (order 4) reflection corresponding

$$V = \mathbb{C}^3 \ni v \mapsto v - (1-i)\langle v, \mathfrak{n}_0 \rangle \mathfrak{n}_0. \tag{8}$$

For its Γ-representation we refer again to the appendix Section 7. The orthogonal reflection disc $\mathbb{D}_0 \subset \mathbb{B}$ has the linear equation $z_1 + z_2 = 1$.

The disc $\mathbb{D}_0$ projects along the quotient projection $\mathbb{B} \to \mathbb{P}^2$ to the quadric C_0, and $\mathbb{D}_1, \mathbb{D}_2$ to the tangents C_1, C_2 of the Apollonius configuration. For more details we refer to [HPV], [BMG].

The reflections σ_0, σ_1, σ_2 generate mod 2 a subgroup of order 8 in $\Gamma(1-i)/\Gamma(2)$, which has the same order. Therefore we found the Galois group together with Galois-Reflection generators of the covering $\Gamma(2)\backslash\mathbb{B} \to \Gamma(1-i)\backslash\mathbb{B}$:

$$Z_2 \times K4 = \langle \bar{\sigma}_0, \bar{\sigma}_1, \bar{\sigma}_2 \rangle = \Gamma(1-i)/\Gamma(2). \tag{9}$$

In the next section we look for fine Kodaira classification of $\widehat{\Gamma(2)\backslash\mathbb{B}}$. This will be managed step by step along Galois-Reflection coverings/towers along the ball

lattices in the following commutative diagram of inclusions:

$$\begin{array}{ccc} \Gamma(2) & \longrightarrow & \langle \Gamma(2), \sigma_1, \sigma_2 \rangle \;=: \Gamma'' \\ \downarrow & & \downarrow \\ \Gamma' := \langle \Gamma(2), \sigma_0 \rangle & \longrightarrow & \Gamma(1-i)\,. \end{array} \tag{10}$$

It reduces $\bmod \Gamma(2)$ to the Galois group diagram of finite Galois coverings (on the right):

$$\begin{array}{ccc} 1 & \longrightarrow & K4 \\ \downarrow & & \downarrow \\ Z_2 & \longrightarrow & Z_2 \times K4 \end{array} \quad , \quad \begin{array}{ccc} \widehat{\Gamma(2)\backslash\mathbb{B}} & \longrightarrow & \widehat{\Gamma''\backslash\mathbb{B}} \\ \downarrow & & \downarrow \\ \widehat{\Gamma'\backslash\mathbb{B}} & \longrightarrow & \widehat{\Gamma(1-i)\backslash\mathbb{B}}\,. \end{array} \tag{11}$$

C) The Galois-Reflection tower of $\Gamma(2) \subset \Gamma$

Composing A) and B) we have the normal series

$$\Gamma(2) \lhd \Gamma'' \lhd \Gamma(1-i) \lhd \Gamma(1) = \Gamma = \mathbb{SU}((2,1), \mathbb{Z}[i]).$$

We can and will also Γ'' substitute by Γ'.

Proposition 3.1.

i) *The full Picard lattice* Γ *is generated by finitely many reflections.*

ii) *The quotient morphism* $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma\backslash\mathbb{B}}$ *is a Galois-Reflection covering.*

iii) *The Galois group* $\Gamma/\Gamma(2)$ *is isomorphic to* $Z_2 \times S_4$*, where* S_4 *is the symmetric group of* 4 *elements.*

iv) *Altogether we dispose on the normal Galois-Reflection series*

$$\Gamma(2) \lhd \Gamma' \lhd \Gamma(1-i) \lhd \Gamma$$

of the Galois-Reflection (*covering*) *tower*

$$\Gamma(2)\backslash\mathbb{B} \longrightarrow \Gamma'\backslash\mathbb{B} \longrightarrow \Gamma(1-i)\backslash\mathbb{B} \longrightarrow \Gamma\backslash\mathbb{B}$$

with normal factors (*Galois groups*) Z_2, $K4$, S_3*, or of compositions:*

$$Z_2 \times K4 \cong Gal(\Gamma(2)\backslash\mathbb{B} \to \Gamma(1-i)\backslash\mathbb{B}) \quad , \quad S_4 \cong Gal(\Gamma'\backslash\mathbb{B} \to \Gamma\backslash\mathbb{B}).$$

Proof. i) We know that $\Gamma(1-i)\backslash\mathbb{B}$ is smooth as open part of $\mathbb{P}^2$. Then, from Theorem 2.11 follows that $\Gamma(1-i)$ is generated by finitely many reflections, say $\rho_1, \ldots, \rho_k$. With A) we get Γ, if we add (generators of) Σ to $\Gamma(1-i)$. With the notations of A) we receive

$$\Gamma = \langle \rho_1, \ldots, \rho_k, \sigma_a, \sigma_b \rangle.$$

ii) Abstractly, this follows immediately from i). Explicitly we dispose on the presentation

$$\Gamma/\Gamma(2) = \langle \bar{\sigma_0}, \bar{\sigma_1}, \bar{\sigma_2}, \bar{\sigma_a}, \bar{\sigma_b} \rangle \tag{12}$$

where $\bar{\sigma}$ denotes the $\Gamma(2)$-coset of σ, and we use the reflections defined in A) and B).

iii) By direct computation using the explicit representations in appendix Section 7 one checks first that $\bar{\sigma}_0$ commutes with all the other four generators in (12). Further direct computations yield isomorphic short exact sequences, where $K4$ below denotes the normal subgroup of all products of two disjoint transpositions in the symmetric group S_4.

$$\begin{tikzcd} \langle \bar{\sigma_1}, \bar{\sigma_2} \rangle \arrow[r] \arrow[d, "||"] & \langle \bar{\sigma_1}, \bar{\sigma_2}, \bar{\sigma_a}, \bar{\sigma_b} \rangle \arrow[r] \arrow[d, "\sim"] & \langle \bar{\sigma_a}, \bar{\sigma_b} \rangle \arrow[d, "\sim"] \\ K4 \arrow[r] & S_4 \arrow[r] & S_3 \,. \end{tikzcd} \tag{13}$$

iv) For the S_3-part look back to A), (7) with proven isomorphy. The $Z_2 \times K4$ -part one can find in B), especially (11). □

For the next corollary we need a further reflection, namely the orthogonal reflection of the norm-1 vector $\mathfrak{n}_3 = (1+i, 0, 1)$. We find the corresponding order-4 reflection σ_3 in a similar manner as σ_0 in B). Its Γ-representation you can find in the appendix Section 7 again.

Remark 3.2. The symmetric group S_4 has a well-known representation as motion group $\mathbb{O}$ of the octahedron. With a 3-dimensionally drawn curve configuration in Section 6 it will be geometrically visible.

Corollary 3.3.

1) *The following two sets coincide:*

$$\{\Gamma(1-i)\text{-reflection discs}\} = \{\mathbb{D}_v;\ v \in \Lambda \text{ a primitive norm-1 vector}\}.$$

2) *The set of $\Gamma(1-i)$-reflection discs on $\mathbb{B}$ coincide with the set of $\Gamma(2)$-reflection discs.*
3) *Each $\Gamma(2)$-reflection is a squares of a $\Gamma(1-i)$ reflection of order 4.*
4) *The reflection disc $\mathbb{D}_0$ of σ_0 projects to the Apollonius quadric C_0 along $p : \mathbb{B} \to \Gamma(1-i)\backslash\mathbb{B}$.*
5) *For $i = 1, 2, 3$ the reflection discs $\mathbb{D}_i$ of σ_i project to the 3 Apollonius tangent lines C_1, C_2, C_3, respectively, along p.*
6) *The branch curve of the Galois covering*

$$\hat{f} : \widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}} = \mathbb{P}^2$$

is the Appollonius curve $C_0 + C_1 + C_2 + C_3$. The covering has ramification index 2 over each component C_i.

For a visualization we refer to Picture 5 in Section 5 again. The key of proof is the following statement presented in [HPV],[BMG]:

Theorem 3.4. *The Apollonius curve $C_0+C_1+C_2+C_3$ is the (Baily-Borel compactified) branch curve of p. More precisely, $4C_0+4C_1+4C_2+4C_3$ is the orbital branch divisor of p. This means that the branch order is 4 over all components C_i. All reflections in $\Gamma\setminus\Gamma(1-i)$ have order 2. Each of them is Γ-conjugated to one of the three reflections of Σ_3.* □

Proof of Corollary 3.3. 1) $\subseteq$: If $\mathbb{D}$ is a $\Gamma(1-i)$-reflection, then it belongs, by definition, to the ramification locus of p on $\mathbb{B}$. This means, that its image C belongs to the branch locus. But then, by Theorem 3.4, it is one of the above C_j, $j\in\{1,\dots,4\}$. It follows that $\mathbb{D}=\mathbb{D}_v$ belongs to the $\Gamma(1-i)$-orbit of the reflection disc $\mathbb{D}_j$ of σ_j. Then the normal vector v of $\mathbb{D}$ belongs to the orbit $\Gamma(1-i)\mathfrak{n}_j$. We conclude that $norm(v)=\ norm(\mathfrak{n}_j)=1$.

$\supseteq$: If we start with a reflection disc $\mathbb{D}_{\mathfrak{n}}$ of a norm-1 vector $\mathfrak{n}\in\Lambda$, then we can construct the order-4 reflection $\sigma_{\mathfrak{n}}$ as we did in (8) for σ_0. It belongs to $\Gamma(1-i)$ because $\Gamma\setminus\Gamma(1-i)$ contains only order-2 reflections.

2) $\subseteq$: A $\Gamma(1-i)$-reflection disc $\mathbb{D}$ has a generating reflection σ of order 4. Its square belongs to $\Gamma(2)$ (easy congruence calculation with a Γ-representation). Therefore $\mathbb{D}$ is also a $\Gamma(2)$-reflection disc.

$\supseteq$: Obviously, by inclusion $\Gamma(2)\subset\Gamma(1-i)$.

3) Let s be a $\Gamma(2)$-reflection with reflection disc $\mathbb{D}$. Since it is a $\Gamma(1-i)$-reflection disc, its reflection group has, by the proof of 1), a generating element σ of order 4. Therefore $s=\sigma^2$.

4) The reflection disc $\mathbb{D}_0$ with p-image C_0 has been constructed in [HPV], see also [BMG].

5) The three other order-4 reflection discs $\mathbb{D}_1,\mathbb{D}_2,\mathbb{D}_3$ are neither $\Gamma(1-i)$-equivalent to $\mathbb{D}_0$ nor to each other, because their ortho-vectors $\mathfrak{n}_i$ are not. You can check it simply with modulo 2 calculations. Therefore their p-images are C_1,C_2,C_3, respectively, for a suitable numeration. Namely, by the Theorem 3.4, there is no other possibility.

6) We omit the cusp points and decompose p in

$$\begin{tikzcd} \mathbb{B} \arrow[d, "p'"'] \arrow[dr, "p"] & \\ \Gamma(2)\backslash\mathbb{B} \arrow[r, "f"'] & \Gamma(1-i)\backslash\mathbb{B}\,. \end{tikzcd}$$

The quotient maps p' and p have the same ramification locus joining all reflection discs of $\Gamma(1-i)$. Let $\mathbb{D}$ be one of them, $C'=p'(\mathbb{D})$, $C=p(\mathbb{D})$. The ramification orders of p' and p at $\mathbb{D}$ coincide with the order of a generating $\Gamma(2)$- or $\Gamma(1-i)$-reflection at $\mathbb{D}$, respectively. The former order is 2, the latter equal to 4; both

by 3) and Theorem 3.4, which restricts the maximal $\Gamma(1-i)$-reflection order to 4. Ramification indices v behave multiplicatively along compositions of coverings. Especially, we have

$$4 = v(\mathbb{D} \to C) = v(\mathbb{D} \to C') \cdot v(C' \to C) = 2 \cdot v(C' \to C).$$

Now it is clear that $v(C' \to C) = 2$. This happens iff C belongs to branch locus of p. This branch locus coincides with $C_0 + C_1 + C_2 + C_3$.

The corollary is proved. □

4. The harmonic model of $\widehat{\Gamma(2)\backslash\mathbb{B}}$

Our next goal is to obtain a fine Kodaira classification of the surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$, based on results of the previous two sections and from the works of K. Matsumoto [Mat], T. Riedel [Ri] and M. Uludag [Ul].

In [Mat] and [Ri], Matsumoto and Riedel study a ball quotient surface $\widehat{\Gamma_M\backslash\mathbb{B}}$, where Γ_M is a subgroup of index 2 of $\Gamma(1-i)$ and the degree 2 covering $\widehat{\Gamma_M\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is ramified exactly over the Apollonius' quadric C_0. On the other hand $\Gamma'' = \langle\Gamma(2), \sigma_1, \sigma_2\rangle$, Diagram (10), is also an index 2 subgroup of $\Gamma(1-i)$ and the covering $\widehat{\Gamma''\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ has C_0 as branch locus, Corollary 3.3. Therefore, according to the Cyclic Cover Theorem, [EPD], the two coverings $\widehat{\Gamma_M\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ and $\widehat{\Gamma''\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$, being both of degree 2 with branch locus C_0, are the same, hence $\Gamma_M = \Gamma''$.

The next ball quotient surface we are interesting in is $\widehat{\Gamma_U\backslash\mathbb{B}}$. In [Ul], M. Uludag has constructed an infinite tower of finite coverings of ball quotient surfaces, all of them equal to $\mathbb{P}^2$. This particular surface, which we call *Uludag's surface*, is a part of the tower and is defined as a degree four covering of the Apollonius' $\mathbb{P}^2$, ramified over the three tangent lines C_1, C_2, C_3. We consider again the group $\Gamma' = \langle\Gamma(2), \sigma_0\rangle$ of index four in $\Gamma(1-i)$, Diagram (10). By Corollary 3.3, $\widehat{\Gamma'\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is a degree four covering with branch locus C_1, C_2, C_3. According to the Extension Theorem of Grauert and Remmert, [GR], the two coverings $\widehat{G_U\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ and $\widehat{\Gamma'\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$, both of degree four with the same unramified (affine) part and the same branch locus, are equal, wherefrom $G_U = \Gamma'$.

Following results from the previous sections there are two ways to construct $\widehat{\Gamma(2)\backslash\mathbb{B}}$ from $\widehat{\Gamma(1-i)\backslash\mathbb{B}}$: as a degree four covering of $\widehat{\Gamma''\backslash\mathbb{B}}$, or as a degree two covering of the surface $\widehat{\Gamma'\backslash\mathbb{B}}$. The two lifts of the Apollonius $\mathbb{P}^2$ are compositions of coverings of degree 8, with corresponding Galois group for the whole covering in each one of the cases $Z_2 \times Z_2 \times Z_2$, and are ramified exactly over the Apollonius configuration. The Galois group $\Gamma(1-i)/\Gamma(2)$ is generated by $\overline{\sigma}_0, \overline{\sigma}_1, \overline{\sigma}_2$. The surface covering $\widehat{\Gamma''\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is of degree 2 with Galois group generated by $\overline{\sigma}_0$ and ramified over C_0, and $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma''\backslash\mathbb{B}}$ is of degree 4 with corresponding Galois group generated by $\overline{\sigma}_1, \overline{\sigma}_2$ and ramified over the preimages of the curves

C_1, C_2, C_3 on $\widehat{\Gamma''\backslash\mathbb{B}}$. On the other hand the covering $\widehat{\Gamma'\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is of degree 4, ramified over C_1, C_2, C_3, with Galois group generated by $\overline{\sigma}_1, \overline{\sigma}_2$, and that of $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma'\backslash\mathbb{B}}$ is generated by $\overline{\sigma}_0$ and the map is ramified over the preimage of C_0 on $\widehat{\Gamma'\backslash\mathbb{B}}$. Hence both paths lift the Apollonius $\mathbb{P}^2$ to the surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$ as visualized by the following diagram:

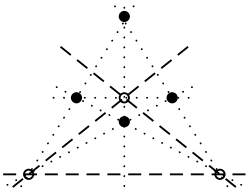
$$\begin{array}{cccc} & \widehat{\Gamma(2)\backslash\mathbb{B}} & \to & \widehat{\Gamma''\backslash\mathbb{B}} & \text{(Matsumoto)} \\ & \downarrow & & \downarrow & \\ \text{(Uludag)} & \widehat{\Gamma'\backslash\mathbb{B}} & \to & \widehat{\Gamma(1-i)\backslash\mathbb{B}} & \text{(Apollonius).} \end{array}$$

In order to obtain the Kodaira classification of the surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$, we need a non singular model which can be obtained by the blow up of the cusps, and which we denote with $(\Gamma(2)\backslash\mathbb{B})'$. The aim is by series of blow downs to obtain from the smooth model a minimal model for the surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$.

In this way we come to the minimal rational surface $\mathbb{P}^2$ together with a line arrangement called the *harmonic configuration*, which is the image of the branch divisor of $(\Gamma(2)\backslash\mathbb{B})'$ with respect to the ball uniformization map. The harmonic configuration is an highly symmetric arrangement, consisting of 9 lines through 7 points. It can be used for the construction of a quadruple of harmonic points in $\mathbb{P}^2$, well studied in the classical projective geometry, as an example in [Har2].

Picture 1

$\mathbb{P}^2$



Harmonic Configuration

To show that $\widehat{\Gamma(2)\backslash\mathbb{B}}$ is a rational surface we use the following technical tools:

1. The Extension Theorem of Grauert and Remmert, [GR], Theorem 8, which we apply in the following situation, where all varieties we consider are complex and normal:
   Let $W^\circ \to V^\circ$ be a finite covering and V be a compactification, then there exists a unique extension of $W^\circ \to V^\circ$ to a finite covering $W \to V$.

$$\begin{array}{ccc} W^\circ & \hookrightarrow & W \\ \downarrow & & \downarrow \\ V^\circ & \hookrightarrow & V \end{array}$$

2. Compatibility of finite coverings and blow ups.
   This property of surface coverings, that finite coverings and blow ups commute, follows from a celebrated theorem of Stein, Stein Factorization Theorem, which can be found in [Har1], p. 280.

Next we come back to our particular surfaces and we consider the tower of finite coverings

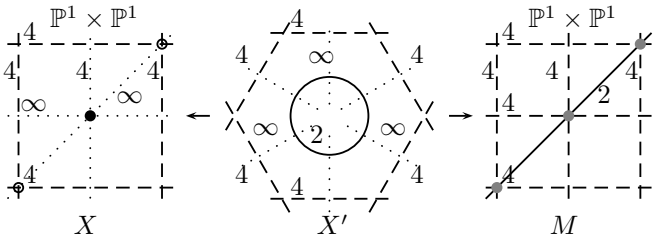
$$\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma''\backslash\mathbb{B}} \to \Gamma(\widehat{1-i})\backslash\mathbb{B},$$

corresponding to the Galois-Reflection tower of $\Gamma(2) \lhd \Gamma(1-i)$ (Diagr. (10), (11)). The Galois groups are $\Gamma(1-i)/\Gamma'' = Z_2$ and $\Gamma''/\Gamma(2) = K4$, as shown in the last chapter, and the branch locus for the composition covering $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \Gamma(\widehat{1-i})\backslash\mathbb{B}$ is the Apollonius curve (Cor. 3.3).

The ball quotient $\widehat{\Gamma''\backslash\mathbb{B}}$, as shown by Matsumoto and Riedel, is the orbital surface $M = (\mathbb{P}^1 \times \mathbb{P}^1, 4V_1' + 4V_2' + 4V_3' + 4H_1' + 4H_2' + 4H_3' + 2D')$ with three cusp points, which are intersection of more than two lines from the orbital divisor. If we blow up the cusps we obtain the surface X'. According to Yoshida, [Yo], (p. 139), this is a projective algebraic surface, which can be also realized by a blow up of four points of $\mathbb{P}^2$ in general position, hence it is the del Pezzo surface of degree 5. Considered as a blow up of four points of $\mathbb{P}^2$, X' has been also studied by Bartels, Hirzebruch and Höfer in [BHH]. There they have shown, by proving the proportionality law, that it is a Baily-Borel compactification of a ball quotient surface (number 20 in their list, (p. 201)). The branch configuration on X' with respect to the natural ball projection is given by a configuration of ten lines, six of them with branch index 4, one with 2, and three with ∞.

If we blow down 3 curves from X', two with branch index 4 and one with 2, we obtain [Yo] the orbital surface $X = (\mathbb{P}^1 \times \mathbb{P}^1, 4V_1 + 4V_2 + 4H_1 + 4H_2)$, where $V_i,\ H_i\ i = 1, 2$ denote vertical resp. horizontal lines. Therefore, X is birationally equivalent to the surface $\widehat{\Gamma''\backslash\mathbb{B}}$.

Picture 2



Let $(\Gamma(2)\backslash\mathbb{B})'$ be the surface obtained from $\widehat{\Gamma(2)\backslash\mathbb{B}}$ by blow up of the cusp points. With *cusp curves* we denote the irreducible exceptional curves plugged in for the cusp points, see [BSA].

Lemma 4.1. *The covering* $(\Gamma(2)\backslash\mathbb{B})' \to X'$ *is unramified over the cusp curves in the Hirzebruch's orbital del Pezzo surface* X'.

Proof. According to [Fe] the surface $\widehat{\Gamma(1)\backslash\mathbb{B}}$ has only one cusp, so the Galois Group $\Gamma(1)/\Gamma(2)$ acts transitively on the cusps set of $\widehat{\Gamma(2)\backslash\mathbb{B}}$, and transforms small neighborhoods of a cusp in a neighborhood of a cusp again. Hence it is enough to consider only the ball cusp point $\kappa = (1:0:1)$. The canonical homomorphism $\phi : \Gamma(1-i) \to G = \Gamma(1-i)/\Gamma(2)$ induces for each point P on $\mathbb{B}$ a surjective homomorphism of isotropy groups $\phi_P : \Gamma(1-i)_P \to G_{P'}$, where P' is the image of P on $\widehat{\Gamma(2)\backslash\mathbb{B}}$ [BSA], (4.6.2). The Galois group $\Gamma(1-i)/\Gamma(2)$ is generated by $\overline{\sigma}_0, \overline{\sigma}_1, \overline{\sigma}_2$ (see (9)). The preimages of the $\overline{\sigma}_0$, $\overline{\sigma}_1$, $\overline{\sigma}_2$ act on κ as $\sigma_0(\kappa) = \kappa$, $\sigma_1(\kappa) = (i:0:1)$, and $\sigma_2(\kappa) = \kappa$. The two cusp points $\kappa = (1:0:1)$ and $(i:0:1)$ are non equivalent modulo 2. Hence the image point κ' of the cusp κ on $\widehat{\Gamma(2)\backslash\mathbb{B}}$ has an isotropy group $\langle\overline{\sigma}_0, \overline{\sigma}_2\rangle \cong Z_2 \times Z_2$.

Following [BSA], (4.5.3), the cusp curve $L_{\kappa'}$ is a rational curve, because the cusp group $\Gamma(2)_\kappa$ is not torsion free, i.e., it contains a reflection, e.g., σ_2^2.

We consider the covering tower

$$(\Gamma(2)\backslash\mathbb{B})' \to (\Gamma'\backslash\mathbb{B})' \to (\Gamma(1-i)\backslash\mathbb{B})',$$

and especially its restriction to the cusp curve $L_{\kappa'}$ in order to show that it is not a ramification curve. For this we study the action of the isotropy group of κ' on $L_{\kappa'}$. $C_0+C_1+C_2+C_3$ is the branch divisor of p, (see Thm. 3.4), and $\widehat{\Gamma'\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is a degree 4 covering branched along $C_1,\ C_2,\ C_3$ [Ul]. According to [Ul] the quadric C_0 has exactly 4 lines as preimages by the whole covering p, and 2 of them intersect $L_{\kappa'}$ in different points. $\overline{\sigma}_0$ acts identically on the preimages of C_0 on $(\Gamma(2)\backslash\mathbb{B})'$, but the extension of the action of $\overline{\sigma}_0$ in the tangential space of the intersection points implies different reflections directions, so $\overline{\sigma}_0$ is not the id on $L_{\kappa'}$.

The group $K4 = \langle\overline{\sigma}_1,\ \overline{\sigma}_2\rangle$ (see Prop. 2.1) acts transitively on the preimages of C_0 on $\widehat{\Gamma'\backslash\mathbb{B}}$. $\overline{\sigma}_0$ fixes the intersection points of these curves with L, where L is the corresponding to κ exceptional curve on $(\Gamma'\backslash\mathbb{B})'$, and $\overline{\sigma}_2$ interchanges these intersection points, so does the composition $\overline{\sigma}_0\overline{\sigma}_2$. The same is true for the preimages of the intersection points on $(\Gamma(2)\backslash\mathbb{B})'$. Hence $L_{\kappa'}$ is not fixed by $\overline{\sigma}_0$, $\overline{\sigma}_2$ or their composition and is not a ramification curve, for the whole covering $(\Gamma(2)\backslash\mathbb{B})' \to \Gamma(1-i)\backslash\mathbb{B})'$ and for every part extension. □

Now, it is clear that the orbital branch locus on $X = \mathbb{P}^1 \times \mathbb{P}^1$, transferred from X', sits on fibres (see above Picture 2). In opposite to the orbital surfaces X' and M it is easy now to find the $K4$-covering of X with prescribed weighted branch curves. For this purpose we consider a rational quadric Q with $Q \to \mathbb{P}^1$ of degree 2, branched over 0 and ∞. The product $Q \times Q \to \mathbb{P}^1 \times \mathbb{P}^1$ is a degree four covering with Galois group $K4$, generated by $g \times id$ and $id \times g$, where $\langle g\rangle$ is the Galois group of $Q \to \mathbb{P}^1$. Because Q is birationally equivalent to the projective line, the above covering is birationally equivalent to $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. The branch locus is the orbital divisor $4V_1 + 4V_2 + 4H_1 + 4H_2$ and is lifted as $2\overline{V}_0 + 2\overline{V}_\infty + 2\overline{H}_0 + 2\overline{H}_\infty$

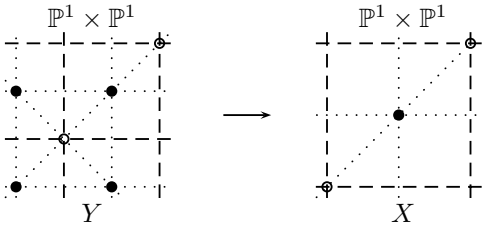
with vertical lines $\overline{V}_0$ and $\overline{V}_\infty$ through 0 and ∞, and the corresponding horizontal lines $\overline{H}_0$ and $\overline{H}_\infty$.

Conversely if we consider a $K4$ quotient of the surface $Q \times Q$ we obtain again the surface X.

$$(Q \times Q)/K4 = (Q/\langle g \rangle) \times (Q/\langle g \rangle) \simeq \mathbb{P}^1 \times \mathbb{P}^1.$$

This $K4$-covering of X is denoted with Y.

Picture 3



We denote with Y', the surface obtained after a blow up of the 6 points, which are intersection of more than 2 lines on Y, as shown in Picture 3.

Proposition 4.2. $\widehat{\Gamma(2)\backslash\mathbb{B}}$ *is birationally equivalent to* Y.

Proof. Consider the following diagram:

$$\begin{tikzcd} Y \arrow[d] \arrow[dr] & \\ X & X'. \arrow[l] \end{tikzcd}$$

Let Y° be the surface Y without the line arrangement of 4 dashed and 6 dotted lines and X° the surface obtained from X by removing the 4 dashed and 3 dotted lines, or from X' again by removing the configuration of 10 curves. From the fact that X' is a compactification of X° it follows by the Extension Theorem of Grauert and Remmert that the finite covering $Y^\circ \to X^\circ$ can be extended in an unique way (up to isomorphism) to the $K4$-covering $Y'' \to X'$. Therefore, $Y'' \to X'$ is the unique extension of the finite covering $Y \to X$, which completes the above diagram.

Because of the compatibility of finite coverings with blow ups, the map $Y \leftarrow Y''$ is exactly the blow up of those points on Y, which lie over the 3 thick points of X, blown up by the map $X \leftarrow X'$. This is exactly the definition of Y', hence $Y'' = Y'$, wherefrom we obtain that Y' is a $K4$-covering of the Hirzebruch's surface X'.

On the other hand let us consider the following diagram:

$$\begin{tikzcd} (\Gamma(2)\backslash\mathbb{B})' \arrow[d] \arrow[dr] & \\ X & M. \arrow[l] \end{tikzcd}$$

The Hirzebruch's list, [BSA], p. 201, gives the branch locus for the $K4$-covering $(\Gamma(2)\backslash\mathbb{B})' \to X'$, consisting of 7 lines, 6 dashed and 1 black, as represented in Picture 2, all of ramification index 2. The 3 dotted lines, which complete the picture are not branch curves according to Lemma 4.1.

Let X° be as above X' without the line configuration of 10 curves and M° be M without the 7 curves (6 dashed and 1 black, Pic. 2), then $X^\circ = M^\circ$. By the Extension Theorem there exists an unique extension of $Y^\circ \to X^\circ$ to a $K4$-covering $Y' \to X'$. On the other hand $(\Gamma(2)\backslash\mathbb{B})^\circ \to X'$, where $(\Gamma(2)\backslash\mathbb{B})^\circ$ is $(\Gamma(2)\backslash\mathbb{B})'$ without the line arrangement obtained by the $K4$-lift of the curve configuration on X', is again an extension of $Y^\circ \to M^\circ = X^\circ$, hence the both extensions are the same, i.e., $Y' = (\Gamma(2)\backslash\mathbb{B})'$.

As a consequence we obtain the following commutative diagram of surfaces, where the vertical maps are $K4$ coverings and the horizontal are birational transformations:

$$\begin{array}{ccccc} Y & \dashleftarrow & (\Gamma(2)\backslash\mathbb{B})' & \dashrightarrow & \widehat{\Gamma(2)\backslash\mathbb{B}} \\ \downarrow & & \downarrow & & \downarrow \\ X & \dashleftarrow & X' & \dashrightarrow & M. \end{array}$$

Therefore, the surfaces Y and $\widehat{\Gamma(2)\backslash\mathbb{B}}$ are birationally equivalent. The line configuration of 10 curves on X' is lifted as the arrangement of 16 lines, four (black) of weight 2, six (dashed) of weight 2, six (dotted) of weight ∞, which come after blow up of the cusp of $\widehat{\Gamma(2)\backslash\mathbb{B}}$. □

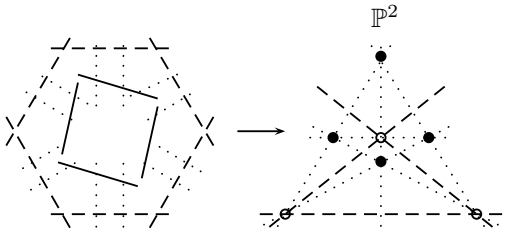
With the results of the former proposition now we are able to prove the following statement.

Theorem 4.3. $(\Gamma(2)\backslash\mathbb{B})'$ *is the surface obtained as a blow up of seven points on* $\mathbb{P}^2$. *The line arrangement on* $(\Gamma(2)\backslash\mathbb{B})'$ *is the preimage of the harmonic configuration.*

Proof. The surface $(\Gamma(2)\backslash\mathbb{B})'$ can be obtained from Y by blow up of the six points, which are intersection of at least three lines.

Y itself is a model of $\widehat{\Gamma(2)\backslash\mathbb{B}}$ given by $\mathbb{P}^1 \times \mathbb{P}^1$ together with the line configuration $2\overline{V}_0 + 2\overline{V}_\infty + 2\overline{H}_0 + 2\overline{H}_\infty$. By blow up of the intersection point of two dashed lines and one dotted, in the line arrangement on Y, and afterwards blow down of the dashed lines $\overline{V}_\infty$ and $\overline{H}_\infty$ going through this point one obtains the projective plane. Hence $(\Gamma(2)\backslash\mathbb{B})'$ can be constructed from $\mathbb{P}^2$ by blowing up the 7 thick points of the harmonic line configuration on $\mathbb{P}^2$ as represented in the following picture.

Picture 4



$(\Gamma(2)\backslash\mathbb{B})'$ Harmonic Configuration

□

At the end of this section we want to remark that the detailed study of the Galois groups of the towers of surface coverings $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma''\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ as well as $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma'\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}}$ proves that the natural congruence subgroup $\Gamma(2)$ is contained in the groups Γ', studied by Hirzebruch, Matsumoto and Riedel, and Γ'', corresponding to the Uludag's surface, which leads to the following result:

Corollary 4.4. *The two groups Γ' and Γ'' are Picard congruence subgroups.*

Corollary 4.5. *The natural Picard congruence subgroup $\Gamma(2)$ is generated by finitely many order-2 reflections.*

Proof. By Theorem 4.3 the quotient surface $\Gamma(2)\backslash\mathbb{B}$ is simply-connected. It is also smooth. Now we apply the second statement of Theorem 2.11 to see that our group is generated by finitely many reflections. At the begin of B) in Section 3 we already remarked that $\Gamma(2)$ contains only reflections of order 2. This finishes the proof. □

5. Numerical space model

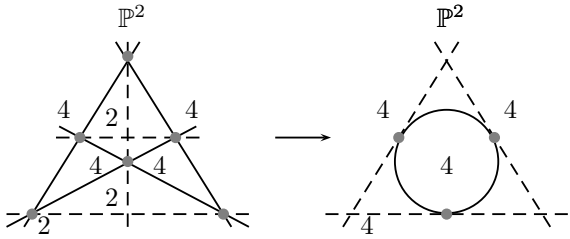
In this section we would like to compute a numerical model for $\widehat{\Gamma(2)\backslash\mathbb{B}}$. For this we consider the covering

$$\widehat{\Gamma(2)\backslash\mathbb{B}} \to \widehat{\Gamma'\backslash\mathbb{B}} \to \widehat{\Gamma(1-i)\backslash\mathbb{B}},$$

from Diagram (11), with Galois groups $\Gamma'/\Gamma(2) = Z_2$ and $\Gamma(1-i)/\Gamma' = K4$ (Diagram (10)).

$\widehat{\Gamma(1-i)\backslash\mathbb{B}}$ is the orbital surface $(\mathbb{P}^2, 4C_0 + 4C_1 + 4C_2 + 4C_3)$. The three tangents $C_1,\ C_2,\ C_3$ can be given for example by the equations $x' = 0,\ y' = 0,\ z' = 0$ and the quadric C_0 by $(x' + y' - z')^2 - 4x'y' = 0$. The Uludag's surface $\widehat{\Gamma'\backslash\mathbb{B}}$ is the orbital surface $(\mathbb{P}^2, 4G_1 + 4G_2 + 4G_3 + 4G_4 + 2B_1 + 2B_2 + 2B_3)$. It is a degree four covering of the Apollonius $\mathbb{P}^2$, ramified along the tangents. C_0 is lifted by this covering as the curve $(x+y-z)(x+y+z)(x-y+z)(x-y-z) = 0$, where each irreducible component is of branch index 4. The tangents, defining the branch locus, are lifted as lines of branch index 2.

Picture 5



Uludag's Configuration Apollonius Configuration

The Picard group of $\mathbb{P}^2$ is generated by a line, hence the divisor class of the four lines $G_1+G_2+G_3+G_4$ is divisible by 2 in $Pic(\mathbb{P}^2)$. Then according to the cyclic cover theorem, see, e.g., [EPD], there exists exactly one degree two covering of the Uludag's surface, ramified along these lines and this surface is exactly $\widehat{\Gamma(2)\backslash\mathbb{B}}$. The covering $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \mathbb{P}^2$-Uludag's is cyclic with Galois group Z_2. The surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$ is obtained as a normalisation of $\mathbb{P}^2$ along the function fields extensions $\mathbb{C}(\mathbb{P}^2) \subset \mathbb{C}(\widehat{\Gamma(2)\backslash\mathbb{B}})$. Using Kummer extensions theory [Ne] we obtain $\mathbb{C}(\widehat{\Gamma(2)\backslash\mathbb{B}}) = \mathbb{C}(x,y)(\sqrt{\delta})$, where $\delta = (x+y-1)(x+y+1)(x-y+1)(x-y-1)$ is the affine divisor corresponding to the branch divisor of the covering $\widehat{\Gamma(2)\backslash\mathbb{B}} \to \mathbb{P}^2$- Uludag's. If we set $u^2 = \delta$, we obtain by projectivisation for the surface $\widehat{\Gamma(2)\backslash\mathbb{B}}$ the following numerical model:

$$\widehat{\Gamma(2)\backslash\mathbb{B}}: \ t^2u^2 + 2x^2t^2 + 2x^2y^2 + 2y^2t^2 - t^4 - x^4 - y^4 = 0.$$

This space model enables the computation of explicit equations for various Shimura curves, important for the coding theory. In the central part of her doctoral thesis [Pet] the second author connects towers of such curves inside of our octahedral Picard surface tower. They are constructed as quotients of "arithmetic subdiscs" of the 2-ball.

6. The octahedral configuration of norm-1 curves

We call an orbital ball quotient surface $\Gamma\backslash\mathbb{B}$ (also its compactification) *neat*, if the ball lattice Γ is neat. In this case $\mathbb{B} \to \Gamma\backslash\mathbb{B}$ is a universal covering.

From Hirzebruch's work in the 1980s, see, e.g., [Hi], and a systematic study in [Ho04] we know that there exist coabelian neat ball lattices Γ. *Coabelian* means that the quotient surface $\Gamma\backslash\mathbb{B}$ has an abelian surface as model. We found the following general situation:

Let A be an abelian surface, $T = T_1 + \cdots + T_k$ a sum of elliptic curves T_i on A with pairwise normal crossings at intersection points. We denote by s the number $\#\,\mathrm{Sing}(T)$ of curve singularities of T and set

$$S_i \ := \mathrm{Sing}(T) \cap T_i \ , \ s_i \ := \#S_i; \ i = 1, \ldots, k.$$

By the adjunction formula for curves on smooth surfaces, it is easy to see that the selfintersection indices of elliptic curves on abelian surfaces vanishes. We assume, that $S_i \neq \emptyset$ for all i. If we blow up each curve singularity of T, we get a surface A' with s exceptional lines of first kind. The proper transforms of the T_i on A' we denote by the same symbol. They do not intersect each other and have negative selfintersections. Therefore we can contract them all to elliptic singularities. On this way we get a surface $\hat{A}$ with k singularities $\hat{\kappa}_i$. We put together the whole construction in the following diagram:

$$\begin{tikzcd} A & A' \arrow[l] \arrow[r] & \hat{A} \\ T_i \arrow[u] \arrow[r, "="] & T_i \arrow[u] \arrow[r] & \hat{\kappa}_i \arrow[u] \end{tikzcd} \tag{14}$$

with vertical inclusions. We proved

Theorem 6.1 ([Ho04], Theorem 2.5). *With the above notations/assumptons, $\hat{A}$ is a neat ball quotient surface $\widehat{\Gamma\backslash\mathbb{B}}$ with cusp singularities $\hat{\kappa}_i$, if and only if the relation*

$$4s = s_1 + \cdots + s_k \tag{15}$$

is satisfied.

Now we start again from the biproduct $\mathbb{P}^1 \times \mathbb{P}^1$, endowed with three horizontal lines and three verticals as drawn in Picture 3 of Section 4 (on the right, without diagonal). We consider the (unique) 4-cyclic cover of $\mathbb{P}^1$ branched over three points: namely the elliptic CM-curve $E = \mathbb{C}/\mathbb{Z}[i]$ with cyclic automorphism group Z_4 of order 4 generated by the i-multiplication. The corresponding Galois covering (with intermediate step)

$$E \longrightarrow E/\langle -id_E \rangle = \mathbb{P}^1 \longrightarrow E/Z_4 = \mathbb{P}^1$$

is ramified at the 2-torsion points $Q_0 = O$, Q_2 of ramification order 4 and Q_1, Q_3 of ramification order 2. Their image points on $\mathbb{P}^1$ are denoted by P_0, P_2 or P_1, respectively, preserving indices. Taking bi-products we get a Galois covering of surfaces with Galois group $Z_4 \times Z_4$

$$E \times E \longrightarrow (E \times E)/(Z_4 \times Z_4) = E/Z_4 \times E/Z_4 = \mathbb{P}^1 \times \mathbb{P}^1$$

with ramification curves $Q_i \times E$, $E \times Q_j$, $i, j = 0, \ldots, 3$, and branch curves $P_i \times \mathbb{P}^1$, $\mathbb{P}^1 \times P_j$, $i, j = 0, \ldots, 2$. More precisely, the orbital branch divisor is

$$4 \cdot P_0 \times \mathbb{P}^1 + 4 \cdot P_2 \times \mathbb{P}^1 + 4 \cdot \mathbb{P}^1 \times P_0 + 4 \cdot \mathbb{P}^1 \times P_2 + 2 \cdot P_1 \times \mathbb{P}^1 + 2 \cdot \mathbb{P}^1 \times P_2.$$

The diagonal curve D of $\mathbb{P}^1 \times \mathbb{P}^1$ has 4 irreducible preimage curves $D_1, \ldots, D_4$ on $E \times E$. These are elliptic curves. So the whole divisor

$$T := D_1 + D_2 + D_3 + D_4 + Q_1 \times E + Q_3 \times E + E \times Q_1 + E \times Q_3$$

is a sum of 8 elliptic curves with

$$\mathrm{Sing}(T) = \{O, Q_2 \times Q_2, Q_1 \times Q_1, Q_1 \times Q_3, Q_3 \times Q_1, Q_3 \times Q_3\}.$$

We count $s = 6$ singular points, 4 of them on each T-component D_i and 2 on each horizontal and vertical component. Altogether we see that the relation (15) is satisfied:

$$4 \cdot 6 = 4 + 4 + 4 + 4 + 2 + 2 + 2 + 2.$$

For more calculation details we refer to [Ho04], Example 4.6. It follows from Theorem 6.1 that $E \times E$ is an abelian model of a neat ball quotient surface of a lattice Γ_E with smooth compactification $(E \times E)' = (\Gamma_E \backslash \mathbb{B})'$ received by blowing up the six points of $\mathrm{Sing}(T) \subset E \times E$. Altogether we have the commutative Galois-covering diagram of blow-ups/contractions:

$$\begin{tikzcd}
E \times E \arrow[d, "\langle -id\rangle \times \langle -id\rangle"] \arrow[dr, phantom, "\cong"] & (E \times E)' \arrow[l] \arrow[r] \arrow[d, "Z_2 \times Z_2"] & \widehat{E \times E} \arrow[d] \\
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[d] & (\Gamma(2)\backslash\mathbb{B})' \arrow[l] \arrow[r] \arrow[d, "K4"'] & \widehat{\Gamma(2)\backslash\mathbb{B}} \arrow[d] \\
\mathbb{P}^1 \times \mathbb{P}^1 & (\Gamma(1-i)\backslash\mathbb{B})' \arrow[l] \arrow[r] & \widehat{\Gamma(1-i)}\backslash\mathbb{B}.
\end{tikzcd}$$

The upper row comes, as already mentioned, from Theorem 6.1. The partial diagram of middle and bottom rows was constructed in Section 4. Both parts are joined as drawn, because the blown-up points of $\mathrm{Sing}(T)$ have as image along $\langle -id\rangle \times \langle -id\rangle$ the six image points blown-up in the middle row to get $(\Gamma(2)\backslash\mathbb{B})'$.

Altogether we have a Galois-Reflection tower

$$\Gamma_E\backslash\mathbb{B} \to \Gamma(2)\backslash\mathbb{B} \to \Gamma_M\backslash\mathbb{B} \to \Gamma(1-i)\backslash\mathbb{B} \to \Gamma\backslash\mathbb{B}$$

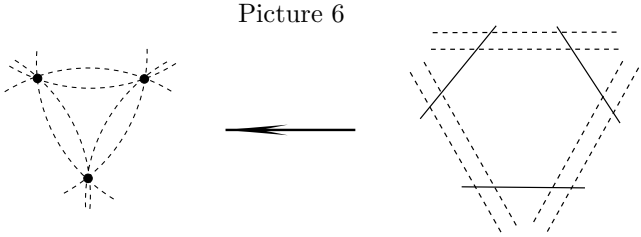
of Picard modular surfaces, which starts with a neat one of abelian type.

Let t be the translation automorphism of $E \times E$ adding to each point $Q \times Q$ the 2-torsion point $Q_1 \times Q_1$. We consider the isogeny

$$E \times E \to (E \times E)/\langle t\rangle \ =: \ B.$$

It is easy to see that t doesn't move the divisor T and the intersection points of their components collected in $\mathrm{Sing}(T)$. The image of the latter points on the abelian surface B consists of three points. The image of T on B consists of 3 elliptic curve pairs. Each of the three points is intersection point of the 4 components of two such pairs. We blow them up, and denote the arising surface by B'. We visualize the transfer of the 6 (here black dotted) elliptic curves along the birational morphism $B \leftarrow B'$:

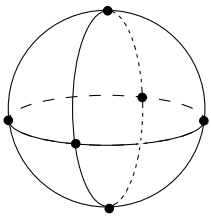
Picture 6



On this way we get the

Globe configuration on the abelian surface model $\hat{B} = \widehat{\Gamma_B \backslash \mathbb{B}}$:
With $s = 3$ and $s_i = 2$, $i = 1, \ldots, 6$ we see that the relation (15) is satisfied again. Therefore, after blowing up the 3 intersection points, we get a neat ball quotient surface compactified by the 6 elliptic curves. Contracting them we get a surface $\hat{B}$ with six cusp singularities painted as black points in Picture 7. Thereby we arrange the (transfers of the) 3 (black) exceptional lines of this picture 3-dimensionally as crossing circles on a globe, reflecting precisely their intersection behaviour. Obviously, the six cusp points span a regular octahedron.

Picture 7



Excercise 6.2. *Find with help of next section the octahedron motion group representations (on $\mathbb{R}^3$) of our Galois-Reflection groups extending $\Gamma(2)$.*

Remark 6.3. The above globe curve configuration is (along our coverings and modifications) a transformation of the Apollonius configuration (consisting of a quadric and 3 tangent lines). By Corollary 3.3, the Apollonius curves are (all) norm-1 curves on $\widehat{\Gamma(1-i)\backslash\mathbb{B}} = \mathbb{P}^2$, defined as quotients of norm-1 subdiscs of $\mathbb{B}$. The latter property doesn't change along correspondence transformations. Therefore the two meridians and the equator on the above globe represent norm-1 curves on $\hat{B}$.

7. Appendix: Explicit unitary representations

For $\Gamma = \Gamma(1) = \mathbb{SU}((2,1),\mathbb{Z}[i])$ we remember to the sequence of normal group extensions by reflections well defined in Sections 3, 4.

$$\begin{aligned} \Gamma' &= \Gamma_U = \langle \Gamma(2), \sigma_0 \rangle, \quad \text{(recognized as Uludag's)}; \\ \Gamma'' &= \Gamma_M = \langle \Gamma(2), \sigma_1, \sigma_2 \rangle, \quad \text{(rec. as Matsumoto's, Hirzebruch's)}; \\ \Gamma(1-i) &= \langle \Gamma(2), \sigma_1, \sigma_2; \sigma_0 \rangle; \\ \Gamma &= \langle \Gamma(2), \sigma_1, \sigma_2, \sigma_0; \sigma_a, \sigma_b \rangle; \end{aligned} \tag{16}$$

with small abelian factor groups

$$\Gamma'/\Gamma(2) \cong Z_2,\ \Gamma''/\Gamma(2) \cong Z_2 \times Z_2;$$
$$\Gamma(1-i)/\Gamma(2) \cong Z_2 \times Z_2 \times Z_2,\ \Gamma/\Gamma(1-i) \cong S_3.$$

As promised we give the special unitary representations of the reflections. One has only to apply their explicit definitions to the canonical basis of $\mathbb{C}^3$:

$$\begin{aligned} \sigma_0 &= -i \cdot \begin{pmatrix} i & -1+i & 1-i \\ -1+i & i & 1-i \\ -1+i & -1+i & 2-i \end{pmatrix}; \\ \sigma_1 &= i \cdot \begin{pmatrix} i & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 = i \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{pmatrix}; \\ \sigma_a &= \begin{pmatrix} -1 & -1-i & 1+i \\ -1+i & 0 & 1 \\ -1+i & -1 & 2 \end{pmatrix}, \sigma_b = -\begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{17}$$

Proposition 7.1. *The factor group* $\Gamma(1)/\Gamma'$ *is isomorphic to the motion group* $\mathbb{O}$ *of the octahedron. The factor group* $\Gamma(1)/\Gamma(2)$ *is* (*isomorphic to*) *the* double octahedron group $Z_2 \times \mathbb{O} \cong Z_2 \times S_4$.

For the proof one uses a presentation of S_4. The corresponding relations are easily checked by the unitary representation of the generating elements (17). The calculations $\mod {}^{\times}\Gamma(2)$ are left to the reader.

Problem. Find explicitly 2-reflections generating $\Gamma(2)$.

Hint. Matsumoto found in [Mat] explicit generators of $\Gamma'' = \Gamma_M$ using the monodromy of a curve family. Try to present them as products of reflections. This is a finite problem. Then take squares of the order-4 reflection among the factors.

The solution of the problem is important for modular function tests for all arithmetic lattices in (16). In [Mat], or better now in [KS], generating modular forms for Γ_M are explicitly known. The interaction with the octahedron group is very interesting, especially for construction of class fields, see [Ri].

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